

NON-ARITHMETICITY OF LENGTH SPECTRA OF SUBGROUPS OF MAPPING CLASS GROUPS

INHYEOK CHOI AND DONGRYUL M. KIM

ABSTRACT. In this paper, we prove that every non-elementary subgroup of the mapping class group of a surface has non-arithmetic Teichmüller length spectrum. Namely, Teichmüller translation lengths of its pseudo-Anosov elements generate a dense additive subgroup of \mathbb{R} . We prove this by introducing the notion of cross-ratios on \mathcal{MF} and \mathcal{PMF} , and studying its geometric and dynamical properties, despite the lack of negatively curved features of the Teichmüller space nor the conformal geometry on \mathcal{PMF} . As an application, we show topological mixing of the Teichmüller geodesic flow for every non-elementary subgroup.

CONTENTS

1. Introduction	1
2. Non-arithmeticity of Teichmüller length spectra	5
3. Topological mixing of Teichmüller geodesic flow	13
References	16

1. INTRODUCTION

Let S be a connected orientable surface of finite type with negative Euler characteristic. Let

$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S) / \sim$$

be the mapping class group of S , the group of isotopy classes of orientation-preserving homeomorphisms on S .

Teichmüller space $\mathcal{T} = \mathcal{T}(S)$ is the space of all marked Riemann surface structures on S , equipped with a natural metric $d_{\mathcal{T}}$, called the Teichmüller metric. The natural $\mathrm{Mod}(S)$ -action on $(\mathcal{T}, d_{\mathcal{T}})$ given by change of markings is properly discontinuous and by isometries. Hence, for each $g \in \mathrm{Mod}(S)$, its *Teichmüller translation length*

$$\mathrm{len}(g) := \lim_{n \rightarrow +\infty} \frac{d_{\mathcal{T}}(o, g^n o)}{n}, \quad o \in \mathcal{T}$$

is well-defined, independent of the choice of $o \in \mathcal{T}$.

For a subgroup $\Gamma < \text{Mod}(S)$, we call

$$\text{len}(\Gamma) := \{\text{len}(g) : g \in \Gamma \text{ pseudo-Anosov}\}$$

the *Teichmüller length spectrum* of Γ . The quotient space $\text{Mod}(S) \setminus \mathcal{T}$ is the moduli space of Riemann surface structures on S , and the Teichmüller length spectrum $\text{len}(\text{Mod}(S))$ is the same as the set of lengths of all closed geodesics in the moduli space $\text{Mod}(S) \setminus \mathcal{T}$, with respect to the induced metric on it. Similarly, the Teichmüller length spectrum $\text{len}(\Gamma)$ of $\Gamma < \text{Mod}(S)$ is the same as the set of lengths of all closed geodesics in the associated cover $\Gamma \setminus \mathcal{T}$ of the moduli space.

We say that $\Gamma < \text{Mod}(S)$ has *non-arithmetic* Teichmüller length spectrum if the additive subgroup

$$\langle \text{len}(\Gamma) \rangle < \mathbb{R}$$

generated by its Teichmüller length spectrum is dense. We also call $\text{len}(\Gamma)$ non-arithmetic in this case.

A subgroup $\Gamma < \text{Mod}(S)$ is called *non-elementary* if Γ contains two independent pseudo-Anosov mapping classes. Our main theorem is the non-arithmeticity of Teichmüller length spectrum of a non-elementary subgroup.

Theorem 1.1 (Non-arithmeticity). *Let $\Gamma < \text{Mod}(S)$ be a non-elementary subgroup. Then its Teichmüller length spectrum $\text{len}(\Gamma)$ is non-arithmetic.*

Note that the non-elementary hypothesis is necessary; if Γ is a cyclic subgroup generated by a single pseudo-Anosov mapping class, then $\langle \text{len}(\Gamma) \rangle \subset \mathbb{R}$ is a scaled copy of \mathbb{Z} .

In general, the length spectrum can be similarly discussed for a general isometric action on a metric space. Its non-arithmeticity is a fundamentally important property from many geometric and dynamical aspects. Indeed, non-arithmeticity is a necessary ingredient for mixing, equidistribution, counting results for geodesic flows, dynamics of horospherical foliations, and measure classification of horospherical-invariant measures. In general negatively curved settings, those geometry and dynamics were studied in ([Dal00], [Rob03], etc). In the settings of subgroups of mapping class groups and Teichmüller spaces, study of geodesic flows was carried in [GM23a], and measure classifications within \mathcal{ML} was studied in our recent work [CK25]. All of those results assume non-arithmeticity of length spectra.

In the following specific cases of a non-elementary discrete subgroup $\Gamma < \text{Isom}(X)$ of isometries on a $\text{CAT}(-1)$ space X , non-arithmeticity of its length spectrum is known:

- (1) $\Gamma \setminus X$ is a negatively curved surface, due to Dal'bo [Dal99],
- (2) X is a rank-one symmetric space, due to Kim¹ [Kim06].
- (3) the limit set of Γ possesses a non-singleton connected component, due to Bourdon [Bou95].
- (4) Γ contains a parabolic isometry, due to Dal'bo–Peigné [DP98].

¹Inkang Kim, not the second-named author.

A natural generalization to Zariski dense discrete subgroups of higher-rank Lie groups is established by Benoist [Ben00].

Remark 1.2. We also note that, to the best of the authors' knowledge, it is an open problem whether the length spectrum of a non-elementary discrete subgroup $\Gamma < \text{Isom}(X)$ is always non-arithmetic when X is a simply connected Riemannian manifold with curvature at most -1 .

In contrast, Teichmüller space is neither Gromov hyperbolic [MW95] nor homogeneous except for sporadic surfaces. Besides the full mapping class group $\text{Mod}(S)$ which contains a Veech group acting on a Teichmüller disk as a non-elementary Fuchsian group, Theorem 1.1 is the first result for non-arithmeticity of Teichmüller length spectra of subgroups of $\text{Mod}(S)$.

1.1. Topological mixing of Teichmüller geodesic flow. Let $\mathcal{Q}^1\mathcal{T}$ be the bundle of unit-area (holomorphic) quadratic differentials over \mathcal{T} . Then the $\text{Mod}(S)$ -action on \mathcal{T} naturally extends to $\mathcal{Q}^1\mathcal{T}$, and moreover $\mathcal{Q}^1\mathcal{T}$ is identified with the unit cotangent bundle over \mathcal{T} , via a $\text{Mod}(S)$ -equivariant map. In this regard, Teichmüller geodesic flow is naturally defined on $\mathcal{Q}^1\mathcal{T}$ and descends to the quotient $\Gamma \backslash \mathcal{Q}^1\mathcal{T}$ for any subgroup $\Gamma < \text{Mod}(S)$.

Especially, the quotient $\text{Mod}(S) \backslash \mathcal{Q}^1\mathcal{T}$ is identified with the cotangent bundle over the moduli space $\text{Mod}(S) \backslash \mathcal{T}$. Dynamical properties of the Teichmüller geodesic flow on $\text{Mod}(S) \backslash \mathcal{Q}^1\mathcal{T}$, such as mixing results, turned out to play a significant role in the study of counting problems for the mapping class group $\text{Mod}(S)$ ([EM11], [ABEM12], [EMR19], [EMM22], [AH23], etc).

As an application of our non-arithmeticity theorem (Theorem 1.1), we prove that the Teichmüller geodesic flow is topologically mixing for a non-elementary subgroup $\Gamma < \text{Mod}(S)$. Unless Γ is the full mapping class group, interesting dynamics only occurs in a certain subset of $\Gamma \backslash \mathcal{Q}^1\mathcal{T}$ in general. To describe this subset, we note that there exists a unique Γ -minimal subset $\Lambda_\Gamma \subset \mathcal{PMF}$, called the limit set of Γ [MP89]. We then consider the subset

$$\Omega_\Gamma \subset \Gamma \backslash \mathcal{Q}^1\mathcal{T}$$

of quadratic differentials whose associated two projective measured foliations belong to Λ_Γ . Then Ω_Γ is a non-wandering domain for the Teichmüller geodesic flow, and we indeed prove its topological mixing.

Theorem 1.3 (Topological mixing). *Let $\Gamma < \text{Mod}(S)$ be a non-elementary subgroup. Then the Teichmüller geodesic flow on $\Omega_\Gamma \subset \Gamma \backslash \mathcal{Q}^1\mathcal{T}$ is topologically mixing.*

It is originally due to Dal'bo [Dal00] that non-arithmeticity implies the topological mixing of geodesic flow, in negatively curved settings. Our deduction of Theorem 1.3 from Theorem 1.1 basically follows ideas of Dal'bo, while it is based on our recent work [CK25] on the dynamics of non-elementary subgroups acting on \mathcal{MF} .

1.2. On the proof. To prove Theorem 1.1, we introduce the notion of cross-ratios on the space \mathcal{MF} of measured foliations and the space \mathcal{PMF} of projective measured foliations on S , and studying its geometric and dynamical properties. See Definition 2.1 for precise definitions of cross-ratios.

Showing the non-arithmeticity using cross-ratios was already done by Kim² [Kim06] for rank-one symmetric spaces. Indeed for instance in real hyperbolic case, Kim proved that if a discrete subgroup of isometries does not have non-arithmetic length spectrum, then its limit set is contained in a discrete union of proper spheres in the boundary. This can also be seen as a consequence of negatively curved geometry and the conformality of boundary action. Kim then deduced that the discrete subgroup cannot be non-elementary, based on a structure theory of rank-one Lie groups.

In contrast, in the setting of Teichmüller geometry, we do not have a negatively curved geometry, homogeneity of the space, and conformality of the boundary action. To overcome this difficulty, after we introduce the notion of cross-ratios on \mathcal{MF} and \mathcal{PMF} , we also observe useful relationships between cross-ratios and geometric and dynamical properties of limit sets. Moreover, we further investigate the piecewise-linear structure of \mathcal{MF} and localize dynamics by using the theory of train tracks, and then complete the proof of the non-arithmeticity by proceeding with certain inductive arguments.

Remark 1.4. We remark that an approach to proving the non-arithmeticity of Teichmüller length spectra (Theorem 1.1) via train tracks and reduction to linear algebra has been attempted with a mistake in an earlier version [GM23b] of the work of Gekhtman–Ma [GM23a] as well as in Gekhtman’s UChicago Ph.D. thesis [Gek14], which was purported to work even for non-elementary semigroups of mapping class groups. In this current paper, we take a different approach, as described above.

Acknowledgements. The authors thank Francisco Arana-Herrera, Mladen Bestvina, Dick Canary, Ursula Hamenstädt, Yair Minsky, Hee Oh, and Saul Schleimer for helpful conversations. The authors also appreciate Ilya Gekhtman and Biao Ma for valuable communication regarding this problem and comments on our earlier draft.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-2424139, while the authors were in residence at the Simons Laufer Mathematical Sciences Institute in Berkeley, California, during the Spring 2026 semester.

Choi was supported by the Mid-Career Researcher Program (RS-2023-00278510) through the National Research Foundation funded by the government of Korea, and by the KIAS individual grant (MG091901) at KIAS.

²Inkang Kim, not the second-named author.

2. NON-ARITHMETICITY OF TEICHMÜLLER LENGTH SPECTRA

In this section, we prove the non-arithmeticity of Teichmüller length spectra of non-elementary subgroups of $\text{Mod}(S)$ (Theorem 1.1).

2.1. Measured foliations and pseudo-Anosov mapping classes. Before proving Theorem 1.1, let us briefly review some basic facts on measured foliations and pseudo-Anosov mapping classes. For more comprehensive overview, we refer to ([FLP79], [FM12]).

We denote by $\mathcal{MF} = \mathcal{MF}(S)$ the space of equivalence classes of (singular) measured foliations on S , where the equivalence is generated by isotopy and Whitehead moves (see [FLP79, Exposé 5] for details). Each element of \mathcal{MF} can be represented by a pair (\mathcal{F}, μ) of (singular) foliation \mathcal{F} on S and a transverse measure μ .

The mapping class group $\text{Mod}(S)$ acts naturally on \mathcal{MF} by homeomorphisms. According to the Nielsen–Thurston classification ([Nie44], [Thu88], cf. [FLP79, Exposé 9]), a mapping class $g \in \text{Mod}(S)$ is called *pseudo-Anosov* if there exist transverse measured foliations $(\mathcal{F}_u, \mu_u), (\mathcal{F}_s, \mu_s) \in \mathcal{MF}$ and a real number $\lambda_g > 1$ such that

$$g \cdot (\mathcal{F}_u, \mu_u) = (\mathcal{F}_u, \lambda_g \cdot \mu_u) \quad \text{and} \quad g \cdot (\mathcal{F}_s, \mu_s) = (\mathcal{F}_s, \lambda_g^{-1} \cdot \mu_s).$$

We call (\mathcal{F}_u, μ_u) and (\mathcal{F}_s, μ_s) the *unstable and stable measured foliations* for g respectively, and they are uniquely determined up to scaling of the transverse measures. The real number λ_g is called the *stretch factor* of g , and it satisfies

$$\text{len}(g) = \log \lambda_g$$

([Ber78], [Thu88]). Note that $\lambda_{g^{-1}} = \lambda_g^{-1}$.

The space $\mathcal{PMF} = \mathcal{PMF}(S)$ of projective measured foliations on S is defined by

$$\mathcal{PMF} := \mathcal{MF} / \sim$$

where the projectivization is given by scaling transverse measures. Thurston compactified the Teichmüller space \mathcal{T} using \mathcal{PMF} as its boundary. The compactification $\mathcal{T} \sqcup \mathcal{PMF}$ is now referred to as the Thurston compactification, and hence \mathcal{PMF} is also referred to as the Thurston boundary. Thurston also showed that the $\text{Mod}(S)$ -action on \mathcal{T} and the induced $\text{Mod}(S)$ -action on \mathcal{PMF} are glued together and give rise to the $\text{Mod}(S)$ -action on $\mathcal{T} \sqcup \mathcal{PMF}$ by homeomorphisms ([Thu88], [Thu97]).

For a pseudo-Anosov $g \in \text{Mod}(S)$, we choose

$$g^+, g^- \in \mathcal{MF}$$

unstable and stable measured foliations for g respectively, which are not unique but we make certain choices. It follows from the above discussion that their projective classes

$$[g^+], [g^-] \in \mathcal{PMF}$$

are distinct fixed points of g acting on $\mathcal{T} \sqcup \mathcal{PMF}$. Moreover, g exhibits the north-south dynamics: for $x \in \mathcal{T} \sqcup \mathcal{PMF} \setminus \{[g^\pm]\}$,

$$g^n x \rightarrow [g^+] \quad \text{as } n \rightarrow +\infty \quad \text{and} \quad g^n x \rightarrow [g^-] \quad \text{as } n \rightarrow -\infty$$

and the convergence is uniform on compact subsets [McC85, Section 4] (cf. [Iva92, Theorem 3.5]). Note also that $[(g^{-1})^+] = [g^-]$ and $[(g^{-1})^-] = [g^+]$.

Two pseudo-Anosov $g, h \in \text{Mod}(S)$ have either the common fixed points (i.e., $\{[g^\pm]\} = \{[h^\pm]\}$) or disjoint fixed points (i.e., $\{[g^\pm]\} \cap \{[h^\pm]\} = \emptyset$) [MP89, Lemma 2.5]. We say that they are *independent* when they have disjoint set of fixed points in \mathcal{PMF} . A subgroup $\Gamma < \text{Mod}(S)$ is called *non-elementary* if there exist independent pseudo-Anosov $g, h \in \Gamma$.

2.2. Cross-ratios on \mathcal{MF} and \mathcal{PMF} . We prove Theorem 1.1 by introducing the notion of cross-ratios on \mathcal{MF} and \mathcal{PMF} .

For two isotopy classes α, β of simple closed curves on S , we denote by $i(\alpha, \beta) \in \mathbb{R}_{\geq 0}$ their *geometric intersection number*, i.e., minimal number of intersection points of their representatives. Isotopy classes of (essential) simple closed curves on S can be regarded as measured foliations whose transverse measures are given by geometric intersection numbers with them, by foliating their annular neighborhoods with closed leaves isotopic to those simple closed curves. In this regard, the function $i(\cdot, \cdot)$ continuously extends to

$$i : \mathcal{MF} \times \mathcal{MF} \rightarrow \mathbb{R}_{\geq 0}$$

which we also call geometric intersection number ([Thu22, Section 9.3], [Ree81, Corollary 1.11]). Then $i(\cdot, \cdot)$ is invariant under the $\text{Mod}(S)$ -action and equivariant under the scaling: for $x, y \in \mathcal{MF}$, $g \in \text{Mod}(S)$, and $t, s > 0$, we have

$$i(g \cdot x, g \cdot y) = i(x, y) \quad \text{and} \quad i(t \cdot x, s \cdot y) = ts \cdot i(x, y)$$

where $t \cdot x$ and $s \cdot y$ are obtained by scaling transverse measures of x and y by t and s respectively.

For a pseudo-Anosov mapping class $g \in \text{Mod}(S)$, we have

$$i(x, g^-) > 0 \quad \text{for every } x \in \mathcal{MF} \text{ with } [x] \neq [g^-]$$

([FLP79, Théorème 1, Exposé 12], [FLP79, Lemme 6, Lemme 16, Exposé 9], [Ree81, Theorem 1.12]). The same property also holds for g^+ .

Definition 2.1 (Cross-ratios). For $x, y, z, w \in \mathcal{MF}$, we define their *cross-ratio*

$$[x, y, z, w] = \frac{i(x, w)i(y, z)}{i(x, z)i(y, w)} \in \mathbb{R}_{\geq 0}$$

whenever $i(x, z)i(y, w) \neq 0$. Note that this is invariant under scaling transverse measures, and hence the cross-ratio is well-defined on \mathcal{PMF} as well.

In Proposition 2.3 below, we relate the cross-ratio to dynamics of pseudo-Anosov mapping classes. We begin with the following lemma, which might be standard to experts; we present the proof for the sake of completeness.

With an extra element “0”, we give a topology on $\mathcal{MF} \sqcup \{0\}$ by setting that a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{MF}$ converges to 0 if the associated sequence of transverse measures converge to the zero measure.

Lemma 2.2. *Let $g \in \text{Mod}(S)$ be pseudo-Anosov. Then the following holds:*

(1) *For any $x \in \mathcal{MF}$, we have*

$$\lambda_g^{-n} g^n x \rightarrow \frac{i(x, g^-)}{i(g^+, g^-)} \cdot g^+ \in \mathcal{MF} \sqcup \{0\} \quad \text{as } n \rightarrow +\infty.$$

(2) *Let $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{MF}$ be a sequence such that, as $n \rightarrow +\infty$, we have $x_n \rightarrow x \in \mathcal{MF}$ with $i(x, g^-) > 0$. Then*

$$\lambda_g^{-n} g^n x_n \rightarrow \frac{i(x, g^-)}{i(g^+, g^-)} \cdot g^+ \quad \text{as } n \rightarrow +\infty.$$

(3) *The convergence in (1) and (2) is uniform on compact subsets of $\mathcal{MF} \setminus \mathbb{R}_{>0} \cdot g^-$ in the sense that for any compact $Q \subset \mathcal{MF} \setminus \mathbb{R}_{>0} \cdot g^-$ and an open neighborhood $\mathcal{U} \subset \mathcal{MF}$ of $\frac{i(Q, g^-)}{i(g^+, g^-)} \cdot g^+ \subset \mathcal{MF}$,*

$$\lambda_g^{-n} g^n Q \subset \mathcal{U} \quad \text{for all large } n \in \mathbb{N}.$$

Proof. We first prove (2). Let $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{MF}$ be a sequence such that $x_n \rightarrow x \in \mathcal{MF}$ with $i(x, g^-) > 0$ as $n \rightarrow +\infty$. Since $i(x, g^-) > 0$, we have $[x] \neq [g^-] \in \mathcal{PMF}$. Then by the north-south dynamics of g on \mathcal{PMF} , we have

$$[g^n x_n] \rightarrow [g^+] \in \mathcal{PMF} \quad \text{as } n \rightarrow +\infty.$$

Hence, there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_{>0}$ such that $t_n g^n x_n \rightarrow g^+$ as $n \rightarrow +\infty$.

As $n \rightarrow +\infty$, we have

$$i(\lambda_g^{-n} g^n x_n, g^-) = \lambda_g^{-n} i(x_n, g^{-n} g^-) = i(x_n, g^-) \rightarrow i(x, g^-) > 0.$$

Note that $\lambda_g^{-n} g^n x_n = (\lambda_g^{-n} t_n^{-1}) t_n g^n x_n$ for all $n \in \mathbb{N}$, and that $t_n g^n x_n \rightarrow g^+$ as $n \rightarrow +\infty$. It follows from the above computation that

$$\lambda_g^{-n} t_n^{-1} = \frac{i(\lambda_g^{-n} g^n x_n, g^-)}{i(t_n g^n x_n, g^-)} \rightarrow \frac{i(x, g^-)}{i(g^+, g^-)} \quad \text{as } n \rightarrow +\infty.$$

Therefore,

$$\lambda_g^{-n} g^n x_n = (\lambda_g^{-n} t_n^{-1}) t_n g^n x_n \rightarrow \frac{i(x, g^-)}{i(g^+, g^-)} \cdot g^+ \quad \text{as } n \rightarrow +\infty,$$

as desired.

We now prove (1). By (2), it suffices to consider the case that $[x] = [g^-] \in \mathcal{PMF}$. Then

$$\lambda_g^{-n} g^n x = \lambda_g^{-2n} x \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Since $i(x, g^-) = 0$ in this case, the desired convergence holds.

Finally, (3) follows from (2). □

We now deduce the following relation between cross-ratios and stretch factors from Lemma 2.2. Note that in the following, $g^n h^n$ is pseudo-Anosov for all large $n \in \mathbb{N}$ since g and h are independent ([McC85], [Iva92]).

Proposition 2.3. *Let $g, h \in \text{Mod}(S)$ be independent pseudo-Anosov mapping classes. Then*

$$\frac{\lambda_{g^n h^n}}{\lambda_g^n \lambda_h^n} \rightarrow [g^+, h^+, g^-, h^-] \quad \text{as } n \rightarrow +\infty.$$

Proof. For each $n \in \mathbb{N}$, let $f_n := g^n h^n \in \text{Mod}(S)$, which is pseudo-Anosov when n is large enough. For such $n \in \mathbb{N}$, we choose $f_n^\pm \in \mathcal{MF}$ such that $i(f_n^+, f_n^-) = 1$. It then follows from the north-south dynamics of g and h that

$$[f_n^+] \rightarrow [g^+] \in \mathcal{PMF} \quad \text{and} \quad [f_n^-] \rightarrow [h^-] \in \mathcal{PMF} \quad \text{as } n \rightarrow +\infty.$$

Hence, by rescaling transverse measures of $f_n^\pm \in \mathcal{MF}$, we may assume that sequences $\{f_n^\pm\}_{n \in \mathbb{N}} \subset \mathcal{MF}$ is convergent. We denote their limits by $x, y \in \mathcal{MF}$, i.e.,

$$f_n^+ \rightarrow x \quad \text{and} \quad f_n^- \rightarrow y \quad \text{as } n \rightarrow +\infty.$$

Since $i(f_n^+, f_n^-) = 1$ for all large $n \in \mathbb{N}$, we have

$$\lambda_{f_n} = i(f_n f_n^+, f_n^-) = i(g^n h^n f_n^+, f_n^-),$$

and hence

$$(2.1) \quad \frac{\lambda_{g^n h^n}}{\lambda_g^n \lambda_h^n} = i((\lambda_g^{-n} g^n) (\lambda_h^{-n} h^n) f_n^+, f_n^-) = i(\lambda_h^{-n} h^n f_n^+, \lambda_g^{-n} g^{-n} f_n^-).$$

Since $[x] = [g^+]$ and $[y] = [h^-]$, we in particular have

$$i(x, h^-) > 0 \quad \text{and} \quad i(y, g^+) > 0$$

by the independence of g and h . Hence, by Lemma 2.2(2), we have as $n \rightarrow +\infty$ that

$$\lambda_h^{-n} h^n f_n^+ \rightarrow \frac{i(x, h^-)}{i(h^+, h^-)} h^+ \quad \text{and} \quad \lambda_g^{-n} g^{-n} f_n^- \rightarrow \frac{i(y, g^+)}{i(g^+, g^-)} g^-.$$

Now since $x = t g^+$ and $y = s h^-$ for some $t, s > 0$, it follows from above convergences and Equation (2.1) that

$$\frac{\lambda_{g^n h^n}}{\lambda_g^n \lambda_h^n} \rightarrow \frac{i(x, h^-) i(y, g^+) i(h^+, g^-)}{i(h^+, h^-) i(g^+, g^-)} = ts \cdot \frac{i(g^+, h^-)^2 i(h^+, g^-)}{i(g^+, g^-) i(h^+, h^-)} \quad \text{as } n \rightarrow +\infty.$$

Since $1 = i(f_n^+, f_n^-) \rightarrow i(x, y) = ts \cdot i(g^+, h^-)$ as $n \rightarrow +\infty$, we also have

$$ts = \frac{1}{i(g^+, h^-)}.$$

Therefore,

$$\frac{\lambda_{g^n h^n}}{\lambda_g^n \lambda_h^n} \rightarrow \frac{i(g^+, h^-) i(h^+, g^-)}{i(g^+, g^-) i(h^+, h^-)} = [g^+, h^+, g^-, h^-] \quad \text{as } n \rightarrow +\infty.$$

This finishes the proof. \square

2.3. Proof of Theorem 1.1. We are now ready to prove Theorem 1.1. Let $\Gamma < \text{Mod}(S)$ be a non-elementary subgroup and suppose to the contrary that its Teichmüller length spectrum $\text{len}(\Gamma)$ is not non-arithmetic. That is, for some $c \geq 0$ we have

$$\langle \text{len}(\Gamma) \rangle \subset c \cdot \mathbb{Z}.$$

Then for any independent pseudo-Anosov $g, h \in \Gamma$, it follows from Proposition 2.3 that

$$(2.2) \quad \log[g^+, h^+, g^-, h^-] \in c \cdot \mathbb{Z}.$$

Denote by

$$(2.3) \quad \Lambda_\Gamma := \overline{\{[g^+] : g \in \Gamma \text{ pseudo-Anosov}\}} \subset \mathcal{PMF}$$

the limit set of Γ . Since $\Gamma < \text{Mod}(S)$ is non-elementary, Λ_Γ is the unique Γ -minimal subset of \mathcal{PMF} and is perfect [MP89, Theorem 4.1, Proposition 5.2]. Moreover, the north-south dynamics of pseudo-Anosov mapping classes also gives rise to that

$$(2.4) \quad \{([g^+], [g^-]) : g \in \Gamma \text{ pseudo-Anosov}\} \subset \Lambda_\Gamma \times \Lambda_\Gamma$$

is a dense subset (cf. proof of Proposition 2.3). Hence, for $x, y, z, w \in \Lambda_\Gamma$ such that $i(x, z)i(y, w)i(x, w)i(y, z) > 0$, it follows from Equation (2.2) that

$$(2.5) \quad \log[x, y, z, w] \in c \cdot \mathbb{Z}.$$

We will deduce a contradiction from Equation (2.5).

Let us now turn to the train track theory. We refer the readers to ([Thu22], [PH92], [Pap83]) for more comprehensive expository on train tracks and their various properties. For a maximal and recurrent train track³ τ on S , let V_τ be the real vector space of weights on the branches of τ satisfying the switch condition, and let $K(\tau) \subset V_\tau$ be the non-empty open convex cone consisting of weights which are positive on every branch of τ . Then there exists a natural $\mathbb{R}_{>0}$ -equivariant map $\phi_\tau : K(\tau) \rightarrow \mathcal{MF}$ which is a homeomorphism onto its image. Moreover, the image $U(\tau) := \phi_\tau(K(\tau)) \subset \mathcal{MF}$ is a non-empty open cone in \mathcal{MF} whose elements are carried by τ (cf. [Pap83, Section II.3]). These $(U(\tau), \phi_\tau)$'s form an atlas of \mathcal{MF} , giving rise to a piecewise-linear structure [Thu22, Proposition 9.5.8] (cf. [Pap08, Proposition 4.3]). Each $(U(\tau), \phi_\tau)$ is called the train track chart associated to τ .

In the rest of the proof,

we fix a pseudo-Anosov $g \in \Gamma$.

Then, after replacing g with its positive power if necessary, there exists a train track chart $(U(\tau), \phi_\tau)$ containing $g^+ \in \mathcal{MF}$ such that, $g^2 U(\tau) \subset U(\tau)$ and the g^2 -action on $U(\tau)$ induces an action on V_τ given by an invertible linear map $A : V_\tau \rightarrow V_\tau$ with the leading eigenvalue $\alpha := \lambda_g^2$ ([Pap83, Section IV], [PP87, Theorem 4.1]).

³We only need the following properties of maximal and recurrent train tracks, not their precise definitions. For this reason, we omit their definitions.

Since Λ_Γ is perfect, there exists a pseudo-Anosov $h \in \Gamma$ such that g and h are independent and $h^+ \in U(\tau)$; in particular, $i(h^+, g^\pm) > 0$. We simply write $z := h^+ \in \mathcal{MF}$. Then for each $n \in \mathbb{N}$,

$$[g^+, g^n z, g^-, g^{-n} z] = \frac{i(g^+, g^{-n} z) i(g^n z, g^-)}{i(g^+, g^-) i(g^n z, g^{-n} z)} = \frac{i(g^+, \lambda_g^{-n} g^{-n} z) i(\lambda_g^{-n} g^n z, g^-)}{i(g^+, g^-) i(\lambda_g^{-n} g^n z, \lambda_g^{-n} g^{-n} z)}.$$

By Lemma 2.2(1), we have $\lambda_g^{-n} g^{\pm n} z \mapsto \frac{i(z, g^\mp)}{i(g^+, g^-)} \cdot g^\pm$, and hence

$$[g^+, g^n z, g^-, g^{-n} z] \rightarrow 1 \quad \text{as } n \rightarrow +\infty.$$

On the other hand, since $i(g^+, g^-) i(g^n z, g^{-n} z) i(g^+, g^{-n} z) i(g^n z, g^-) > 0$, it follows from Equation (2.5) that

$$[g^+, g^n z, g^-, g^{-n} z] = 1 \quad \text{for all large } n \in \mathbb{N}.$$

In other words,

$$i(g^n z, g^{-n} z) = \frac{i(g^+, g^{-n} z) i(g^n z, g^-)}{i(g^+, g^-)} \quad \text{for all large } n \in \mathbb{N}.$$

Since $i(g^n z, g^{-n} z) = i(z, g^{2n} z)$, $i(g^+, g^{-n} z) = i(g^n g^+, z) = \lambda_g^n i(g^+, z)$, and $i(g^n z, g^-) = i(z, g^{-n} g^-) = \lambda_g^n i(z, g^-)$, we have

$$(2.6) \quad i(z, g^{2n} z) = \lambda_g^{2n} \cdot \frac{i(g^+, z) i(z, g^-)}{i(g^+, g^-)} \quad \text{for all large } n \in \mathbb{N}.$$

Now noting that $[z] = [h^+] \neq [g^+]$, it follows from ([Pap86, Proposition 3], [PH92, Section 3.4]) that there exists a maximal and recurrent train track σ on S and associated train track chart $(U(\sigma), \phi_\sigma : K(\sigma) \subset V_\sigma \rightarrow U(\sigma))$ such that $g^+ \in U(\sigma)$ and that the function

$$i(z, \phi_\sigma(\cdot)) \quad \text{is linear on } K(\sigma).$$

Let $\psi : V_\sigma \rightarrow \mathbb{R}$ be the linear form such that

$$\psi(\cdot) = i(z, \phi_\sigma(\cdot)) \quad \text{on } K(\sigma).$$

By [Pap08, Proposition 4.3], there exists an open cone neighborhood $W \subset K(\tau)$ of $\phi_\tau^{-1}(g^+)$ such that $\phi_\tau(W) \subset U(\sigma)$ and there exist convex cones $W_1, \dots, W_m \subset W$ with non-empty interiors such that $W = \bigcup_{j=1}^m W_j$ and the transition map

$$\phi_\sigma^{-1} \circ \phi_\tau \quad \text{is linear on } W_j \text{ for each } 1 \leq j \leq m.$$

For each j , let $\phi_j : V_\tau \rightarrow V_\sigma$ be the linear map such that

$$\phi_j = \phi_\sigma^{-1} \circ \phi_\tau \quad \text{on } W_j.$$

Define a linear form $L_j : V_\tau \rightarrow \mathbb{R}$ by

$$L_j := \psi \circ \phi_j.$$

Now define the function $f_\tau : K(\tau) \rightarrow \mathbb{R}$ by

$$f_\tau(\cdot) = i(z, \phi_\tau(\cdot)).$$

Then we have

$$f_\tau = L_j \quad \text{on } W_j.$$

We simply write

$$v := \phi_\tau^{-1}(z) \in K(\tau).$$

The following observation is crucial in deducing the desired contradiction. See Figure 1 for its name.

Lemma 2.4 (Bottom of Iceberg Lemma). *For each $1 \leq j \leq m$, we have*

$$L_j(v) \leq 0.$$

Proof. Let $u \in \text{int } W_j$ and for $t \in [0, 1]$, let $u_t := (1-t)u + tv \in K(\tau)$. Then by the convexity of f_τ proven in [Mir04, Theorem A.1] (cf. [AH24, Theorem 4.1]),

$$f_\tau(u_t) \leq (1-t)f_\tau(u) + tf_\tau(v).$$

On the other hand, for small enough $t > 0$, $u_t \in W_j$ and hence

$$f_\tau(u_t) = L_j(u_t) = (1-t)L_j(u) + tL_j(v) = (1-t)f_\tau(u) + tL_j(v).$$

Combining them, we have

$$L_j(v) \leq f_\tau(v) = i(z, z) = 0.$$

□

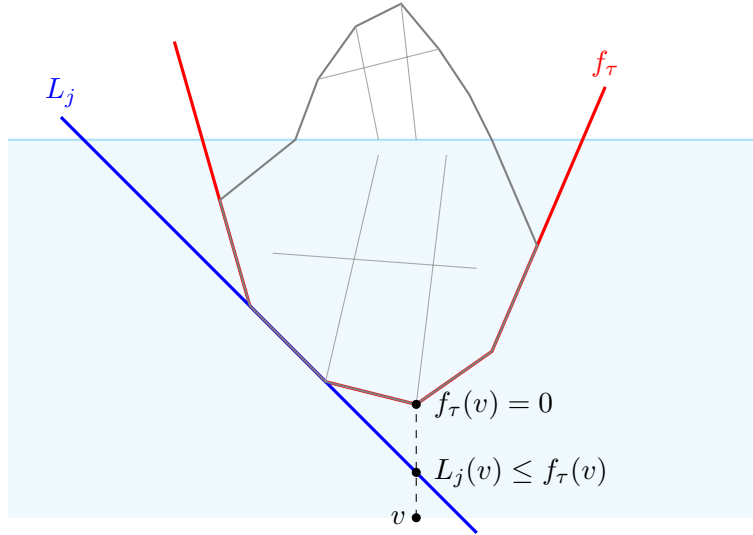


FIGURE 1. Iceberg on f_τ

Now we set

$$C := \frac{i(g^+, z)i(z, g^-)}{i(g^+, g^-)} > 0.$$

Recalling $\alpha = \lambda_g^2$, it follows from Equation (2.6) that

$$f_\tau(A^n v) = C \cdot \alpha^n \quad \text{for all large } n \in \mathbb{N}.$$

Since $[g^{2n}z] \rightarrow [g^+]$ in \mathcal{PMF} as $n \rightarrow +\infty$, we have $[A^n v] \rightarrow [\phi_\tau^{-1}(g^+)]$ as $n \rightarrow +\infty$, after projecting to some unit sphere in $K(\tau) \subset V_\tau$. Hence,

$$A^n v \in W \quad \text{for all large } n \in \mathbb{N}.$$

For each $n \geq 0$, set

$$P_n := \prod_{j=1}^m (L_j(A^n v) - C \cdot \alpha^n).$$

Then for all large $n \in \mathbb{N}$, it follows from $A^n v \in W$ that $L_j(A^n v) = f_\tau(A^n v) = C \cdot \alpha^n$ for some $1 \leq j \leq m$. This implies

$$(2.7) \quad P_n = 0 \quad \text{for all large } n \in \mathbb{N}.$$

Now for each $1 \leq j \leq m$, define the vector space $V_j := V_\tau \times \mathbb{R}$ and consider the linear form $\widehat{L}_j : V_j \rightarrow \mathbb{R}$ defined by

$$\widehat{L}_j(u, s) := L_j(u) - C \cdot s.$$

We also define $A_j : V_j \rightarrow V_j$ by

$$A_j(u, s) := (Au, \alpha s),$$

which is an invertible linear map.

We then consider tensor products

$$\widehat{V} := \bigotimes_{j=1}^m V_j, \quad \widehat{L} := \bigotimes_{j=1}^m \widehat{L}_j, \quad \text{and} \quad \widehat{A} := \bigotimes_{j=1}^m A_j.$$

Then $\widehat{L} : \widehat{V} \rightarrow \mathbb{R}$ is a linear form and $\widehat{A} : \widehat{V} \rightarrow \widehat{V}$ is an invertible linear map.

Let

$$\chi_{\widehat{A}}(t) := t^d + c_{d-1}t^{d-1} + \cdots + c_1t + c_0$$

be the characteristic polynomial of \widehat{A} . Since \widehat{A} is invertible, $c_0 \neq 0$. By Cayley–Hamilton theorem,

$$\widehat{A}^d + c_{d-1}\widehat{A}^{d-1} + \cdots + c_1\widehat{A} + c_0 \text{Id} = 0.$$

In other words, for each $n \geq 0$,

$$\widehat{A}^{n+d} + c_{d-1}\widehat{A}^{n+d-1} + \cdots + c_1\widehat{A}^{n+1} + c_0\widehat{A}^n = 0.$$

In particular, setting

$$\widehat{v} := \bigotimes_{j=1}^m (v, 1) \in \widehat{V},$$

we have

$$\widehat{A}^{n+d}\widehat{v} + c_{d-1}\widehat{A}^{n+d-1}\widehat{v} + \cdots + c_1\widehat{A}^{n+1}\widehat{v} + c_0\widehat{A}^n\widehat{v} = 0,$$

and hence

$$\widehat{L}(\widehat{A}^{n+d}\widehat{v}) + c_{d-1}\widehat{L}(\widehat{A}^{n+d-1}\widehat{v}) + \cdots + c_1\widehat{L}(\widehat{A}^{n+1}\widehat{v}) + c_0\widehat{L}(\widehat{A}^n\widehat{v}) = 0.$$

Note that for each $n \geq 0$,

$$\widehat{A}^n \widehat{v} = \bigotimes_{j=1}^m (A^n v, \alpha^n)$$

and hence

$$\widehat{L}(\widehat{A}^n \widehat{v}) = \prod_{j=1}^m (L_j(A^n v) - C \cdot \alpha^n) = P_n.$$

Therefore, it follows that

$$P_{n+d} + c_{d-1}P_{n+d-1} + \cdots + c_1P_{n+1} + c_0P_n = 0 \quad \text{for each } n \geq 0.$$

On the other hand, as in Equation (2.7), $P_n = 0$ for all large $n \geq 0$. Therefore, since $c_0 \neq 0$, backward induction gives $P_n = 0$ for all $n \geq 0$. In particular,

$$P_0 = 0.$$

Since $P_0 = \prod_{j=1}^m (L_j(v) - C)$, this implies

$$L_j(v) - C = 0 \quad \text{for some } 1 \leq j \leq m.$$

However, by Bottom of Iceberg Lemma (Lemma 2.4), $L_j(v) \leq 0$. This is a contradiction to $C > 0$, completing the proof of Theorem 1.1. \square

3. TOPOLOGICAL MIXING OF TEICHMÜLLER GEODESIC FLOW

In this section, we prove the topological mixing of Teichmüller geodesic flow for non-elementary subgroups of $\text{Mod}(S)$ (Theorem 1.3), as an application of our non-arithmeticity result. We begin with recalling the setup.

3.1. Bundle of quadratic differentials. A (holomorphic) quadratic differential q on a (marked) Riemann surface $X \in \mathcal{T}$ is a tensor locally given by $\phi(z)dz^2$ where ϕ is a holomorphic function with simple poles at the punctures. The vector space $Q(X)$ of quadratic differentials on X is identified with the cotangent space of \mathcal{T} at $X \in \mathcal{T}$, and hence the bundle

$$\mathcal{QT} := \{(X, q) : X \in \mathcal{T} \text{ and } q \in Q(X)\}$$

of quadratic differentials is identified with the cotangent bundle of \mathcal{T} .

Noting that the Teichmüller metric is Finsler, the norm of $q \in Q(X)$, $X \in \mathcal{T}$, is given by

$$\|q\| := \int_X |q| = \int_X |\phi(z)| |dz|^2,$$

which represents the area of q . Then the bundle

$$\mathcal{Q}^1\mathcal{T} := \{(X, q) \in \mathcal{QT} : \|q\| = 1\}$$

of unit-norm quadratic differentials is identified with the unit cotangent bundle of \mathcal{T} . The $\text{Mod}(S)$ -action on \mathcal{T} naturally extends to the actions on \mathcal{QT} and $\mathcal{Q}^1\mathcal{T}$. We refer the readers to [FM12] for more comprehensive exposition.

In this regard, the Teichmüller geodesic flow is given as a certain \mathbb{R} -action on \mathcal{QT} . For a more explicit description, we note that a non-zero quadratic

differential $(X, q) \in \mathcal{QT}$ determines real and imaginary measured foliations $\text{Re}(q^{1/2})$ and $\text{Im}(q^{1/2})$ on S , respectively, omitting the basepoint X : $\text{Re}(q^{1/2})$ consists of a foliation obtained by taking the union of the zeros of q with the set of smooth paths in X whose tangent vectors at each point evaluate to negative real numbers under q , and a transverse measure given by $\mu(\alpha) := \int_\alpha \left| \text{Re} \left(\sqrt{\phi(z)} dz \right) \right|$ where we write a local expression $q = \phi(z) dz^2$. The measured foliation $\text{Im}(q^{1/2})$ is defined by taking the paths in X whose tangent vectors evaluate to positive real numbers under q instead, and considering the transverse measure defined by integrating $\left| \text{Im} \left(\sqrt{\phi(z)} dz \right) \right|$. Those measured foliations $\text{Re}(q^{1/2})$ and $\text{Im}(q^{1/2})$ are also referred to as vertical and horizontal measured foliations of q , respectively. See ([FM12], [ABEM12]) for details.

The above correspondence defines a $\text{Mod}(S)$ -equivariant homeomorphism

$$\begin{aligned} \mathcal{QT} \setminus \{0\} &\longrightarrow (\mathcal{MF} \times \mathcal{MF}) \setminus \tilde{\Delta} \\ q &\longmapsto \left(\text{Re}(q^{1/2}), \text{Im}(q^{1/2}) \right) \end{aligned}$$

where $\tilde{\Delta} := \{(x, y) \in \mathcal{MF} \times \mathcal{MF} : \exists z \in \mathcal{MF} \text{ s.t. } i(x, z) = i(y, z) = 0\}$ [GM91]. Noting that the Teichmüller geodesic flow commutes with the $\text{Mod}(S)$ -action on \mathcal{QT} , we denote the Teichmüller geodesic flow by time t by the right-action a_t , $t \in \mathbb{R}$. Then in terms of the above homeomorphism, we have

$$qa_t \longmapsto \left(e^t \text{Re}(q^{1/2}), e^{-t} \text{Im}(q^{1/2}) \right)$$

for $t \in \mathbb{R}$.

The above discussion descends to $\mathcal{Q}^1\mathcal{T}$ as follows: we have a $\text{Mod}(S)$ -equivariant homeomorphism

$$\begin{aligned} \mathcal{Q}^1\mathcal{T} &\longrightarrow (\mathcal{MF} \times \mathcal{PMF}) \setminus \Delta \\ (3.1) \quad q &\longmapsto \left(\text{Re}(q^{1/2}), [\text{Im}(q^{1/2})] \right) \end{aligned}$$

where $\Delta := \{(x, [y]) \in \mathcal{MF} \times \mathcal{PMF} : (x, y) \in \tilde{\Delta}\}$, and the Teichmüller geodesic flow can be described via

$$qa_t \longmapsto \left(e^t \text{Re}(q^{1/2}), [\text{Im}(q^{1/2})] \right)$$

for $t \in \mathbb{R}$.

3.2. Non-wandering domains. Let $\Gamma < \text{Mod}(S)$ be a non-elementary subgroup. Since $\text{Mod}(S)$ -action on $\mathcal{Q}^1\mathcal{T}$ commutes with the Teichmüller geodesic flow, the Teichmüller geodesic flow descends to the bundle

$$\Gamma \backslash \mathcal{Q}^1\mathcal{T}$$

which is identified with the unit cotangent bundle over $\Gamma \backslash \mathcal{T}$. We also denote this induced Teichmüller geodesic flow on $\Gamma \backslash \mathcal{Q}^1 \mathcal{T}$ by $\{a_t\}_{t \in \mathbb{R}}$.

We set

$$\tilde{\Omega}_\Gamma := \left\{ q \in \mathcal{Q}^1 \mathcal{T} : \left[\operatorname{Re}(q^{1/2}) \right], \left[\operatorname{Im}(q^{1/2}) \right] \in \Lambda_\Gamma \right\} \quad \text{and} \quad \Omega_\Gamma := \Gamma \backslash \tilde{\Omega}_\Gamma.$$

Then it follows from the density of fixed points of pseudo-Anosov elements in Γ (Equation (2.4)) that Ω_Γ is a non-wandering domain for the Teichmüller geodesic flow. Namely, for any non-empty open subset $\mathcal{O} \subset \Omega_\Gamma$, there exists a diverging sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that

$$\mathcal{O} \cap \mathcal{O} a_{t_n} \neq \emptyset \quad \text{for all } n \in \mathbb{N}.$$

The rest of this section is devoted to the proof of *topological mixing* of the Teichmüller geodesic flow on Ω_Γ , which is stated as Theorem 1.3 in the introduction. That is, for any non-empty open subsets $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega_\Gamma$, we have

$$\mathcal{O}_1 \cap \mathcal{O}_2 a_t \neq \emptyset \quad \text{for all large } t.$$

3.3. Proof of Theorem 1.3. Suppose to the contrary that there exist non-empty open subsets $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega_\Gamma$ and a diverging sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_{>0}$ such that $\mathcal{O}_1 \cap \mathcal{O}_2 a_{t_n} = \emptyset$ for all $n \in \mathbb{N}$. In other words,

$$\mathcal{O}_1 a_{-t_n} \cap \mathcal{O}_2 = \emptyset \quad \text{for all } n \in \mathbb{N}.$$

Let $\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2 \subset \tilde{\Omega}_\Gamma$ be the Γ -invariant lifts of \mathcal{O}_1 and \mathcal{O}_2 , respectively. In the rest of the proof, we identify $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$ with subsets of $(\mathcal{MF} \times \mathcal{PMF}) \setminus \Delta$ as in Equation (3.1).

Using the density of fixed points of pseudo-Anosov elements of Γ in Equation (2.4) and the identification in Equation (3.1), we fix a pseudo-Anosov $g \in \Gamma$ such that $(g^+, [g^-]) \in \tilde{\mathcal{O}}_1$, with some choice of $g^+ \in \mathcal{MF}$. The corresponding quadratic differential generates the axis of g in \mathcal{T} .

We also fix an open neighborhood $U \subset \Lambda_\Gamma$ of $[g^-]$ such that

$$\{g^+\} \times U \subset \tilde{\mathcal{O}}_1.$$

Then for any $k, n \in \mathbb{N}$,

$$\left\{ \lambda_g^k e^{-t_n} g^+ \right\} \times g^k U = g^k (\{g^+\} \times U) a_{-t_n} \subset \tilde{\mathcal{O}}_1 a_{-t_n}.$$

Now for each $n \in \mathbb{N}$, let $k_n \in \mathbb{N}$ be such that $\lambda_g^{k_n} e^{-t_n} \in [1, \lambda_g]$. After passing to a subsequence, we may assume that

$$(3.2) \quad \lambda_g^{k_n} e^{-t_n} \rightarrow \lambda \in [1, \lambda_g] \quad \text{as } n \rightarrow +\infty.$$

Then by [CK25, Theorem 10.1],

$$(3.3) \quad \overline{\Gamma \cdot (\lambda g^+)} = \{x \in \mathcal{MF} : [x] \in \Lambda_\Gamma\}$$

which uses our non-arithmeticity result (Theorem 1.1). Note that while [CK25, Theorem 10.1] is stated only for convex cocompact subgroups, this implies the above since for any pseudo-Anosov $h \in \Gamma$ independent from g ,

there exists a non-elementary convex cocompact subgroup of Γ containing powers of g and h [FM02, Theorem 1.4], together with Equation (2.3).

Now fix a point $(x, [y]) \in \tilde{\mathcal{O}}_2$. Since $[x] \in \Lambda_\Gamma$, it follows from Equation (3.3) that there exists a sequence $\{\gamma_j\}_{j \in \mathbb{N}} \subset \Gamma$ such that

$$(3.4) \quad \gamma_j(\lambda g^+) \rightarrow x \quad \text{as } j \rightarrow +\infty.$$

Noting that \mathcal{PMF} is metrizable (cf. [FLP79, Théorème V.4, Exposé 6]), denote by $B_{1/j}([y]) \subset \mathcal{PMF}$ the open ball of radius $1/j$ centered at y , for each $j \in \mathbb{N}$, fixing a metric on \mathcal{PMF} . Since Λ_Γ is perfect [MP89, Proposition 5.2], for each $j \in \mathbb{N}$ there exists $[z_j] \in \gamma_j^{-1} B_{1/j}([y]) \cap \Lambda_\Gamma \setminus \{[g^+]\}$. Then we have a diverging sequence $\{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ so that $[z_j] \in g^{k_{n_j}} U$ for all $j \in \mathbb{N}$. In other words,

$$(3.5) \quad [z_j] \in \gamma_j^{-1} B_{1/j}([y]) \cap g^{k_{n_j}} U \quad \text{for all } j \in \mathbb{N}.$$

Now consider the sequence of points

$$(3.6) \quad \left(\gamma_j \left(\lambda g^{k_{n_j}} e^{-t_{n_j}} g^+ \right), \gamma_j [z_j] \right) \in \gamma_j g^{k_{n_j}} (\{g^+\} \times U) a_{-t_{n_j}} \subset \tilde{\mathcal{O}}_1 a_{-t_{n_j}},$$

for $j \in \mathbb{N}$. By Equation (3.5), we have

$$\gamma_j [z_j] \mapsto [y] \quad \text{as } j \rightarrow +\infty.$$

Since the $\text{Mod}(S)$ -action on \mathcal{MF} commutes with the $\mathbb{R}_{>0}$ -action on \mathcal{MF} by scaling transverse measures, it follows from Equation (3.2) and Equation (3.4) that

$$\gamma_j \left(\lambda g^{k_{n_j}} e^{-t_{n_j}} g^+ \right) = \frac{\lambda g^{k_{n_j}} e^{-t_{n_j}}}{\lambda} \cdot \gamma_j(\lambda g^+) \rightarrow x \quad \text{as } j \rightarrow +\infty.$$

Therefore, the sequence of points in Equation (3.6) converges to $(x, [y]) \in \tilde{\mathcal{O}}_2$ as $j \rightarrow +\infty$. Since $\tilde{\mathcal{O}}_2$ is open, this implies that

$$\tilde{\mathcal{O}}_1 a_{-t_{n_j}} \cap \tilde{\mathcal{O}}_2 \neq \emptyset \quad \text{for all large } j \in \mathbb{N}.$$

This is a contradiction to that $\mathcal{O}_1 a_{-t_n} \cap \mathcal{O}_2 = \emptyset$ for all $n \in \mathbb{N}$, and the topological mixing follows. \square

REFERENCES

- [ABEM12] Jayadev Athreya, Alexander Bufetov, Alex Eskin, and Maryam Mirzakhani. Lattice point asymptotics and volume growth on Teichmüller space. *Duke Math. J.*, 161(6):1055–1111, 2012.
- [AH23] Francisco Arana-Herrera. Effective mapping class group dynamics, I: counting lattice points in Teichmüller space. *Duke Math. J.*, 172(8):1437–1529, 2023.
- [AH24] Francisco Arana-Herrera. Effective mapping class group dynamics III: counting filling closed curves on surfaces. *Camb. J. Math.*, 12(3):563–622, 2024.
- [Ben00] Yves Benoist. Propriétés asymptotiques des groupes linéaires. II. In *Analysis on homogeneous spaces and representation theory of Lie groups, Okayama-Kyoto (1997)*, volume 26 of *Adv. Stud. Pure Math.*, pages 33–48. Math. Soc. Japan, Tokyo, 2000.

- [Ber78] Lipman Bers. An extremal problem for quasiconformal mappings and a theorem by Thurston. *Acta Math.*, 141(1-2):73–98, 1978.
- [Bou95] M. Bourdon. Structure conforme au bord et flot géodésique d’un $CAT(-1)$ -espace. *Enseign. Math. (2)*, 41(1-2):63–102, 1995.
- [CK25] Inhyeok Choi and Dongryul M. Kim. Invariant measures on the space of measured laminations for subgroups of mapping class group. *arXiv preprint arXiv:2510.23256*, 2025.
- [Dal99] Françoise Dal’bo. Remarques sur le spectre des longueurs d’une surface et comptages. *Bol. Soc. Brasil. Mat. (N.S.)*, 30(2):199–221, 1999.
- [Dal00] Françoise Dal’bo. Topologie du feuilletage fortement stable. *Ann. Inst. Fourier (Grenoble)*, 50(3):981–993, 2000.
- [DP98] Françoise Dal’bo and Marc Peigné. Some negatively curved manifolds with cusps, mixing and counting. *J. Reine Angew. Math.*, 497:141–169, 1998.
- [EM11] Alex Eskin and Maryam Mirzakhani. Counting closed geodesics in moduli space. *J. Mod. Dyn.*, 5(1):71–105, 2011.
- [EMM22] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi. Effective counting of simple closed geodesics on hyperbolic surfaces. *J. Eur. Math. Soc. (JEMS)*, 24(9):3059–3108, 2022.
- [EMR19] Alex Eskin, Maryam Mirzakhani, and Kasra Rafi. Counting closed geodesics in strata. *Invent. Math.*, 215(2):535–607, 2019.
- [FLP79] Albert Fathi, François Laudenbach, and Valentin Poénaru. *Travaux de Thurston sur les surfaces*, volume 66 of *Astérisque*. Société Mathématique de France, Paris, 1979. Séminaire Orsay, With an English summary.
- [FM02] Benson Farb and Lee Mosher. Convex cocompact subgroups of mapping class groups. *Geom. Topol.*, 6:91–152, 2002.
- [FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
- [Gek14] Ilya Gekhtman. *Dynamics of convex cocompact subgroups of mapping class groups*. ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)—The University of Chicago.
- [GM91] Frederick P. Gardiner and Howard Masur. Extremal length geometry of Teichmüller space. *Complex Variables Theory Appl.*, 16(2-3):209–237, 1991.
- [GM23a] Ilya Gekhtman and Biao Ma. Dynamics of subgroups of mapping class groups. *arXiv preprint arXiv:2311.03779*, 2023.
- [GM23b] Ilya Gekhtman and Biao Ma. Dynamics of subgroups of mapping class groups. *arXiv preprint arXiv:2311.03779v1*, 2023.
- [Iva92] Nikolai V. Ivanov. *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.
- [Kim06] In Kang Kim. Length spectrum in rank one symmetric space is not arithmetic. *Proc. Amer. Math. Soc.*, 134(12):3691–3696, 2006.
- [McC85] John McCarthy. A “Tits-alternative” for subgroups of surface mapping class groups. *Trans. Amer. Math. Soc.*, 291(2):583–612, 1985.
- [Mir04] Maryam Mirzakhani. *Simple geodesics on hyperbolic surfaces and the volume of the moduli space of curves*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)—Harvard University.
- [MP89] John McCarthy and Athanase Papadopoulos. Dynamics on Thurston’s sphere of projective measured foliations. *Comment. Math. Helv.*, 64(1):133–166, 1989.
- [MW95] Howard A. Masur and Michael Wolf. Teichmüller space is not Gromov hyperbolic. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 20(2):259–267, 1995.

- [Nie44] Jakob Nielsen. Surface transformation classes of algebraically finite type. *Danske Vid. Selsk. Mat.-Fys. Medd.*, 21(2):89, 1944.
- [Pap83] Athanase Papadopoulos. *Réseaux ferroviaires, difféomorphismes pseudo-Anosov et automorphismes sympléctique de l'homologie d'une surface*, volume 83-3 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1983.
- [Pap86] Athanase Papadopoulos. Geometric intersection functions and Hamiltonian flows on the space of measured foliations on a surface. *Pacific J. Math.*, 124(2):375–402, 1986.
- [Pap08] Athanase Papadopoulos. Measured foliations and mapping class groups of surfaces. *Balkan J. Geom. Appl.*, 13(1):93–106, 2008.
- [PH92] R. C. Penner and J. L. Harer. *Combinatorics of train tracks*, volume 125 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992.
- [PP87] Athanase Papadopoulos and Robert C. Penner. A characterization of pseudo-Anosov foliations. *Pacific J. Math.*, 130(2):359–377, 1987.
- [Ree81] Mary Rees. An alternative approach to the ergodic theory of measured foliations on surfaces. *Ergodic Theory Dynam. Systems*, 1(4):461–488 (1982), 1981.
- [Rob03] Thomas Roblin. Ergodicité et équidistribution en courbure négative. *Mém. Soc. Math. Fr. (N.S.)*, (95):vi+96, 2003.
- [Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.
- [Thu97] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.
- [Thu22] William P. Thurston. *The geometry and topology of three-manifolds. Vol. IV*. American Mathematical Society, Providence, RI, [2022] ©2022. Edited and with a preface by Steven P. Kerckhoff and a chapter by J. W. Milnor.

SCHOOL OF MATHEMATICS, KIAS, HOEGI-RO 85, DONGDAEMUN-GU, SEOUL 02455, SOUTH KOREA

SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE, BERKELEY, CA 94720
Email address: inhyeokchoi48@gmail.com

SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE, BERKELEY, CA 94720
Email address: dongryul.kim97@gmail.com