HOROCYCLES IN HYPERBOLIC 3-MANIFOLDS WITH ROUND SIERPIŃSKI LIMIT SETS

DONGRYUL M. KIM AND MINJU LEE

ABSTRACT. Let M be a geometrically finite hyperbolic 3-manifold whose limit set is a round Sierpiński gasket, i.e. M is geometrically finite and acylindrical with a compact, totally geodesic convex core boundary. In this paper, we classify orbit closures of the 1-dimensional horocycle flow on the frame bundle of M. As a result, the closure of a horocycle in M is a properly immersed submanifold. This extends the work of McMullen-Mohammadi-Oh where M is further assumed to be convex cocompact.

1. INTRODUCTION

Let M be a complete hyperbolic 3-manifold, and let $\chi \subset \mathsf{M}$ be an isometrically immersed copy of $\mathbb{R}$ with torsion zero and geodesic curvature 1, referred to as a *1-dimensional horocycle* or simply a *horocycle*. Shah [14] and Ratner [13] classified the closure $\overline{\chi} \subset \mathsf{M}$ in the case $\mathrm{Vol}(\mathsf{M}) < \infty$, proving that $\overline{\chi}$ is a properly immersed submanifold of M. This classification was generalized to infinite-volume hyperbolic 3-manifolds by McMullen-Mohammadi-Oh in [9], where they considered convex cocompact hyperbolic 3-manifolds with round Sierpiński limit sets.

We call M *convex cocompact* if its convex core $\mathrm{core}(\mathsf{M})$ is compact, and *geometrically finite* if the unit neighborhood of $\mathrm{core}(\mathsf{M})$ has finite volume. We say that M has a *round Sierpiński limit set* if the limit set $\Lambda \subset \widehat{\mathbb{C}}$ of the Kleinian group $\pi_1(\mathsf{M}) < \mathrm{PSL}_2(\mathbb{C})$ is a round Sierpiński gasket, i.e.,

$$\widehat{\mathbb{C}} - \Lambda = \bigcup_{i=1}^{\infty} B_i$$

is a countable union of round open disks $B_i \subset \widehat{\mathbb{C}}$ with disjoint closures (see Figure 1).

A geometrically finite hyperbolic 3-manifold M has a round Sierpiński limit set if and only if $\mathrm{core}(\mathsf{M})$ has a non-empty interior and a compact, totally geodesic boundary. Moreover, such M is acylindrical[1] and has no rank-1 cusps (Lemma 3.1). Indeed, as shown by Thurston [15] and McMullen [8, Corollary 4.3], every geometrically finite, acylindrical hyperbolic 3-manifold M with compact $\partial \mathrm{core}(\mathsf{M})$ is quasiconformally conjugate to a unique one with a round Sierpiński limit set.

---

[1] A 3-manifold is called acylindrical if its compact core (also called Scott core) has incompressible boundary and every essential cylinder therein is boundary-parallel.
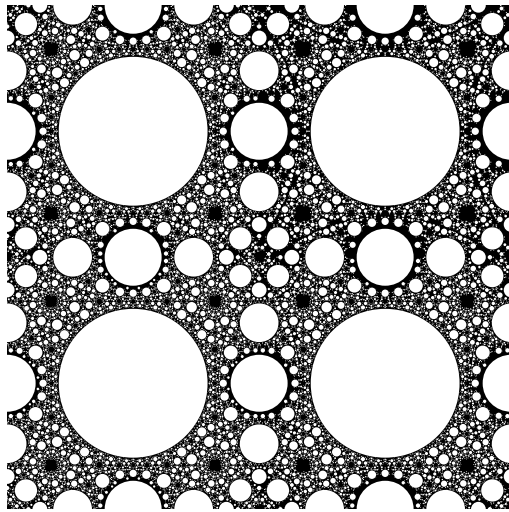
FIGURE 1. A round Sierpiński limit set with rank-2 para-
bolic limit points, drawn on $\widehat{\mathbb{C}} - \{\infty\}$. The point ∞ is also a
parabolic limit point. [2]

Convex cocompact hyperbolic 3-manifolds with round Sierpiński limit sets
have been the only known infinite-volume examples where the topological
behavior of closures of horocycles is fully understood [9]. In this paper, we
extend the classification to geometrically finite 3-manifolds:

Theorem 1.1. *Let* M *be a geometrically finite hyperbolic 3-manifold with a
round Sierpiński limit set. For any 1-dimensional horocycle* $\chi \subset$ M, *one of
the following holds:*

   (1) $\chi = \overline{\chi}$ *is closed.*
   (2) $\overline{\chi}$ *is a 2-dimensional compact horosphere.*
   (3) $\overline{\chi}$ *is a properly immersed 2-manifold, parallel to a totally geodesic
       surface* S $\subset$ M.
   (4) $\overline{\chi}$ *is the entire 3-manifold* M.

Horocycle flows on frame bundles. As in [9], Theorem 1.1 is a con-
sequence of the classification of orbit closures of the horocycle flow on the
frame bundle of M. To be precise, let $G = \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ and
consider the following subgroups:

$$H = \mathrm{PSL}_2(\mathbb{R}), \qquad N = \left\{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\},$$

$$U = \{n_s : s \in \mathbb{R}\}, \text{ and } \quad V = \{n_{is} : s \in \mathbb{R}\}.$$

---

[2]Image credit: Yongquan Zhang

For a hyperbolic 3-manifold $\mathsf{M} = \Gamma \backslash \mathbb{H}^3$ with an associated Kleinian group $\Gamma < G$, we have the identification of its frame bundle FM with $\Gamma \backslash G$:

$$\mathsf{FM} = \Gamma \backslash G.$$

Then every (oriented) horocycle $\chi \subset \mathsf{M}$ lifts uniquely to a U-orbit $xU \subset \mathsf{FM}$ for some $x \in \mathsf{FM}$ and vice versa.

We denote by

$$\mathrm{RF}_+\mathsf{M} \subset \mathsf{FM}$$

the set of all frames directed toward $\mathrm{core}(\mathsf{M})$ under the frame flow; that is, their forward trajectories project to geodesic rays in M that remain within a bounded distance of $\mathrm{core}(\mathsf{M})$. For a precise definition, see (3.1). This set $\mathrm{RF}_+\mathsf{M}$ is U-invariant, and any U-orbit outside $\mathrm{RF}_+\mathsf{M}$ is a properly immersed copy of $\mathbb{R}$. Hence, interesting dynamics appear only within $\mathrm{RF}_+\mathsf{M}$. We now state our classification of U-orbit closures in FM.

Theorem 1.2. *Let M be a geometrically finite hyperbolic 3-manifold with a round Sierpiński limit set. Then for any $x \in \mathsf{FM}$, one of the following holds:*

(1) *xU is closed.*
(2) *$\overline{xU} = xN$ which is compact.*
(3) *$\overline{xU} = xvHv^{-1} \cap \mathrm{RF}_+\mathsf{M}$ for some $v \in V$.*
(4) *$\overline{xU} = \mathrm{RF}_+\mathsf{M}$.*

Note that N-orbits and H-orbits in FM project to 2-dimensional horo-spheres in M and the images of totally geodesic immersions of a hyperbolic plane into M, respectively. In particular, a compact N-orbit in FM corresponds to a compact horosphere in M, and a closed H-orbit in FM corresponds to a totally geodesic plane in M. Therefore, Theorem 1.1 follows from Theorem 1.2.

When M is convex cocompact, Thoerem 1.2 was proved by McMullen-Mohammadi-Oh [9] and (2) does not occur in that case. This was extended by Lee-Oh [7] to higher-dimensional convex cocompact hyperbolic manifolds whose convex cores have non-empty interiors and totally geodesic bound-aries. In their work, they classified orbit closures of any connected, closed subgroup of $\mathrm{SO}^\circ(n, 1) = \mathrm{Isom}^+(\mathbb{H}^n)$ generated by unipotent elements.

Remark 1.3. We emphasize that $U < G$ is a *non-maximal* unipotent sub-group, i.e. U is not a horospherical subgroup of G. This non-maximality introduces a fundamental difficulty in studying behavior of U-orbits. For these reasons, manifolds with round Sierpiński limit sets, or equivalently, convex cores with compact, totally geodesic boundaries, were considered in [9] and [7] as well as in this paper.

Indeed, the orbit closure of a horospherical subgroup has been classified for *any* geometrically finite hyperbolic manifold by Dal'bo [2], Ferte [4], and Winter [17], without requiring any additional geometric assumption.

On the proof. We briefly outline our strategy. We adapt the idea of McMullen-Mohammadi-Oh [9], which uses the classification of closures of geodesic planes to classify closures of horocycles, to our setting. On the other hand, the presence of cusps poses an obstacle to directly applying the arguments in [9]. To address this, we incorporate techniques developed by Dani-Margulis [3] and Shah [14] as well as the specific features of a limit set, which ensure that every compact horocycle is contained within a compact horosphere.

Given a horocycle $\chi \subset \mathsf{M}$, we assume that χ is neither closed nor is its closure $\overline{\chi}$ a compact horosphere. We then show that its closure $\overline{\chi}$ is a surface parallel to a closed geodesic plane or equal to M. Our proof proceeds in two major steps:

The first step is to prove that, under the given hypothesis, $\overline{\chi}$ contains a surface in M equidistant from a geodesic plane (Proposition 10.4). This step relies on the classification of closures of geodesic planes by Benoist-Oh [1]. To achieve this, we consider the following two cases:

(a) $\overline{\chi}$ contains a compact horocycle $\chi_0 \subset \mathsf{M}$;
(b) $\overline{\chi}$ does not contain any compact horocycle in M.

A key observation in handling case (a) is that every compact horocycle is contained in a compact horosphere due to the specific feature of a limit set. Using the unipotent blowup developed in ([3], [14]), we scatter χ_0 along geodesics or horospheres and deduce that $\overline{\chi}$ contains either a surface or a compact horosphere (Corollary 8.3). Then employing the expansion of horospheres, we obtain that $\overline{\chi}$ always contains a surface (Theorem 8.1). We remark that this is the place where the presence of cusp introduces an obstruction to directly adapting the arguments in [9].

To address case (b), we utilize the notion of relatively U-minimal sets introduced in [10]. As shown in [1], horocycles in M exhibit recurrence to certain compact subsets. Based on the recurrence, we adapt the approach of [9] to our setting and show that $\overline{\chi}$ is scattered along a geodesic ray or a horosphere (Lemma 9.7). From this, we deduce that $\overline{\chi}$ contains a surface (Theorem 9.1).

As the second step, we show that if $\overline{\chi}$ contains a surface in M, then either $\overline{\chi}$ is entirely contained within the surface, or the surface is scattered along a horosphere within $\overline{\chi}$, from which the conclusion follows. In this step, we use the recurrence properties of horocycles established in [1] and the unipotent blowup for such horocycles ([10], [9]).

Geodesic planes. As mentioned above, we heavily use Besnoist-Oh's classification of closures of geodesic planes [1] to classify closures of horocycles. For other works classifying closures of geodesic planes in infinite-volume hyperbolic manifolds, see ([10], [7], [11], [18], [1], [6], [16], [5]).

Open question. It is an open question whether a similar classification of closures of horocycles holds for a general geometrically finite (acylindrical) hyperbolic 3-manifold.

Structure of the paper.

- In Section 2 and Section 3, we fix notations and terminologies used throughout the paper.
    - Section 2 is about subgroups of $\mathrm{PSL}_2(\mathbb{C})$, and
    - Section 3 is about Kleinian groups and geometrically finite 3-manifolds.
- Section 4 is devoted to the recurrence of horocycle flows and the unipotent blowup lemmas.
- In Section 5, we explain the expansion of horospheres and deduce properties of horocycles contained in an expanding sequence of horospheres by applying unipotent blowup.
- Closed geodesic planes and the horocycles contained within them are discussed in Section 6.
- In Section 7, we prove Theorem 1.2 for the closure of a horocycle that contains a closed geodesic plane.
- In Section 8, we classify closures of horocycles intersecting compact horospheres.
- The classification of closures of horocycles that do not contain any compact horocycle is addressed in Section 9.
- Finally, we prove Theorem 1.2 in Section 10.

2. Subgroups of $\mathrm{PSL}_2(\mathbb{C})$

In this section, we introduce basic notions and fix notations for subgroups of $\mathrm{PSL}_2(\mathbb{C})$ that we use throughout the paper. We mainly use the upper half-space model for the hyperbolic 3-space $\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$ whose boundary is the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We fix a basepoint $o = (0,1) \in \mathbb{H}^3$. The group of orientation-preserving isometries on $\mathbb{H}^3$ is identified with $\mathrm{PSL}_2(\mathbb{C})$, and its action on $\mathbb{H}^3$ extends to a conformal action of $\mathrm{PSL}_2(\mathbb{C})$ on $\widehat{\mathbb{C}}$ given by linear fractional transformations. Including the ones in the introduction, we fix the following notations for subgroups of

$\mathrm{PSL}_2(\mathbb{C})$:

$$G := \mathrm{PSL}_2(\mathbb{C})$$

$$K := \mathrm{stab}_G(o) \cong \mathrm{PSU}(2)$$

$$H := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : a, b, c, d \in \mathbb{R} \right\} \cong \mathrm{PSL}_2(\mathbb{R})$$

$$A := \left\{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}$$

(2.1) $$M := \left\{ a_{i\theta} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong \mathrm{PSU}(1) \cong \mathbb{S}^1$$

$$N := \left\{ n_z := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$$

$$U := \left\{ u_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\} < N$$

$$V := \left\{ v_s := \begin{pmatrix} 1 & is \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\} < N$$

We then have the identification

$$G/K = \mathbb{H}^3 \quad \text{and} \quad G/MAN = \widehat{\mathbb{C}}$$

where identity cosets $K \in G/K$ and $MAN \in G/MAN$ correspond to $o \in \mathbb{H}^3$ and $\infty \in \widehat{\mathbb{C}}$ respectively. Moreover, equipping G/K and G/MAN with the left-multiplications by G, the above identifications are G-equivariant.

Let $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subset \widehat{\mathbb{C}}$ be the standard circle given by the real axis. Then H is the orientation-preserving stabilizer of $\widehat{\mathbb{R}}$ in G. In addition, $\widehat{\mathbb{R}}$ is the boundary of the $\mathbb{H}^2$-copy in $\mathbb{H}^3$ invariant under H. Observes also that

$$AU \subset H.$$

Note that A and M commute, and U and V commute. For a subgroup $S < G$, we denote by $\mathrm{N}_G(S) < G$ the subgroup consisting of the normalizers of S in G. Then

$$AM \subset \mathrm{N}_G(N), \quad AV \subset \mathrm{N}_G(U), \quad \text{and} \quad AU \subset \mathrm{N}_G(V).$$

Moreover, for $n_z \in N$, we have

$$a_t^{-1} n_z a_t = n_{ze^{-t}} \to e \quad \text{as } t \to +\infty.$$

Throughout the paper, we use the notations a_t, n_z, u_s, u_t, v_s, and v_t to represent matrices as defined in (2.1). Abusing notations, we occasionally use a_n, u_n, or v_n to represent sequences in A, U, or V respectively, where n serves as an index rather than a matrix value. When sequences explicitly track the values of matrices, we use sequences in $\mathbb{R}$ and notations in (2.1). For example, we take a sequence $t_n \in \mathbb{R}$ and consider $a_{t_n} = \begin{pmatrix} e^{t_n/2} & 0 \\ 0 & e^{-t_n/2} \end{pmatrix} \in A$.

Frame bundle. Denote by $\mathrm{F}\mathbb{H}^3$ the frame bundle over $\mathbb{H}^3$, the space of all (positively oriented, orthonormal) frames on $\mathbb{H}^3$. The induced action of G on $\mathrm{F}\mathbb{H}^3$ is transitive, and hence we identify

$$G = \mathrm{F}\mathbb{H}^3$$

so that the quotient map $G \to G/K$ becomes the basepoint projection $\mathrm{F}\mathbb{H}^3 \to \mathbb{H}^3$, and the right-multiplication by A and U give the (geodesic) frame flow and horocycle flow on $\mathrm{F}\mathbb{H}^3$, in directions of the first and the second components of a frame respectively.

For $g \in G$, let

$$g^+ := g(\infty) \in \widehat{\mathbb{C}} \quad \text{and} \quad g^- := g(0) \in \widehat{\mathbb{C}}.$$

Via the map $G \to G/K$, the orbit gA projects to the bi-infinite geodesic $gAo \subset \mathbb{H}^3$ with endpoints $g^\pm \in \widehat{\mathbb{C}}$. Similarly, the orbit $g \cdot \{a_t \in A : t \geq 0\}$ projects to the geodesic ray in $\mathbb{H}^3$ based at $go \in \mathbb{H}^3$ and toward $g^+ \in \widehat{\mathbb{C}}$. Moreover, noting that $N < G$ projects to the horizontal plane $No = \{(z, 1) \in \mathbb{H}^3 : z \in \mathbb{C}\}$, the orbit $gNo \subset \mathbb{H}^3$ is the horosphere passing through $go \in \mathbb{H}^3$ and resting at $g^+ \in \widehat{\mathbb{C}}$. For each $n \in N$, the frame gn is based at the horosphere gNo and its first component is toward $(gn)^+ = g^+ \in \widehat{\mathbb{C}}$.

Finally, $Ho \subset \mathbb{H}^3$ is the oriented copy of $\mathbb{H}^2$ whose boundary is $\widehat{\mathbb{R}} \subset \widehat{\mathbb{C}}$ and H corresponds to the set of all frames whose first two components are restricted to positively oriented frames on the geodesic plane Ho. Hence, $gHo \subset \mathbb{H}^3$ is the geodesic plane spanned by the first two components of the frame $g \in G$ and gH is the set of all frames whose first two components are restricted to positively oriented frames on gHo. The boundary of the geodesic plane gHo is the circle $g\widehat{\mathbb{R}} = \{gh^+ \in \widehat{\mathbb{C}} : h \in H\}$.

3. Hyperbolic 3-manifolds

A discrete subgroup $\Gamma < G$ is called a Kleinian group, and the quotient $\mathsf{M} := \Gamma\backslash\mathbb{H}^3$ is a complete hyperbolic orbifold (manifold if Γ is torsion-free). We denote by $\Lambda \subset \widehat{\mathbb{C}}$ the limit set of Γ, which is defined as the set of accumulation points of $\Gamma o \subset \mathbb{H}^3$ in the compactification $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$. When $\#\Lambda \geq 3$, Γ is called non-elementary. In this case, the Γ-action on Λ is minimal and $\#\Lambda = \infty$.

The convex core of M is defined as

$$\mathrm{core}(\mathsf{M}) := \Gamma\backslash \mathrm{hull}(\Lambda) \subset \mathsf{M}$$

where $\mathrm{hull}(\Lambda) \subset \mathbb{H}^3$ is the convex hull of Λ in $\mathbb{H}^3$. We call Γ and M *geometrically finite* if the unit neighborhood of $\mathrm{core}(\mathsf{M})$ has finite volume.

In this paper, we are interested in a geometrically finite Kleinian group Γ such that Λ is the round Sierpiński gasket. In other words,

$$\widehat{\mathbb{C}} - \Lambda = \bigcup_{i=1}^{\infty} B_i$$

is a countable union of round open disks $B_i \subset \widehat{\mathbb{C}}$ with disjoint closures (Figure 1). It is clear from the definition that Γ is non-elementary, and moreover Zariski dense in $G \cong \mathrm{SO}^\circ(3,1)$. We say that such Γ and $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$ have *round Sierpiński limit set*.

We remark that these conditions imply that M is acylindrical in the sense of [15], and $\partial\,\mathrm{core}(\mathsf{M})$ is compact and totally geodesic. Indeed, every geometrically finite, acylindrical hyperbolic 3-manifold M with $\partial\,\mathrm{core}(\mathsf{M})$ compact is a quasiconformal deformation of a unique geometrically finite hyperbolic 3-manifold with a round Sierpiński limit set ([15], [8, Corollary 4.3]).

Conical and parabolic limit points. Let $\Gamma < G$ be a Kleinian group with the limit set $\Lambda \subset \widehat{\mathbb{C}}$. A limit point $x \in \Lambda$ is called *conical* if any geodesic ray in $\mathbb{H}^3$ toward x has an accumulation point in the quotient $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$ and *parabolic* if x is fixed by a parabolic element of Γ, an element conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For a parabolic limit point $x \in \Lambda$, its stabilizer $\mathrm{stab}_\Gamma(x)$ in Γ is virtually abelian, and its rank is called *rank* of x and is either 1 or 2. A parabolic limit point $x \in \Lambda$ is called *bounded parabolic* if the $\mathrm{stab}_\Gamma(x)$-action on $\Lambda - \{x\}$ is cocompact.

When Γ is geometrically finite, the limit set Λ is a disjoint union of conical limit points and bounded parabolic limit points. Moreover, there are finitely many bounded parabolic limit points $x_1, \cdots, x_n \in \Lambda$ so that

$$\Lambda = \{\text{conical limit points}\} \cup \bigcup_{i=1}^n \Gamma x_i.$$

In particular, there are at most countably many parabolic limit points.

Lemma 3.1 (cf. [1, Lemma 11.2]). *Let $\Gamma < G$ be geometrically finite with a round Sierpiński limit set Λ. Then every parabolic limit point is of rank 2.*

Proof. Suppose that there exists a parabolic limit point of rank 1, say $\infty \in \Lambda$ without loss of generality. Since Γ is geometrically finite, ∞ is bounded parabolic, and hence $\mathrm{stab}_\Gamma(\infty)$ acts cocompactly on $\Lambda - \{\infty\}$. Since $\mathrm{stab}_\Gamma(\infty)$ is virtually conjugate to the subgroup $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$, there are two parallel lines $L_1, L_2 \subset \mathbb{C}$ such that $\Lambda - \{\infty\}$ is contained in the region bounded by L_1 and L_2. Then there exist two components $B_1, B_2 \subset \widehat{\mathbb{C}} - \Lambda$ such that $L_1 \subset B_1$ and $L_2 \subset B_2$. Since $\Lambda - \{\infty\}$ is bounded by L_1 and L_2, $B_1 \neq B_2$. On the other hand, since L_1 and L_2 are lines in $\mathbb{C}$, their closures in the Riemann sphere $\widehat{\mathbb{C}}$ are circles passing through $\infty \in \widehat{\mathbb{C}}$, and hence $\overline{B_1} \cap \overline{B_2} \neq \emptyset$. This contradicts the hypothesis that Λ is a round Sierpiński limit set. $\square$

Renormalized frame bundle. Let $\Gamma < G$ be a Kleinian group and $\mathsf{M} := \Gamma\backslash G$. Since the identification $\mathrm{F}\mathbb{H}^3 = G$ is G-equivariant, this induces the

identification of the frame bundle FM of M with $\Gamma\backslash G$:

$$\text{FM} = \Gamma\backslash G.$$

Then frame flow and horocycle flow on $F\mathbb{H}^3$ descend to FM and they are given as the right-multiplications of A and U on $\Gamma\backslash G$ respectively. We denote by the projection

$$\pi : \Gamma\backslash G \to \Gamma\backslash G/K$$

which is the basepoint projection FM $\to$ M. Throughout the paper, we denote by $[g] \in \Gamma\backslash G$ the coset Γg for $g \in G$, and we refer to elements of $\Gamma\backslash G$ and G as frames in M and $\mathbb{H}^3$, respectively.

Interesting dynamics arise in certain subsets of FM. We define the renormalized frame bundle over M as

$$\text{RFM} := \{[g] \in \Gamma\backslash G : g^\pm \in \Lambda\} \subset \text{FM}.$$

This is the closed set consisting of all frames in M whose orbits under the frame flow are based at $\text{core}(M)$. It is clear that RFM is AM-invariant. We also set

(3.1) $$\text{RF}_+\text{M} := \text{RFM} \cdot N = \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\}$$

which is MAN-invariant. The projection $\pi|_{\text{RF}_+\text{M}} : \text{RF}_+\text{M} \to \text{M}$ is surjective.

Boundary frames. In the rest of the section, let $\Gamma < G$ be a geometrically finite Kleinian group with a round Sierpiński limit set, and $M := \Gamma\backslash G$. Since $\partial\,\text{core}(M)$ is totally geodesic and compact, there are finitely many elements $z_1, \cdots, z_n \in \Gamma\backslash G$ with compact H-orbits so that the set of frames whose first two components are tangent to $\partial\,\text{core}(M)$ is equal to $\bigcup_{i=1}^n z_i H \subset \Gamma\backslash G$. We set

(3.2) $$\text{BFM} := \bigcup_{i=1}^n z_i H$$

and call boundary frames. Note that $\text{BFM} \subset \text{RFM}$ and $\text{BFM} \cdot H = \text{BFM}$.

Due to the specific feature of the limit set Λ, we observe the following:

Lemma 3.2. *For any $x \in \text{RF}_+\text{M}$,*

$$x \in \text{RFM} \cdot U \quad or \quad x \in \text{BFM} \cdot V.$$

Proof. Let $x = [g] \in \text{RF}_+\text{M} - \text{RFM} \cdot U$ for $g \in G$. Then for any $u \in U$, $(gu)^- \notin \Lambda$ since $(gu)^+ = g^+ \in \Lambda$. Then the set $\{(gu)^- : u \in U\}$ is a connected subset of $\widehat{\mathbb{C}} - \Lambda$, and hence for some component Ω_0 of $\widehat{\mathbb{C}} - \Lambda$, $\{(gu)^- : u \in U\} \subset \Omega_0$ and $g^+ \in \partial\Omega_0$. Then $\{g^+\} \cup \{(gu)^- : u \in U\}$ is the round circle in $\overline{\Omega_0}$ tangent to $\partial\Omega_0$ at g^+. This implies that for some $v \in V$, $\{g^+\} \cup \{(gvu)^- : u \in U\} = \partial\Omega_0$. Noting that $(gv)^+ = g^+$, it follows that $xv = [gv] \in \text{BFM}$. $\square$

Proposition 3.3. [9, Theorem 4.1] *Let $x_n \in \text{RFM} \cdot U$ be a sequence such that $x_n \to y \in \text{RFM}$ as $n \to \infty$.*

(1) *if $y \notin$ BFM, then there exists a sequence $u_n \to e$ in U such that $x_n u_n \in$ RFM for all $n \geq 1$. In particular, $x_n u_n \to y$ as $n \to \infty$.*

(2) *if $y \in$ BFM, then passing to a subsequence of x_n, there exists a sequence $u_n \in U$ such that $x_n u_n \in$ RFM and $x_n u_n$ converges to an element of BFM as $n \to \infty$, which is potentially different from y.*

Proof. This was proved in [9] under an extra assumption that M is convex cocompact. On the other hand, the proof only uses the fact that Λ is a round Sierpiński gasket. Therefore, the same proof works verbatim in our setting. □

Volume of horospheres. Since every parabolic limit point is of rank two (Lemma 3.1), every compact U-orbit in $\Gamma \backslash G$ is contained in a compact N-orbit (Lemma 5.1). This observation is useful in our classification of closures of horocycles.

Let $x \in \Gamma \backslash G$ be such that xN is compact. We denote by $\mathsf{V}(x)$ the volume of xN with respect to the Haar measure of N. We then have

(3.3) $\mathsf{V}(xa_t n) = e^{-2t}\mathsf{V}(x)$ for all $t \in \mathbb{R}, \ n \in N$.

For each $\xi > 0$, we define the following closed subsets of $\Gamma \backslash G$:

$$F_\xi(N) := \{x \in \Gamma \backslash G : xN \text{ is compact and } \mathsf{V}(x) \leq \xi\}$$
$$F_\xi := F_\xi(N) \cdot K.$$

Geometrically, the set $F_\xi(N)$ consists of frames tangent to horospheres in a horoball, while F_ξ consists of all frames based in the horoball. Note that $\pi^{-1}(\mathrm{core}(\mathsf{M})) - \mathrm{int}(F_\xi)$ is a compact subset of $\Gamma \backslash G$, since M is geometrically finite and every parabolic limit point is of rank 2 (Lemma 3.1).

Cusp neighborhoods. For an (open) horoball h in $\mathbb{H}^3$ and $\rho \geq 0$, let $h_\rho \subset h$ be the horoball in h with distance ρ from ∂h. Since M is geometrically finite, there exists $\xi_\mathsf{M} > 0$ and finitely many horoballs $h^1, \cdots, h^n \subset \mathbb{H}^3$ with disjoint closures so that

(3.4) $\mathrm{int}(\pi(F_{\xi_\mathsf{M}})) = \bigcup_{i=1}^{n} \Gamma h^i$

where $\Gamma h^i \subset \mathsf{M}$ is the image of h^i under the quotient map $\mathbb{H}^3 \to \mathsf{M}$. Fixing such $\xi_\mathsf{M} > 0$ and h^i's, we simply write

$$\mathcal{H} := \mathrm{int}(F_{\xi_\mathsf{M}}) = \left\{x \in \Gamma \backslash G : \pi(x) \in \bigcup_{i=1}^{n} \Gamma h^i\right\}$$

For $\rho \geq 0$, we similarly define

$$\mathcal{H}_\rho := \left\{x \in \Gamma \backslash G : \pi(x) \in \bigcup_{i=1}^{n} \Gamma h_\rho^i\right\}.$$

Since every parabolic limit point is of rank two (Lemma 3.1), it follows from (3.3) that

$$(3.5) \qquad\qquad \mathcal{H}_\rho = \mathrm{int}(F_{e^{-2\rho}\xi_{\mathsf{M}}}).$$

We also define

$$W_\rho := \mathrm{RFM} - \mathcal{H}_\rho$$

which is a compact subset of $\Gamma\backslash G$. Note that $W_0 = \mathrm{RFM} - \mathcal{H}$.

4. Recurrence of horocycle flows and unipotent blowup

Let $\Gamma < G$ be a geometrically finite Kleinian group with a round Sierpiński limit set and $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$. In this section, we discuss recurrence of horocycle flows on FM, i.e., U-action on $\Gamma\backslash G$. We also collect lemmas concerning the sequence of non-trivial elements in the double coset space $S\backslash G/U$ that converges to the identity coset where $S = U$, N, or H. In [10], they were referred to as "unipotent blowup".

Recurrence of horocycle flows. To study the recurrence, we use the notion of thickness.

Definition 4.1 (Thickness). Let $k > 1$ and $T \subset \mathbb{R}$. We say that

- T is *k-thick* if
$$T \cap ([-ks, -s] \cup [s, ks]) \neq \emptyset \quad \text{for all } s > 0;$$

- T is *k-thick at ∞* if there exists $s_T \geq 0$ such that
$$T \cap ([-ks, -s] \cup [s, ks]) \neq \emptyset \quad \text{for all } s > s_T.$$

Recall from (2.1) that

$$U = \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} \quad \text{and} \quad V = \left\{ v_t = \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Via the maps $u_t \leftrightarrow t$ and $v_t \leftrightarrow t$, we identify U and V with $\mathbb{R}$ and define the thickness of subsets of U and V as well: $T \subset U$ is called k-thick (resp. k-thick at ∞) if $\{t \in \mathbb{R} : u_t \in T\}$ is k-thick (resp. k-thick at ∞). Similarly, $T \subset V$ is called k-thick (resp. k-thick at ∞) if $\{t \in \mathbb{R} : v_t \in T\}$ is k-thick (resp. k-thick at ∞).

We use the term "thickness" for subsets of $\mathbb{R}$, U, and V, with the specific choice made for convenience in each context. In many cases, we measure the thickness of the recurrence time for horocycle flows.

Definition 4.2 (Recurrence time). For $x \in \Gamma\backslash G$ and $W \subset \Gamma\backslash G$, set

$$\mathsf{T}_W(x) := \{u \in U : xu \in W\}$$

The following recurrence of horocycle flows was established by McMullen-Mohammadi-Oh:

Proposition 4.3. [10, Lemma 9.2] *There exists $k_{\mathsf{M}} > 1$ depending only on M such that for all $x \in \mathrm{RFM}$, the set $\mathsf{T}_{\mathrm{RFM}}(x)$ is k_{M}-thick.*

Proof. The proposition is stated in [10] for the case when Γ is further assumed to be convex cocompact. On the other hand, their proof only relies on the fact that Λ is a round Sierpiński gasket. Therefore, the same proof works in our setting. $\qquad\square$

For practical applications, it is desirable for the set W in Definition 4.2 to be compact. Recall that $W_\rho = \mathrm{RFM} - \mathcal{H}_\rho$ is compact for all $\rho \geq 0$. When M admits a cusp and hence RFM is not compact, W_ρ will replace the role of RFM in Proposition 4.3. We deduce the following from the work of Benoist-Oh, where general geometrically finite, acylindrical Kleinian groups were considered.

Proposition 4.4. [1, Corollary 5.5, Proposition 5.4] *Let $\xi_\mathsf{M} > 0$ and $k_\mathsf{M} > 1$ be as in (3.4) and Proposition 4.3 respectively. Then there exists $R > 0$ such that the following holds:*

(1) *for any $\rho \geq 0$ and $x \in W_\rho$, the set $\mathsf{T}_{W_{\rho+R}}(x)$ is $4k_\mathsf{M}$-thick.*
(2) *for any $\rho \geq 0$ and $x \in W_{\rho+R}$, the set $\mathsf{T}_{W_{\rho+R}}(x)$ is $4k_\mathsf{M}$-thick at ∞.*
(3) *for any $x \in \mathrm{RFM} - F_{e^{-2R}\xi_\mathsf{M}}(N)$, the set $\mathsf{T}_{W_R}(x)$ is $4k_\mathsf{M}$-thick at ∞.*

Proof. Let us explain how the statement can be deduced from [1], accounting for the differences in formulation. In [1, Corollary 5.5, Proposition 5.4], they considered the set

$$\mathrm{RF}_k\mathsf{M} := \left\{ x \in \mathrm{RFM} : \begin{array}{l} \exists\, \mathsf{T}' \subset \mathsf{T}_{\mathrm{RFM}}(x) \text{ s.t.} \\ e \in \mathsf{T}' \text{ and } \mathsf{T}' \cdot u^{-1} \text{ is } k\text{-thick } \forall u \in \mathsf{T}' \end{array} \right\}$$

for each $k > 1$ and showed that for some $R > 0$,

(4.1)
$$\mathsf{T}_{\mathrm{RF}_k\mathsf{M}-\mathcal{H}_R}(x) \text{ is } 4k\text{-thick for all } x \in \mathrm{RF}_k\mathsf{M} - \mathcal{H};$$
$$\mathsf{T}_{\mathrm{RF}_k\mathsf{M}-\mathcal{H}_R}(x) \text{ is } 4k\text{-thick at } \infty \text{ for all } x \in \mathrm{RF}_k\mathsf{M} - \mathcal{H}_R.$$

Now we take $k = k_\mathsf{M}$ given by Proposition 4.3. Then for any $x \in \mathrm{RFM}$, $e \in \mathsf{T}_{\mathrm{RFM}}(x)$. Moreover, for any $u \in \mathsf{T}_{\mathrm{RFM}}(x)$, $xu \in \mathrm{RFM}$ and $\mathsf{T}_{\mathrm{RFM}}(xu) = \mathsf{T}_{\mathrm{RFM}}(x) \cdot u^{-1}$. Hence, it follows from Proposition 4.3 that $\mathsf{T}_{\mathrm{RFM}}(x) \cdot u^{-1}$ is k_M-thick for all $u \in \mathsf{T}_{\mathrm{RFM}}(x)$. This verifies that $\mathrm{RF}_k\mathsf{M} = \mathrm{RFM}$ with $k = k_\mathsf{M}$ in our setting, and therefore (1) and (2) for $\rho = 0$ follow from (4.1), noting that $W_0 = \mathrm{RFM} - \mathcal{H}$ and $W_R = \mathrm{RFM} - \mathcal{H}_R$. In fact, only the fact that $\mathcal{H}$ consists of disjoint horoballs was used in the proof of [1, Corollary 5.5, Proposition 5.4]. Hence, we can replace $\mathcal{H}$ and $\mathcal{H}_R$ with deeper horoballs $\mathcal{H}_\rho \subset \mathcal{H}$ and $\mathcal{H}_{\rho+R} \subset \mathcal{H}_R$ for arbitrary $\rho \geq 0$ in (4.1). Therefore, (1) and (2) hold for general $\rho \geq 0$.

As for item (3), recall that $\mathcal{H} = \mathrm{int}(F_{\xi_\mathsf{M}})$ and $\mathcal{H}_R = \mathrm{int}(F_{e^{-2R}\xi_\mathsf{M}})$ from the identification we made in (3.5). Hence in terms of F_{ξ_M}, item (2) with the choice of $\rho = 0$ translates into the statement that for $R > 0$ given in the previous statement, $\mathsf{T}_{W_R}(x)$ is $4k_\mathsf{M}$-thick at ∞ for all $x \in \mathrm{RFM} - \mathrm{int}(F_{e^{-2R}\xi_\mathsf{M}})$. In the proof of [1, Proposition 5.4], the condition that $x \notin \mathrm{int}(F_{e^{-2R}\xi_\mathsf{M}})$ was used to have that xU is not contained in $\mathrm{int}(F_{e^{-2R}\xi_\mathsf{M}})$. However, it is enough to have $x \notin F_{e^{-2R}\xi_\mathsf{M}}(N)$ to guarantee that xU is

not contained in $F_{e^{-2R}\xi_\mathsf{M}}$. Therefore, the argument therein works and (3) follows.   □

Unipotent blowup: thick sets. The notion of topological limsup will be repeatedly used throughout the paper. We mainly consider a sequence of sets parametrized by a subset of $\mathbb{R}$.

Definition 4.5. Let $\mathcal{X}$ be a metric space. For $T \subset \mathbb{R}$ and family of subsets $\{Y_t \subset \mathcal{X} : t \in T\}$, we define

$$\limsup_{t \in T, t \to \infty} Y_t := \left\{ x \in \mathcal{X} : \begin{array}{l} \exists \text{ sequences } t_n \in T \text{ and } y_{t_n} \in Y_{t_n} \\ \text{s.t. } t_n \to \infty \text{ and } y_{t_n} \to x \text{ as } n \to \infty. \end{array} \right\}$$

In other words,

(4.2)
$$\limsup_{t \in T, t \to \infty} Y_t = \bigcap_{t_0 > 0} \overline{\bigcup_{t \in T, t > t_0} Y_t}.$$

When $T = \mathbb{N}$ or $(t_0, \infty) \subset T$ for some $t_0 \in \mathbb{R}$, we simply write $\limsup_{n \to \infty} Y_n$ or $\limsup_{t \to \infty} Y_t$ respectively.

We use the above definition for $\mathcal{X} = G$ or $\mathcal{X} = \Gamma \backslash G$. Note that by (4.2), the topological limsup is closed. We first record unipotent blowup lemmas that take thick sets into account.

Lemma 4.6. [1, Lemma 6.1] *Let $T \subset U$ be k-thick at ∞ for some $k > 1$. If $g_n \in G - AN$ is a sequence such that $g_n \to e$, then $\limsup_{n \to \infty} Tg_nU$ contains a sequence $\ell_n \to e$ in $AV - \{e\}$.*

Lemma 4.7. [9, Theorem 3.1] *Let $T_n \subset U$ be a sequence of k-thick sets for some $k > 1$. If $g_n \in G - HV$ is a sequence such that $g_n \to e$, then there exists $k' > 1$ depending only on k, and a k'-thick set $V_0 \subset V$ such that*

$$V_0 \subset \limsup_{n \to \infty} Hg_nT_n.$$

Unipotent blowup: polynomials. We now discuss the unipotent blowup lemma involving some polynomials, which was studied by Dani-Margulis [3] and Shah [14]. For simplicity, we use the following notations: for $z \in \mathbb{C} - \{0\}$ and $s \in \mathbb{R}$,

$$d(z) = \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \in AM \quad \text{and} \quad v(s) = \begin{pmatrix} 1 & is \\ 0 & 1 \end{pmatrix} \in V.$$

Using the notations in (2.1), $d(z) = a_{-2\log|z|}a_{-2i\arg(z)}$ and $v(s) = v_s$. We will consider real polynomials $\sigma, \nu \in \mathbb{R}[t]$ and $d(\sigma(t))v(\nu(t)) \in AV$, or a complex polynomial $\sigma \in \mathbb{C}[t]$ and $d(\sigma(t)) \in AM$, for $t \in \mathbb{R}$ such that $\sigma(t) \neq 0$. The following is the unipotent blowup with polynomials:

Lemma 4.8. [14, Proposition 4.3.2] *Let $S < G$ be either U or N. Let $g_n \in G - \mathrm{N}_G(S)$ be a sequence converging to the identity $e \in G$.*

(1) *If $S = U$, then there exist polynomials $\sigma, \nu \in \mathbb{R}[t]$ such that at least one of them is non-constant, $\sigma(0) = 1$, $\nu(0) = 0$, and*

$$d(\sigma(t))v(\nu(t)) \in \limsup_{n \to \infty} U g_n U \quad \text{for all } t \in \mathbb{R} \text{ with } \sigma(t) \neq 0.$$

(2) *If $S = N$, then there exists a non-constant polynomial $\sigma \in \mathbb{C}[t]$ such that $\sigma(0) = 1$ and*

$$d(\sigma(t)) \in \limsup_{n \to \infty} N g_n U \quad \text{for all } t \in \mathbb{R} \text{ with } \sigma(t) \neq 0.$$

Proof. As our formulation is slightly different from [14, Proposition 4.3.2], let us explain how we deduce the desired statement. Given a subgroup $S < G$ which is either U or N and a sequence $g_n \to e$ in $G - \mathrm{N}_G(S)$, what directly follows from [14, Proposition 4.3.2] is that

(1) if $S = U$, then there exist polynomials $\sigma, \nu \in \mathbb{R}[t]$ such that at least one of them is non-constant, $\sigma(0) = 1$, $\nu(0) = 0$, and

$$d(\sigma(t))v(\nu(t)) \in \overline{\bigcup_{n \in \mathbb{N}} U g_n U} \quad \text{for all } t \in \mathbb{R} \text{ with } \sigma(t) \neq 0.$$

(2) if $S = N$, then there exists a non-constant polynomial $\sigma \in \mathbb{C}[t]$ such that $\sigma(0) = 1$ and

$$d(\sigma(t)) \in \overline{\bigcup_{n \in \mathbb{N}} N g_n U} \quad \text{for all } t \in \mathbb{R} \text{ with } \sigma(t) \neq 0.$$

We handle both cases simultaneously by setting

$$\Phi(t) := \begin{cases} d(\sigma(t))v(\nu(t)) & \text{if } S = U \\ d(\sigma(t)) & \text{if } S = N \end{cases}$$

for $t \in \mathbb{R}$ with $\sigma(t) \neq 0$, where σ and ν are polynomials given above.

Then it suffices to deduce that for each $t \in \mathbb{R}$ with $\sigma(t) \neq 0$,

(4.3) $$\Phi(t) \in \limsup_{n \to \infty} S g_n U.$$

Suppose not. Then $\Phi(t) \notin \limsup_{n \to \infty} S g_n U$ for some $t \in \mathbb{R}$ with $\sigma(t) \neq 0$. Since $\Phi(t) \in \overline{\bigcup_{n \in \mathbb{N}} S g_n U}$, there exists $n \in \mathbb{N}$ and sequences $s_k \in S$ and $u_k \in U$ such that

$$\Phi(t) = \lim_{k \to \infty} s_k g_n u_k.$$

Since $g_n \notin \mathrm{N}_G(S)$, there exists $s \in S$ such that $g_n s g_n^{-1} \notin S$. Noting that S and U commute, we have

(4.4) $$\lim_{k \to \infty} s_k (g_n s g_n^{-1}) s_k^{-1} = \lim_{k \to \infty} (s_k g_n u_k) s (s_k g_n u_k)^{-1} = \Phi(t) s \Phi(t)^{-1} \in S$$

since $\Phi(t) \in \mathrm{N}_G(S)$.

For some $a, b, c, d \in \mathbb{C}$ and a sequence $c_k \in \mathbb{C}$, we write

$$g_n s g_n^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad s_k = \begin{pmatrix} 1 & c_k \\ 0 & 1 \end{pmatrix} \quad \text{for all } k \in \mathbb{N}.$$

We then have

$$s_k(g_n s g_n^{-1})s_k^{-1} = \begin{pmatrix} a + cc_k & b - ac_k + dc_k - cc_k^2 \\ c & d - cc_k \end{pmatrix}$$

which converges to an element in S by (4.4). This implies $c = 0$, and hence

$$s_k(g_n s g_n^{-1})s_k^{-1} = \begin{pmatrix} a & b + c_k(d - a) \\ 0 & d \end{pmatrix}.$$

Again, since this sequence converges to an element in S, we must have $a = d$, and hence

$$s_k(g_n s g_n^{-1})s_k^{-1} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \quad \text{for all } k \in \mathbb{N}.$$

Hence, we now have that the constant sequence $s_k(g_n s g_n^{-1})s_k^{-1}$ converges to an element of S as $k \to \infty$. In particular,

$$s_k(g_n s g_n^{-1})s_k^{-1} \in S \quad \text{for all } k \in \mathbb{N}.$$

This implies $g_n s g_n^{-1} \in S$, which is a contradiction. Therefore, (4.3) follows.

$\square$

The following proposition was proved by Dani-Margulis when σ is non-constant, and by Shah in general:

Proposition 4.9 ([3, Proposition 2.4], [14, Proposition 4.4.3]). *Let $\sigma, \nu \in \mathbb{R}[t]$ be polynomials such that one of them is non-constant. Let $t_0 \geq 0$ be such that $\sigma(t) \neq 0$ for all $t > t_0$. Defining a function $\Phi : (t_0, \infty) \to AV$ as*

(4.5) $$\Phi(t) := d(\sigma(t))v(\nu(t)),$$

there exists a non-trivial one-parameter subgroup $L < AV$, and for every $\ell \in L$, there exists a function $f_\ell : (0, \infty) \to \mathbb{R}$ such that as $t \to \infty$,

$$t + f_\ell(t) \to \infty \quad \text{and} \quad \Phi(t)^{-1}\Phi(t + f_\ell(t)) \to \ell.$$

Recalling Definition 4.5, we have:

Corollary 4.10. *Let $\Phi(t) = d(\sigma(t))v(\nu(t))$ be as in (4.5). For any $Y \subset \Gamma \backslash G$, there exists a one-parameter subgroup $L < AV$ such that*

$$\limsup_{t \to \infty} Y\Phi(t) \text{ is invariant under } L.$$

Proof. Let $X_0 := \limsup_{t \to \infty} Y\Phi(t)$ and $x_0 \in X_0$ be arbitrary. Then $x_0 = \lim_{n \to \infty} y_n \Phi(t_n)$ for some sequences $y_n \in Y$ and $t_n \to \infty$ as $n \to \infty$. We then apply Proposition 4.9: let $L < AV$ and $\{f_\ell : \ell \in L\}$ be as in the proposition associated to $\Phi(t)$. Then for every $\ell \in L$, $t_n + f_\ell(t_n) \to \infty$ and

$$y_n \Phi(t_n + f_\ell(t_n)) = y_n \Phi(t_n)(\Phi(t_n)^{-1}\Phi(t_n + f_\ell(t_n))) \to x_0 \ell.$$

This implies that

$$x_0 \ell = \lim_{n \to \infty} y_n \Phi(t_n + f_\ell(t_n)) \in X_0.$$

Since $x_0 \in X_0$ and $\ell \in L$ are arbitrary, we get $X_0 L \subset X_0$, and hence $X_0 L = X_0$. $\square$

The following is standard (cf. [14, Lemma 2.2.2]):

Lemma 4.11. *If L is a one-parameter subgroup of AV, then either $L = V$ or $L = vAv^{-1}$ for some $v \in V$.*

5. EXPANSION OF N-ORBITS

In this section, we discuss some properties of horospheres in a complete hyperbolic 3-manifold $\Gamma\backslash\mathbb{H}^3$, which correspond to N-orbits in $\mathsf{FM} = \Gamma\backslash G$. We begin with the classification of N-orbit closures. The following was proved by Ferte, and the last claim follows from Lemma 3.1.

Lemma 5.1. [4, Theorem A, Theorem B] *Let Γ be a non-elementary Kleinian group and $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$. Let $x = [g] \in \mathrm{RF}_+\mathsf{M}$ for $g \in G$.*

(1) *If g^+ is a conical limit point, then*

$$\overline{xN} = \mathrm{RF}_+\mathsf{M}.$$

(2) *If g^+ is a parabolic limit point of rank 1, then xN is closed and not compact.*

(3) *If g^+ is a parabolic limit point of rank 2, then xN is compact.*

In particular, if Γ is geometrically finite with a round Sierpiński limit set, then for any $x \in \mathrm{RF}_+\mathsf{M}$, either (1) or (3) occurs.

One key feature of horospheres is the expansion along frame flows in negative time as in the following lemma, which is a consequence of the equidistribution result due to Winter [17]:

Lemma 5.2. [17, Theorem 6.1] *Let $\Gamma < G$ be a Zariski dense geometrically finite Kleinian group and $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$. Let $x \in \mathrm{RF}_+\mathsf{M}$. For any sequence $t_n \to +\infty$ in $\mathbb{R}$ and $m_n \in M$, we have*

$$\limsup_{n\to\infty} xNm_n a_{-t_n} = \mathrm{RF}_+\mathsf{M}.$$

In the rest of the section, let $\Gamma < G$ be a geometrically finite Kleinian group with a round Sierpiński limit set Λ, and $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$. Note that both Lemma 5.1 and Lemma 5.2 apply to M. From the expansion of horospheres above, the classification of N-invariant subsets follows:

Proposition 5.3. *Let $X_0 \subset \mathrm{RF}_+\mathsf{M}$ be a closed N-invariant set. Then either $X_0 = \mathrm{RF}_+\mathsf{M}$, or there exists $\eta > 0$ such that $X_0 \subset F_\eta(N)$.*

Proof. Suppose $X_0 \neq \mathrm{RF}_+\mathsf{M}$. By Lemma 5.1, for $x = [g] \in \mathrm{RF}_+\mathsf{M}$ such that $g^+ \in \Lambda$ is a conical limit point of Γ, the orbit $xN \subset \mathrm{RF}_+\mathsf{M}$ is dense. Therefore, for any $x = [g] \in X_0$, we have that $g^+ \in \Lambda$ is a parabolic limit point of Γ. This implies that there are finitely many $z_1, \cdots, z_k \in \mathrm{RF}_+\mathsf{M}$ such that

$$X_0 \subset \bigcup_{i=1}^{k} z_i N M A.$$

Suppose to the contrary that $X_0 \not\subset F_n(N)$ for all $n \geq 1$. Then there exists a sequence $x_n \in X_0 - F_n(N)$ for all $n \geq 1$. For each $n \geq 1$, write

$$x_n = z_{i_n} p_n m_n a_{-t_n} \in z_{i_n} NMA.$$

After passing to a subsequence, we may assume that $z_{i_n} = z$ is constant. Since $\mathsf{V}(x_n) = e^{2t_n}\mathsf{V}(zp_n m_n) = e^{2t_n}\mathsf{V}(z)$ by (3.3), we must have $t_n \to +\infty$ as $n \to \infty$. We then have that X_0 contains $x_n N = zNm_n a_{-t_n}$ for all $n \geq 1$. Therefore, $\mathrm{RF}_+\mathsf{M} \subset X_0$ by Lemma 5.2, which is a contradiction. This proves the lemma. $\qquad\square$

U-orbits in expanding N-orbits. We now consider a sequence of U-orbits contained in an expanding sequence of N-orbits. We will show that we can find a $vAUv^{-1}$-orbit for some $v \in V$ in the set of accummulation points of such U-orbits (Proposition 5.5). We first prove the following lemma:

Lemma 5.4. *Let $x_n \in \mathrm{RFM}\cdot U$ be a sequence such that for any subsequence $\{x_{n_j} : x_{n_j}N \text{ is compact}\}$, we have $\mathsf{V}(x_{n_j}) \to \infty$. Then for any $\eta > 0$, there exists a neighborhood $O_\eta(N) \subset \Gamma\backslash G$ of $F_\eta(N)$ such that $\mathrm{RFM} - O_\eta(N)$ is compact and*

$$\left(\limsup_{n \to \infty} x_n U\right) \cap \mathrm{RFM} - O_\eta(N) \neq \emptyset.$$

In particular, $(\limsup_{n \to \infty} x_n U) \cap \mathrm{RFM} - F_\eta(N) \neq \emptyset$.

Proof. Let $\xi_\mathsf{M}, R > 0$ be as in (3.4) and Proposition 4.4, respectively, and $\xi_1 := e^{-2R}\xi_\mathsf{M}$. Given any $\eta > 0$, choose $s > 0$ satisfying $e^{-2s}\eta < 0.5\xi_1$.

We claim that $x_n U a_s \cap \mathrm{RFM} - F_{\xi_1}(N) \neq \emptyset$ for all sufficiently large $n \geq 1$. Note that by (3.3),

$$x_n U a_s \cap \mathrm{RFM} - F_{\xi_1}(N) = (x_n U \cap \mathrm{RFM} - F_{e^{2s}\xi_1}(N))a_s,$$

and it suffices to check that $x_n U \cap \mathrm{RFM} - F_{e^{2s}\xi_1}(N) \neq \emptyset$. Since $x_n \in \mathrm{RFM}\cdot U$, there exists $\tilde{x}_n \in x_n U \cap \mathrm{RFM}$ for each $n \geq 1$. If $x_n N$ is not compact, then neither is $\tilde{x}_n N$, and hence $\tilde{x}_n \notin F_{e^{2s}\xi_1}(N)$ trivially. Hence it suffices to consider the case that $x_n N$ is compact for infinitely many $n \geq 1$. If $x_n N$ is compact, then $\tilde{x}_n N$ is compact as well and $\mathsf{V}(\tilde{x}_n) = \mathsf{V}(x_n)$, which diverges as $n \to \infty$ by the hypothesis. Hence $\tilde{x}_n \notin F_{e^{2s}\xi_1}(N)$ for all sufficiently large $n \geq 1$. This proves the claim.

Note that $W_R = \mathrm{RFM} - \mathrm{int}(F_{\xi_1})$ using notations in Proposition 4.4. It then follows from the above claim and Proposition 4.4(3) that for $k_\mathsf{M} > 1$ given in Proposition 4.3 and all sufficiently large $n \geq 1$, there exists $y_n \in x_n U$ such that $\mathsf{T}_{W_R}(y_n a_s)$ is $4k_\mathsf{M}$-thick at ∞. In particular,

$$x_n U a_s \cap W_R = y_n U a_s \cap W_R = y_n a_s U \cap W_R \neq \emptyset.$$

Denote by $O_\eta(N) := \mathrm{int}(F_{\xi_1})a_s^{-1}$. From the definition of W_R and $\mathrm{RFM} = \mathrm{RFM} \cdot a_s$, it follows that

$$(5.1) \qquad\qquad x_n U \cap \mathrm{RFM} - O_\eta(N) \neq \emptyset$$

for all large enough $n \geq 1$. Note that $O_\eta(N)$ is an open neighborhood of $F_\eta(N)$ because

$$\text{int}(F_{\xi_1})a_s^{-1} \supset F_{0.5\xi_1}a_s^{-1} \supset F_{0.5\xi_1}(N)a_s^{-1} = F_{0.5e^{2s}\xi_1}(N) \supset F_\eta(N)$$

by (3.3). Since $\text{RFM}-O_\eta(N) = (\text{RFM}-\text{int}(F_{\xi_1}))a_s^{-1}$ is compact, the lemma follows from (5.1). $\qquad\qquad\square$

We now prove the existence of a $vAUv^{-1}$-orbit for some $v \in V$ mentioned above. Recall that for $z \in \mathbb{C} - \{0\}$ and $s \in \mathbb{R}$,

$$d(z) = \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \in AM \quad \text{and} \quad v(s) = \begin{pmatrix} 1 & is \\ 0 & 1 \end{pmatrix} \in V.$$

Proposition 5.5. *Let $y \in \text{RF}_+\text{M}$. Let $\sigma, \nu \in \mathbb{R}[t]$ be polynomials and set $\Phi(t) = d(\sigma(t))v(\nu(t))$ for $t \in \mathbb{R}$ with $\sigma(t) \neq 0$. Suppose that σ is non-constant and $y\Phi(t) \in \text{RFM} \cdot U$ for all sufficiently large $t > 0$. Then there exists $v \in V$ such that*

$$\limsup_{t\to\infty} yU\Phi(t) \text{ contains a } vAUv^{-1}\text{-orbit.}$$

Proof. Let $X_0 := \limsup_{t\to\infty} yU\Phi(t)$. We first claim that for any $\eta > 0$,

$$(5.2) \qquad\qquad X_0 \cap \text{RFM} - F_\eta(N) \neq \emptyset.$$

Fix a sequence $t_n \to \infty$ and for each $n \geq 1$, let $x_n := y\Phi(t_n) \in \text{RFM} \cdot U$. If x_{n_j} is a subsequence such that $x_{n_j}N$ is compact for all $n \geq 1$, then yN is compact as well, and moreover it follows from (3.3) that

$$\mathsf{V}(x_{n_j}) = \mathsf{V}(yd(\sigma(t_{n_j}))) = |\sigma(t_{n_j})|^4 \cdot \mathsf{V}(y).$$

Since σ is non-constant and $\lim_{j\to\infty} t_{n_j} = \infty$, we have $|\sigma(t_{n_j})| \to \infty$, from which we deduce $\mathsf{V}(x_{n_j}) \to \infty$ as $j \to \infty$. Therefore, the sequence $x_n \in \text{RFM} \cdot U$ satisfies the condition of Lemma 5.4, and hence for any $\eta > 0$,

$$\left(\limsup_{n\to\infty} x_nU\right) \cap \text{RFM} - F_\eta(N) \neq \emptyset.$$

Since $\Phi(t_n) \in AV < \mathrm{N}_G(U)$,

$$\limsup_{n\to\infty} yU\Phi(t_n) = \limsup_{n\to\infty} y\Phi(t_n)U = \limsup_{n\to\infty} x_nU,$$

and therefore

$$X_0 \cap \text{RFM} - F_\eta(N) \supset \left(\limsup_{n\to\infty} yU\Phi(t_n)\right) \cap \text{RFM} - F_\eta(N) \neq \emptyset.$$

This shows the claim.

Now by Corollary 4.10, X_0 is invariant under a one-parameter subgroup $L < AV$, and it follows from Lemma 4.11 that either $L = V$ or $L = vAv^{-1}$ for some $v \in V$. Moreover, since $\Phi(t) \in AV < \mathrm{N}_G(U)$, X_0 is U-invariant. Together with the commutativity of U and V, we now have that

$$X_0 \text{ is invariant under } N = UV \text{ or } vAUv^{-1}.$$

Suppose first that X_0 is N-invariant. By Proposition 5.3, we have $X_0 = \mathrm{RF}_+\mathsf{M}$ or $X_0 \subset F_\eta(N)$ for some $\eta > 0$. The latter is forbidden by the claim (5.2), and hence

$$X_0 = \mathrm{RF}_+\mathsf{M}$$

in this case. Then X_0 is a non-empty AU-invariant set, and therefore X_0 contains an AU-orbit as desired.

Now suppose that X_0 is invariant under $vAUv^{-1}$ for some $v \in V$. By (5.2), we in particular have that X_0 is non-empty. Therefore, there exists a $vAUv^{-1}$-orbit in X_0. This completes the proof. $\square$

6. A closed H-orbit and U-orbits therein

Let $\Gamma < G$ be a geometrically finite Kleinian group with a round Sierpiński limit set and $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$. We will also employ geometry and dynamics appearing in geodesic planes in $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$, or H-orbits in $\mathrm{FM} = \Gamma\backslash G$. In this section, we discuss properties of a closed H-orbit and U-orbits therein. Recall that $H < G$ is a copy of $\mathrm{PSL}_2(\mathbb{R})$ which is an orientation-preserving stabilizer of $\widehat{\mathbb{R}} \subset \widehat{\mathbb{C}}$, the boundary of $\mathbb{H}^2$-copy in $\mathbb{H}^3$ invariant under H. The following is the classification of H-orbit closures by Benoist-Oh:

Theorem 6.1. [1, Theorem 11.10] *Let* $y \in \mathrm{RFM}$. *Either*

$$yH \text{ is closed} \quad \text{or} \quad \overline{yH} = \mathrm{RF}_+\mathsf{M} \cdot H.$$

Remark 6.2. We note that Benoist-Oh showed Theorem 6.1 in a more general setting that M is geometrically finite and acylindrical with $\partial\,\mathrm{core}(\mathsf{M})$ totally geodesic.

When yH is closed and $y = [g]$ for $g \in G$, the conjugate $\Gamma^g := g^{-1}\Gamma g$ is the stabilizer of $y \in \Gamma\backslash G$ for the right-multiplication, and the orbit map

$$\phi : (H \cap \Gamma^g)\backslash H \to \Gamma\backslash G$$
(6.1)
$$(H \cap \Gamma^g)h \mapsto yh$$

is a proper embedding [12, Section 4.2]. Via ϕ, the closed H-orbit xH can be identified with the unit tangent bundle $(H \cap \Gamma^g)\backslash H$ of the hyperbolic surface $(H \cap \Gamma^g)\backslash\mathbb{H}^2$. In this regard, we recall the following, which is a special case of the work of Dal'bo:

Lemma 6.3. [2, Proposition B] *Let* $\Gamma_H < H$ *be a non-elementary discrete subgroup with the limit set* Λ_{Γ_H}. *Let* $y = [h] \in \Gamma_H\backslash H$ *be such that* $h^+ \in \Lambda_{\Gamma_H}$.

(1) *If* h^+ *is a conical limit point of* Γ_H, *then*

$$\overline{yU} = \{z = [\ell] \in \Gamma_H\backslash H : \ell^+ \in \Lambda_{\Gamma_H}\}.$$

(2) *If* h^+ *is a parabolic limit point of* Γ_H, *then* yU *is compact.*

In particular, if Γ_H *is geometrically finite, then for any* $y = [h] \in \Gamma_H\backslash H$ *with* $h^+ \in \Lambda_{\Gamma_H}$, *either (1) or (2) occurs.*

Note that this is a surface version of Lemma 5.1. Although we can try to apply Lemma 6.3 to $H \cap \Gamma^g < H$, the limit set we are interested in is the limit set Λ of Γ or the limit set $g^{-1}\Lambda$ of Γ^g, not the limit set of $H \cap \Gamma^g$. We first need to handle this subtlety.

U-orbits in a closed H-orbit. We deduce from the work of Oh-Shah [12] that in our setting, the limit set of $H \cap \Gamma^g$ is precisely equal to the intersection of the limit set of Γ^g with the circle $\widehat{\mathbb{R}}$ stabilized by H.

Proposition 6.4. *Let $y = [g] \in$ RFM for some $g \in G$ be such that yH is closed. Then $H \cap \Gamma^g < H$ is a non-elementary geometrically finite subgroup and its limit set is equal to $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$.*

Proof. It was shown in [12, Theorem 4.7] that $H \cap \Gamma^g$ is a geometrically finite subgroup of H. Moreover, it is clear that the limit set of $H \cap \Gamma^g$ is contained in $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$. Since $y \in$ RFM, $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$ is perfect; otherwise, $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$ has an isolated point z. Since $g^{-1}\Lambda \cap \widehat{\mathbb{R}} \subset \widehat{\mathbb{C}}$ is contained in a circle $\widehat{\mathbb{R}}$, this means that there exists an open segment $I \subset \widehat{\mathbb{R}}$ such that $I \cap g^{-1}\Lambda = \{z\}$. Since components of $\widehat{\mathbb{C}} - g^{-1}\Lambda$ are round open disks with mutually disjoint closures, there exists a component $B \subset \widehat{\mathbb{C}} - g^{-1}\Lambda$ containing $I - \{z\}$. This implies $g^{-1}\Lambda \cap \widehat{\mathbb{R}} - \{z\} \subset B$, contradicting to $y \in$ RFM.

In particular, $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$ is uncountable. Since $g^{-1}\Lambda$ is the union of conical limit points of Γ^g and countably many parabolic limit points of Γ^g, $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$ contains infinitely many conical limit points of Γ^g. By [12, Lemma 4.5], all conical limit points of Γ^g in $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$ are conical limit points of $H \cap \Gamma^g$. Therefore, $H \cap \Gamma^g$ is non-elementary.

Finally, we show that the limit set of $H \cap \Gamma^g$ is equal to $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$. Without loss of generality, we may assume that $0 \in g^{-1}\Lambda \cap \widehat{\mathbb{R}}$ and it suffices to show that 0 is in the limit set of $H \cap \Gamma^g$. There are two cases:

- Suppose first that there are sequences $\varepsilon_n', \varepsilon_n > 0$ such that $\varepsilon_n, \varepsilon_n' \to 0$ as $n \to \infty$ and $\varepsilon_n, -\varepsilon_n' \notin g^{-1}\Lambda$ for all $n \geq 1$. Then $g^{-1}\Lambda \cap (-\varepsilon_n', \varepsilon_n)$ is compact.

  We claim that $g^{-1}\Lambda \cap (-\varepsilon_n', \varepsilon_n)$ has no isolated point for all $n \geq 1$. Suppose to the contrary that $z \in g^{-1}\Lambda \cap (-\varepsilon_n', \varepsilon_n)$ is isolated. Then for some $\delta > 0$, open segments $(z - \delta, z)$ and $(z, z + \delta)$ in $(-\varepsilon_n', \varepsilon_n)$ are disjoint from $g^{-1}\Lambda$. Since $z \in g^{-1}\Lambda$ and $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$ is uncountable, this implies that there are two distinct components B_1, B_2 of $\widehat{\mathbb{C}} - g^{-1}\Lambda$ such that $(z - \delta, z) \subset B_1$ and $(z, z + \delta) \subset B_2$. On the other hand, $z \in \overline{B_1} \cap \overline{B_2}$, which is a contradiction to the hypothesis that Λ, and hence $g^{-1}\Lambda$, is a round Sierpiński limit set.

  By the above claim, $g^{-1}\Lambda \cap (-\varepsilon_n', \varepsilon_n)$ is perfect for all $n \geq 1$. In particular, $g^{-1}\Lambda \cap (-\varepsilon_n', \varepsilon_n)$ is uncountable, and hence contains a conical limit point of Γ^g for all $n \geq 1$. Since $\varepsilon_n, \varepsilon_n' \to 0$ as $n \to \infty$, this implies that there is a sequence of conical limit points of Γ^g in

$g^{-1}\Lambda \cap \widehat{\mathbb{R}}$ that converges to 0. Since every conical limit point of Γ^g is a conical limit point of $H \cap \Gamma^g$ [12, Lemma 4.5], it follows that 0 is a limit point of $H \cap \Gamma^g$, as desired.

- Otherwise, there exists $\delta > 0$ such that at least one of the segments $[-\delta, 0]$ or $[0, \delta]$ is contained in $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$. This implies that there is a sequence of conical limit points of Γ^g in $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$ that converges to 0. As in the previous case, it follows that 0 is a limit point of $H \cap \Gamma^g$.

In any case, 0 is a limit point of $H \cap \Gamma^g$, finishing the proof. $\qquad\square$

We are now able to apply Lemma 6.3 to $H \cap \Gamma^g$ and obtain the following:

Corollary 6.5. *Let $y \in \mathrm{RFM}$ be such that yH is closed. Let $z = [g_z] \in yH \cap \mathrm{RF}_+\mathsf{M}$ for $g_z \in G$.*

- *If g_z^+ is a conical limit point of Γ, then*
$$\overline{zU} = yH \cap \mathrm{RF}_+\mathsf{M}.$$

- *Otherwise, zU is compact.*

In particular, for any $y \in \mathrm{BFM}$, we have $\overline{yU} = yH$.

Proof. We first prove the claim for $z = y$. Let $g \in G$ be such that $y = [g]$. By Proposition 6.4, $H \cap \Gamma^g < H$ is a non-elementary geometrically finite subgroup and its limit set is equal to $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$. Since $e^+ = \infty \in g^{-1}\Lambda \cap \widehat{\mathbb{R}}$, we can apply Lemma 6.3 to the identity coset $[e] \in (H \cap \Gamma^g)\backslash H$. As mentioned in the proof of Proposition 6.4, e^+ is a conical limit point of $H \cap \Gamma^g$ if and only if g^+ is a conical limit point of Γ by [12, Lemma 4.5]. Hence, applying Lemma 6.3, we obtain the following dichotomy:

- if g^+ is a conical limit point of Γ, then

(6.2) $$\overline{[e]U} = \{z = [\ell] \in (H \cap \Gamma^g)\backslash H : \ell^+ \in g^{-1}\Lambda \cap \widehat{\mathbb{R}}\}.$$

- otherwise, $[e]U$ is compact.

Recall the proper embedding $\phi : (H \cap \Gamma^g)\backslash H \to \Gamma\backslash G$ from (6.1). Since $\phi([e]U) = yU$ and the right hand side of (6.2) has the image $yH \cap \mathrm{RF}_+\mathsf{M}$ under ϕ, the claim follows for y.

Now let $z \in yH \cap \mathrm{RF}_+\mathsf{M}$ be arbitrary. Since yH meets RFM, there exists $u \in U$ such that $zu \in \mathrm{RFM}$. We then have that $zuH = yH$ is closed, and hence the above claim applies to zu. Since $zuU = zU$, this finishes the proof. $\qquad\square$

The following AU-minimality is a direct consequence of Proposition 6.4:

Corollary 6.6. *Let $y \in \mathrm{RFM}$ be such that yH is closed. For any $z \in yH \cap \mathrm{RF}_+\mathsf{M}$,*
$$\overline{zAU} = yH \cap \mathrm{RF}_+\mathsf{M}.$$

Proof. We first prove the claim for $z = y$. Let $g \in G$ be such that $y = [g]$. By Proposition 6.4, $H \cap \Gamma^g < H$ is non-elementary and geometrically finite, and its limit set is $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$. Since $g^+ \in \Lambda$, $e^+ = \infty$ is contained in the limit set of $H \cap \Gamma^g$. As $H \cap \Gamma^g$ is non-elementary, it acts minimally on its limit set, and hence

$$\overline{(H \cap \Gamma^g)e^+} = g^{-1}\Lambda \cap \widehat{\mathbb{R}}.$$

Since $\widehat{\mathbb{R}} = H/AU$, the above identity is equivalent to the following identity in $(H \cap \Gamma^g)\backslash H$:

$$\overline{[e]AU} = \{[h] \in (H \cap \Gamma^g)\backslash H : h^+ \in g^{-1}\Lambda \cap \widehat{\mathbb{R}}\}.$$

As in the proof of Corollary 6.5, taking ϕ implies the claim for y. The claim for general $z \in yH \cap \mathrm{RF}_+\mathsf{M}$ can be deduced by the same argument as in the proof of Corollary 6.5. □

Expansion of U-orbits within a closed H-orbit. In the rest of this section, we discuss expanding behaviors of compact U-orbits in a closed H-orbit. The following may be standard, and can be shown by arguments in the proof Proposition 5.3:

Lemma 6.7. *Let $\Gamma_H < H$ be a non-elementary geometrically finite subgroup with the limit set Λ_{Γ_H}. Let $y_n \in \Gamma_H\backslash H$ be a sequence such that y_nU is compact for all $n \geq 1$ and the length of y_nU with respect to the Haar measure of U diverges as $n \to \infty$. Then*

$$\limsup_{n\to\infty} y_nU = \{[h] \in \Gamma_H\backslash H : h^+ \in \Lambda_{\Gamma_H}\}.$$

Proof. We sketch the argument. Since Γ_H is geometrically finite, there are finitely many elements $z_1, \cdots, z_k \in \Gamma_H\backslash H$ such that all compact U-orbits are contained in the union $\bigcup_{i=1}^{k} z_iUA$. After passing to a subsequence, we may assume that $y_n \in z_1UA$ for all $n \geq 1$, and hence there exists a sequence $t_n \in \mathbb{R}$ such that $y_nU = z_1Ua_{t_n}$ for all $n \geq 1$. Since the length of y_nU diverges, we must have $t_n \to -\infty$ as $n \to \infty$, by the scaling property of the U-Haar measure along the geodesic flow, which is similar to (3.3). Then the equidistribution result of Winter [17, Theorem 6.1] applies and finishes the proof as in Proposition 5.3. □

We now obtain the following expansion of compact U-orbits within a closed H-orbit:

Lemma 6.8. *Let $y \in \mathrm{RFM}$ be such that yH is closed. Let $\eta_n \in \mathbb{R}$ be a sequence such that $\eta_n \to \infty$ as $n \to \infty$. Let $y_n \in yH$ be a sequence such that y_nU is compact and $y_n \notin F_{\eta_n}(N)$ for all $n \geq 1$. Then*

$$\limsup_{n\to\infty} y_nU = yH \cap \mathrm{RF}_+\mathsf{M}.$$

Proof. Let $g \in G$ be such that $y = [g] \in \Gamma \backslash G$ and let $\Gamma^g := g^{-1}\Gamma g$ be its stabilizer. By Proposition 6.4, $H \cap \Gamma^g$ is a non-elementary geometrically finite subgroup of H and its limit set is $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$. Recall that the orbit map

$$\phi : (H \cap \Gamma^g) \backslash H \to \Gamma \backslash G$$
$$(H \cap \Gamma^g)h \mapsto yh$$

is a proper embedding [12, Section 4.2].

For each $n \geq 1$, let $z_n \in (H \cap \Gamma^g)\backslash H$ be such that $\phi(z_n) = y_n$. Since $y_n U$ is compact and ϕ is proper, $z_n U \subset (H \cap \Gamma^g)\backslash H$ is compact as well. Since $y_n \notin F_{\eta_n}(N)$ and $\eta_n \to \infty$, the length of $y_n U$ diverges as $n \to \infty$, and hence the length of $z_n U$ does so. We now apply Lemma 6.7 to the sequence $z_n \in (H \cap \Gamma^g)\backslash H$. Since the limit set of $H \cap \Gamma^g < H$ is $g^{-1}\Lambda \cap \widehat{\mathbb{R}}$, it follows from Lemma 6.7 that

$$\limsup_{n \to \infty} z_n U = \{[h] \in (H \cap \Gamma^g)\backslash H : h^+ \in g^{-1}\Lambda \cap \widehat{\mathbb{R}}\}.$$

Therefore, applying ϕ finishes the proof. $\qquad\qquad\square$

7. A U-ORBIT CLOSURE CONTAINING A CLOSED H-ORBIT

Let $\Gamma < G$ be a geometrically finite Kleinian group with a round Sierpiński limit set and $\mathsf{M} = \Gamma \backslash \mathbb{H}^3$. The goal of this section is to classify a U-orbit closure $X = \overline{xU}$ in the case when X contains a closed H-orbit meeting RFM.

Theorem 7.1. *Let $x \in \mathrm{RFM}$ and $X = \overline{xU}$. Suppose that there exists $y \in \mathrm{RFM}$ such that yH is closed and $yH \cap \mathrm{RF}_+\mathsf{M} \subset X$. Then either*

$$X = yH \cap \mathrm{RF}_+\mathsf{M} \quad \text{or} \quad X = \mathrm{RF}_+\mathsf{M}.$$

Recall the notion of boundary frames from (3.2). We first make the following observation on the dichotomy of closed H-orbits:

Lemma 7.2. *Let $y \in \Gamma \backslash G$ be such that yH is closed. Then either*

$$yH \subset \mathrm{BFM} \quad \text{or} \quad yH \cap \mathrm{BFM} \cdot V = \emptyset.$$

Proof. Suppose that the closed H-orbit yH intersects $\mathrm{BFM} \cdot V$. We then have $y_0 \in yH$, $z \in \mathrm{BFM}$, and $v \in V$ such that

$$y_0 = zv.$$

For $t > 0$, we have $y_0 a_t = z a_t(a_t^{-1} v a_t)$. Since $z a_t$ belongs to a compact set BFM, there exists a sequence $t_n \to \infty$ as $n \to \infty$ so that $z a_{t_n}$ converges to a point in BFM. We denote by $z_0 \in \mathrm{BFM}$ its limit. Since $a_t^{-1} v a_t \to e$ as $t \to \infty$,

$$y_0 a_{t_n} = z a_{t_n}(a_{t_n}^{-1} v a_{t_n}) \to z_0 \in \mathrm{BFM}.$$

On the other hand, $y_0 a_{t_n} \in yH$ and yH is closed, and hence $z_0 \in yH$ as well. Since BFM is H-invariant, we have

$$yH = z_0 H \subset \mathrm{BFM}.$$

This finishes the proof. $\qquad\qquad\square$

The following is the key lemma of this section:

Lemma 7.3. *Let $y \in \mathrm{RFM}$ be such that yH is closed. Let $v_n \in V$ be a sequence such that $v_n \to \infty$ as $n \to \infty$. Suppose that $yHv_n \cap \mathrm{RFM} \cdot U \neq \emptyset$ for all $n \geq 1$. Then*

$$\limsup_{n \to \infty}(yHv_n \cap \mathrm{RF}_+\mathsf{M}) = \mathrm{RF}_+\mathsf{M}.$$

Proof. For simplicity, we set $Y := yH \cap \mathrm{RF}_+\mathsf{M}$ which is AU-invariant and $X_0 := \limsup_{n\to\infty} Yv_n$. Then the lemma is equivalent to $X_0 = \mathrm{RF}_+\mathsf{M}$.

Since Y is AU-invariant and U and V commute,

$$X_0 = \limsup_{n \to \infty} Yv_n(v_n^{-1}Av_n)U.$$

Since $V = \limsup_{n\to\infty} v_n^{-1}Av_n$ and $N = VU$, we have that

$$X_0 = X_0 \cdot N.$$

By (4.2), X_0 is closed as well. Now it follows from Proposition 5.3 that either

(7.1) $X_0 = \mathrm{RF}_+\mathsf{M}$ or $X_0 \subset F_\eta(N)$ for some $\eta > 0$.

We apply Lemma 5.4 to finish the proof, by finding a sequence $y_n \in Y$ such that $y_nv_n \in Yv_n$ satisfies the condition therein. By Lemma 7.2, there are two cases:

$$yH \subset \mathrm{BFM} \quad \text{or} \quad yH \cap \mathrm{BFM} \cdot V = \emptyset.$$

Suppose first that $yH \subset \mathrm{BFM}$. In this case, for each $n \geq 1$, we choose any element $y_n \in yH$ such that $y_nv_n \in \mathrm{RFM} \cdot U$, which exists by the hypothesis. This in particular implies $y_n \in \mathrm{RF}_+\mathsf{M}$ as well, and hence $y_n \in Y$. Since $y_n \in \mathrm{BFM}$, y_nH is compact, and hence $y_nv_nN = y_nN$ is not compact for all $n \geq 1$ by Lemma 5.1. Therefore, the sequence $y_nv_n \in \mathrm{RFM} \cdot U$ satisfies the condition in Lemma 5.4.

We now consider the case that $yH \cap \mathrm{BFM} \cdot V = \emptyset$. In this case, we fix any sequence $t_n > 0$ such that $t_n \to \infty$ as $n \to \infty$, and set $y_n := ya_{t_n}^{-1} \in Y$ for each $n \geq 1$, where $a_{t_n} = \begin{pmatrix} e^{t_n/2} & 0 \\ 0 & e^{-t_n/2} \end{pmatrix}$. Since $Y \cap \mathrm{BFM} \cdot V = \emptyset$, we also have that $y_nv_n \notin \mathrm{BFM} \cdot V$, and hence $y_nv_n \in \mathrm{RFM} \cdot U$ for all $n \geq 1$ by Lemma 3.2. If $y_nv_nN = y_nN$ is compact, then

$$\mathsf{V}(y_nv_n) = \mathsf{V}(ya_{t_n}^{-1}v_n) = e^{2t_n}\mathsf{V}(y).$$

Since $t_n \to \infty$ as $n \to \infty$, the sequence $y_nv_n \in \mathrm{RFM} \cdot U$ also satisfies the condition in Lemma 5.4.

In any case, we obtain a sequence $y_nv_n \in Yv_n \cap \mathrm{RFM} \cdot U$ to which Lemma 5.4 applies. Therefore, we have for any $\eta > 0$ that

$$\left(\limsup_{n \to \infty} y_nv_nU\right) \cap \mathrm{RFM} - F_\eta(N) \neq \emptyset.$$

Since $y_n v_n U \subset Y v_n U = Y v_n$, this implies that

$$X_0 \cap \mathrm{RFM} - F_\eta(N) = \left(\limsup_{n \to \infty} Y v_n U \right) \cap \mathrm{RFM} - F_\eta(N) \neq \emptyset \quad \text{for any } \eta > 0.$$

Together with (7.1), we must have

$$X_0 = \mathrm{RF}_+\mathsf{M},$$

completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

A closed H-orbit in BFM**.** To prove Theorem 7.1, we first consider the case when the U-orbit closure contains an H-orbit in BFM $\subset$ RFM.

Proposition 7.4. *Let $x \in \mathrm{RFM}$ and $X = \overline{xU}$. Suppose that there exists $y \in \mathrm{BFM}$ such that $yH \subset X$. Then either*

$$X = yH \quad \text{or} \quad X = \mathrm{RF}_+\mathsf{M}.$$

Proof. Note that $y \in \mathrm{BFM}$ implies that yH is closed. Since $y \in X$, there exists a sequence $x_n \in xU \subset \mathrm{RFM} \cdot U$ such that $x_n \to y$ as $n \to \infty$. By Proposition 3.3(2), after passing to a subsequence, there exists a sequence $u_n \in U$ such that $x_n u_n \in \mathrm{RFM}$ and $x_n u_n \to y_0$ for some $y_0 \in \mathrm{BFM}$. Replacing x_n with $x_n u_n$, we assume that $x_n \in \mathrm{RFM}$ and $x_n \to y_0$.

Then there exists a sequence $g_n \in G$ such that $g_n \to e$ and $x_n = y_0 g_n$ for all $n \geq 1$. After passing to a subsequence, $g_n \in HV$ for all $n \geq 1$ or $g_n \notin HV$ for all $n \geq 1$.

If $g_n \in HV$ for all $n \geq 1$, we write $g_n = h_n v_n$ for some $h_n \in H$ and $v_n \in V$, and hence

$$X = \overline{x_n U} = \overline{y_0 g_n U} = \overline{y_0 h_n v_n U} = \overline{y_0 h_n U} v_n \quad \text{for all } n \geq 1.$$

Since $\overline{y_0 h_n U} \subset y_0 H$ and $y_0 H$ is compact as $y_0 \in \mathrm{BFM}$, it follows from Corollary 6.5 that $\overline{y_0 h_n U} = y_0 H$. Therefore,

$$X = y_0 H v_n \quad \text{for all } n \geq 1.$$

Since X already contains a closed H-orbit yH, we must have $v_n = e$ and

$$X = y_0 H = yH$$

in this case.

Now consider the case that $g_n \notin HV$ for all $n \geq 1$. Let $k_\mathsf{M} > 1$ and $R > 0$ as in Proposition 4.3 and Proposition 4.4 respectively. Let $\rho > 0$ be such that $x_n \to y_0$ in W_ρ. Then by Proposition 4.4(1), the set $T_n := \mathsf{T}_{W_{\rho+R}}(x_n)$ is $4k_\mathsf{M}$-thick. Applying Lemma 4.7 to $g_n \in G - HV$ and $T_n \subset U$, we have an unbounded subset $V_0 \subset V$ such that

$$V_0 \subset \limsup_{n \to \infty} H g_n T_n.$$

In particular, for any $v \in V_0$, there exist sequences $h_n \in H$ and $u_n \in T_n$ such that $h_n g_n u_n \to v$ as $n \to \infty$. Since $y_0 H$ is compact, $y_0 h_n^{-1} \in y_0 H$ converges to some element, say $y_1 \in y_0 H$. Then for each $n \geq 1$,

$$y_0 h_n^{-1}(h_n g_n u_n) = x_n u_n \in X \cap W_{\rho+R}$$

and hence taking $n \to \infty$, we have $y_1 v \in X \cap W_{\rho+R}$. Therefore,

$$(7.2) \qquad y_0 H v = \overline{y_1 U} v = \overline{y_1 v U} \subset X$$

where the first equality is due to Corollary 6.5. Moreover, since $y_1 v \in W_{\rho+R} \subset \mathrm{RFM}$ and $y_1 v \in y_0 H v$, we in particular have that $y_0 H v \cap \mathrm{RFM} \cdot U \neq \emptyset$. Since this holds for any $v \in V_0$ and $V_0 \subset V$ is unbounded, we take any sequence $v_n \to \infty$ in V_0 and apply Lemma 7.3 to conclude that

$$\limsup_{n \to \infty} (y_0 H v_n \cap \mathrm{RF}_+ \mathsf{M}) = \mathrm{RF}_+ \mathsf{M}.$$

Since $y_0 H V_0 \subset X$ as in (7.2), this implies

$$X = \mathrm{RF}_+ \mathsf{M},$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

A closed H-orbit outside BFM. We next turn to the case that yH is disjoint from BFM. Unlike the case that the H-orbit is contained in BFM, yH can contain a compact U-orbit. The proof of the following lemma is similar to that of Lemma 5.4:

Lemma 7.5. *Let $x_n \in \mathrm{RFM}$ be a sequence converging to $y \in \mathrm{RFM}$ such that the A-orbit yA is compact. Let $k_\mathsf{M} > 1$ be as in Proposition 4.3. Then for any $\eta > 0$, there exists a neighborhood $O_\eta(N) \subset \Gamma \backslash G$ of $F_\eta(N)$ such that $\mathrm{RFM} - O_\eta(N)$ is compact and*

$$\mathsf{T}_{\mathrm{RFM}-O_\eta(N)}(x_n) = \{u \in U : x_n u \in \mathrm{RFM} - O_\eta(N)\}$$

is $4k_\mathsf{M}$-thick for all sufficiently large $n \geq 1$.

Proof. Let $R > 0$ be the constant as in Proposition 4.4. We choose $\rho \geq 0$ and set $\xi := e^{-2\rho} \xi_\mathsf{M}$ such that

$$yA \subset W_{2\rho} \subset \mathrm{int}(W_\rho) = \mathrm{int}(\mathrm{RFM} - \mathrm{int}(F_\xi))$$

which is possible by the compactness of yA.

We set $\xi_1 := e^{-2R}\xi = e^{-2(\rho+R)}\xi_\mathsf{M}$. Let $\eta > 0$ be arbitrary and fix $s > 0$ such that $e^{-2s}\eta < 0.5\xi_1$. We set $O_\eta(N) := \mathrm{int}(F_{\xi_1})a_s^{-1}$, which is an open neighborhood of $F_\eta(N)$ since

$$\mathrm{int}(F_{\xi_1})a_s^{-1} \supset F_{0.5\xi_1}a_s^{-1} \supset F_{0.5\xi_1}(N)a_s^{-1} = F_{0.5e^{2s}\xi_1}(N) \supset F_\eta(N).$$

Moreover, $\mathrm{RFM} - O_\eta(N) = (\mathrm{RFM} - \mathrm{int}(F_{\xi_1}))a_s^{-1}$ is compact.

To see the thickness, note that for $u \in U$,

$$x_n u \in \mathrm{RFM} - O_\eta(N) \Leftrightarrow x_n u \in \mathrm{RFM} - \mathrm{int}(F_{\xi_1})a_s^{-1}$$

$$\Leftrightarrow (x_n a_s)(a_s^{-1} u a_s) \in \mathrm{RFM} - \mathrm{int}(F_{\xi_1})$$

$$\Leftrightarrow a_s^{-1} u a_s \in \mathsf{T}_{W_{\rho+R}}(x_n a_s)$$

since $W_{\rho+R} = \mathrm{RFM} - \mathrm{int}(F_{\xi_1})$. Observing that $a_s^{-1} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} a_s = \begin{pmatrix} 1 & te^{-s} \\ 0 & 1 \end{pmatrix}$ for all $t \in \mathbb{R}$, it suffices to show that $\mathsf{T}_{W_{\rho+R}}(x_n a_s)$ is $4k_\mathsf{M}$-thick.

On the other hand, since yA is contained in the open subset $\operatorname{int}(W_\rho) \subset$ RFM, it follows from $x_n a_s \to y a_s$ that $x_n a_s \in W_\rho$ for all large $n \geq 1$. Therefore, $\mathsf{T}_{W_{\rho+R}}(x_n a_s)$ is $4k_{\mathsf{M}}$-thick by Proposition 4.4(1). This completes the proof. $\qquad\square$

Recall that $v_t = \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix} \in V$ for $t \in \mathbb{R}$.

Lemma 7.6. *Let $x \in \mathrm{RF}_+\mathsf{M}$ and $X = \overline{xU}$. Let $y \in \mathrm{RFM}$ be such that yH is closed. Let $I \subset \mathbb{R}$ be a compact subset. Suppose that for any $\eta > 0$, there exist $t_\eta \in I$ and $y_\eta \in yH \cap \mathrm{RF}_+\mathsf{M}$ such that $y_\eta v_{t_\eta} \in X - F_\eta(N)$. Then there exists $T_I \in I$ such that*

$$(yH \cap \mathrm{RF}_+\mathsf{M})v_{T_I} \subset X.$$

Proof. Since U and V commute and X is U-invariant, $\overline{y_\eta U}v_{t_\eta} \subset X$ for all $\eta > 0$. Hence, if $\overline{y_{\eta_0} U} = yH \cap \mathrm{RF}_+\mathsf{M}$ for some $\eta_0 > 0$, setting $T_I := t_{\eta_0}$ verifies the claim.

We now assume that $\overline{y_\eta U} \neq yH \cap \mathrm{RF}_+\mathsf{M}$ for all $\eta > 0$. By Corollary 6.5, $\overline{y_\eta U} \neq yH \cap \mathrm{RF}_+\mathsf{M}$ implies that $y_\eta U$ is compact for all $\eta > 0$. Since I is compact, there exists a sequence $\eta_n > 0$ such that $\eta_n \to \infty$ and $t_n := t_{\eta_n} \in I$ converges to some $T_I \in I$ as $n \to \infty$. Since $y_{\eta_n} v_{t_n} \notin F_\eta(N)$, or equivalently $y_{\eta_n} \notin F_\eta(N)$, we have

$$\limsup_{n \to \infty} y_{\eta_n} U = yH \cap \mathrm{RF}_+\mathsf{M}$$

by Lemma 6.8. Since $v_{t_n} \to v_{T_I}$ as $n \to \infty$, this implies

$$\limsup_{n \to \infty} y_{\eta_n} v_{t_n} U = \limsup_{n \to \infty} y_{\eta_n} U v_{t_n} = (yH \cap \mathrm{RF}_+\mathsf{M})v_{T_I}.$$

Since X is a closed U-invariant set and $y_{\eta_n} v_{t_n} \in X$ for all $n \geq 1$, it follows that

$$(yH \cap \mathrm{RF}_+\mathsf{M})v_{T_I} \subset X,$$

as desired. $\qquad\square$

We now classify the U-orbit closure containing a closed H-orbit outside of BFM.

Proposition 7.7. *Let $x \in \mathrm{RFM}$ and $X = \overline{xU}$. Suppose that there exists $y \in \mathrm{RFM} - \mathrm{BFM}$ such that yH is closed and $yH \cap \mathrm{RF}_+\mathsf{M} \subset X$. Then either*

$$X = yH \cap \mathrm{RF}_+\mathsf{M} \quad or \quad X = \mathrm{RF}_+\mathsf{M}.$$

Proof. By Proposition 6.4, there exists a compact A-orbit in yH. Hence, by replacing y with an element of yH, we may assume that yA is compact. Note that we still have $y \in \mathrm{RFM} - \mathrm{BFM}$ after the replacement due to Lemma 7.2.

Since $y \in X$, there exists a sequence $x_n \in xU$ such that $x_n \to y$ as $n \to \infty$. Since $y \notin \mathrm{BFM}$, by Proposition 3.3(1), we may further assume that $x_n \in \mathrm{RFM}$ by modifying $x_n \in xU$, without changing y.

We next write $x_n = yg_n$ for some sequence $g_n \in G$ such that $g_n \to e$ as $n \to \infty$. After passing to a subsequence, $g_n \in HV$ for all $n \geq 1$ or $g_n \notin HV$ for all $n \geq 1$. Suppose first that $g_n \in HV$ for all $n \geq 1$. For each $n \geq 1$, write $g_n = h_n v_n$ for some $h_n \in H$ and $v_n \in V$. We then have

$$X = \overline{x_n U} = \overline{y h_n U} v_n \quad \text{for all } n \geq 1.$$

Note that $yh_n \in yH \cap \mathrm{RF}_+\mathsf{M}$ and hence either $\overline{yh_n U}$ is compact or $\overline{yh_n U} = yH \cap \mathrm{RF}_+\mathsf{M}$ by Corollary 6.5. Since $yH \cap \mathrm{RF}_+\mathsf{M} \subset X$ by hypothesis, necessarily $v_n = e$ and $X = yH \cap \mathrm{RF}_+\mathsf{M}$ in this case.

Now consider the case that $g_n \notin HV$ for all $n \geq 1$. Let $k_\mathsf{M} > 1$ be given in Proposition 4.3 and $k' > 1$ the constant given in Lemma 4.7 but associated to $k = 4k_\mathsf{M}$ instead of k_M.

We claim that for every $\eta > 0$ and $r > 0$,

$$\exists t_{\eta,r} \in [-k'r, -r] \cup [r, k'r] \quad \text{and} \quad y_{\eta,r} \in yH \cap \mathrm{RF}_+\mathsf{M}$$

$$\text{such that } y_{\eta,r} v_{t_{\eta,r}} \in X - F_\eta(N).$$

Let us verify the claim. Fix arbitrary η, $r > 0$ and let $O_\eta(N)$ be the open set in Lemma 7.5 associated to $\eta > 0$. Then $T_n := \mathsf{T}_{\mathrm{RFM} - O_\eta(N)}(x_n)$ is $4k_\mathsf{M}$-thick for all sufficiently large $n \geq 1$ by Lemma 7.5. Next, apply Lemma 4.7 to obtain a k'-thick set $V_0 \subset V$ with $k' = k'(4k_\mathsf{M})$ such that

$$V_0 \subset \limsup_{n \to \infty} H g_n T_n.$$

Since V_0 is k'-thick, we can find $t_{\eta,r} \in [-k'r, -r] \cup [r, k'r]$ such that $v_{t_{\eta,r}} \in V_0$. Hence, there exist sequences $h_n \in H$, $u_n \in T_n$ such that $h_n g_n u_n \to v_{t_{\eta,r}}$ as $n \to \infty$. Note that

$$(yh_n^{-1})(h_n g_n u_n) = x_n u_n \in X \cap \mathrm{RFM} - O_\eta(N) \quad \text{for all } n \geq 1.$$

Since $\mathrm{RFM} - O_\eta(N)$ is compact, $x_n u_n$ is convergent after passing to a subsequence, and so is yh_n^{-1}, to some element $y_{\eta,r} \in yH$. Since $O_\eta(N)$ is a neighborhood of $F_\eta(N)$, taking the limit $n \to \infty$, we obtain $y_{\eta,r} v_{t_{\eta,r}} \in X - F_\eta(N)$. Since $X \subset \mathrm{RF}_+\mathsf{M}$, we have $y_{\eta,r} \in \mathrm{RF}_+\mathsf{M}$ and the claim follows.

Fixing $r > 0$ and by varying $\eta > 0$, we apply Lemma 7.6 to the compact set $[-k'r, -r] \cup [r, k'r]$. The previous claim and Lemma 7.6 imply that there exists $T_r \in [-k'r, -r] \cup [r, k'r]$ such that

(7.3) $(yH \cap \mathrm{RF}_+\mathsf{M}) v_{T_r} \subset X$

Repeating this for increasing values of $r > 0$, we obtain a sequence $|T_r| \to \infty$.

Since $y \notin \mathsf{BFM}$, we in particular have $y \notin \mathsf{BFM} \cdot V$ by Lemma 7.2. This implies that $y v_{T_r} \notin \mathsf{BFM} \cdot V$ for all $r > 0$. Since $y \in \mathrm{RFM}$, we have $y v_{T_r} \in \mathrm{RF}_+\mathsf{M}$, and hence $y v_{T_r} \in \mathrm{RFM} \cdot U$ by Lemma 3.2. In particular,

$$yH v_{T_r} \cap \mathrm{RFM} \cdot U \neq \emptyset \quad \text{for all } r > 0.$$

Since $v_{T_r} \to \infty$ as $r \to \infty$, it follows from Lemma 7.3 that

$$\limsup_{r \to \infty} (yH v_{T_r} \cap \mathrm{RF}_+\mathsf{M}) = \mathrm{RF}_+\mathsf{M}.$$

Together with (7.3), $X = \mathrm{RF}_+\mathsf{M}$. This finishes the proof.   □

Proof of Theorem 7.1. Let $x \in \mathrm{RFM}$ and $X = \overline{xU}$. Suppose that there exists $y \in \mathrm{RFM}$ such that yH is closed and $yH \cap \mathrm{RF}_+\mathsf{M} \subset X$. If $y \in \mathrm{RFM} - \mathrm{BFM}$, then the claim follows from Proposition 7.7. If $y \in \mathrm{BFM}$, then $X = yH$ or $X = \mathrm{RF}_+\mathsf{M}$ by Proposition 7.4. Since $yH \subset \mathrm{BFM} \subset \mathrm{RFM}$ in this case, this completes the proof.   □

8. A U-ORBIT CLOSURE MEETING A COMPACT N-ORBIT

Let $\Gamma < G$ be a geometrically finite Kleinian group with a round Sierpiński limit set and $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$. The goal of the section is to prove the following dichotomy for U-orbit closures meeting compact N-orbits.

Theorem 8.1. *Let* $x \in \mathrm{RF}_+\mathsf{M}$ *and* $X := \overline{xU}$. *Suppose that* X *meets a compact* N-*orbit. Then one of the following holds:*

   (1) xN *is compact.*
   (2) X *contains a* $vAUv^{-1}$-*orbit for some* $v \in V$.

The proof is based on unipotent blowup involving polynomials. Recall that for $z \in \mathbb{C} - \{0\}$ and $s \in \mathbb{R}$,

$$d(z) = \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \in AM \quad \text{and} \quad v(s) = \begin{pmatrix} 1 & is \\ 0 & 1 \end{pmatrix} \in V.$$

Lemma 8.2. *Let* $x \in \mathrm{RF}_+\mathsf{M}$ *and* $X := \overline{xU}$. *Suppose that there exists* $y \in X$ *such that* yN *is compact. Then one of the following holds:*

   (1) xN *is compact.*
   (2) *there exist polynomials* $\sigma, \nu \in \mathbb{R}[t]$ *such that at least one of them is non-constant,* $\sigma(0) = 1$, $\nu(0) = 0$, *and*

$$\overline{yU}\Phi(t) \subset X \quad \text{for all } t \in \mathbb{R} \text{ s.t. } \sigma(t) \neq 0$$

   where $\Phi(t) = d(\sigma(t))v(\nu(t))$.

Proof. Since $y \in X$, there exists a sequence $x_n \in xU$ such that $x_n \to y$ as $n \to \infty$. We may write $x_n = yg_n$ for some sequence $g_n \to e$ in G. After passing to a subsequence, we have either $g_n \in \mathrm{N}_G(U)$ for all $n \geq 1$ or $g_n \notin \mathrm{N}_G(U)$ for all $n \geq 1$.

Suppose first that $g_n \in \mathrm{N}_G(U)$ for all $n \geq 1$. Then for each $n \geq 1$,

$$xU = yg_nU = yUg_n,$$

and hence there exists $u_n \in U$ such that $x = yu_ng_n$. Since $yu_n \in yN$ and yN is compact, after passing to a subsequence, we may assume that yu_n converges to some $z \in yN$. It then follows from $g_n \to e$ that $x = z$, and therefore

$$xN = zN = yN \text{ is compact.}$$

Now assume that $g_n \notin \mathrm{N}_G(U)$ for all $n \geq 1$. Applying Lemma 4.8(1) with $S = U$, we obtain polynomials $\sigma, \nu \in \mathbb{R}[t]$ such that at least one of them is non-constant, $\sigma(0) = 1$, $\nu(0) = 0$, and

$$(8.1) \quad \Phi(t) = d(\sigma(t))v(\nu(t)) \in \limsup_{n \to \infty} U g_n U \quad \text{for all } t \in \mathbb{R} \text{ s.t. } \sigma(t) \neq 0.$$

Let $t \in \mathbb{R}$ be such that $\sigma(t) \neq 0$. By (8.1), there exist sequences $\widehat{u}_n, u'_n \in U$ such that $\widehat{u}_n g_n u'_n \to \Phi(t)$ as $n \to \infty$, after passing to a subsequence. Observe that

$$y\widehat{u}_n^{-1}(\widehat{u}_n g_n u'_n) = x_n u'_n \in X \quad \text{for all } n \geq 1.$$

Since yN is compact, passing to a subsequence, there exists $y_t \in \overline{yU}$ such that $y\widehat{u}_n^{-1} \to y_t$ as $n \to \infty$. This implies

$$y_t \Phi(t) \in X.$$

Since $\overline{yU}$ is a U-minimal subset of a compact N-orbit yN, it follows from $\Phi(t) \in AV < \mathrm{N}_G(U)$ that

$$\overline{yU}\Phi(t) = \overline{y_t U}\Phi(t) = \overline{y_t \Phi(t) U} \subset X.$$

Therefore, (2) holds. □

As a corollary, we obtain:

Corollary 8.3. *Let $x \in \mathrm{RF}_+\mathsf{M}$, and $X = \overline{xU}$. Suppose that X meets a compact N-orbit. Then one of the following holds:*

(1) *xN is compact.*
(2) *for any $y \in X$,*

$$yN \text{ is compact} \Longrightarrow yN \subset X.$$

(3) *X contains a $vAUv^{-1}$-orbit for some $v \in V$.*

Proof. Assume that (1) and (2) do not hold. We will prove that (3) is the case. Since (2) does not hold, there exists $y \in X$ such that yN is compact and $yN \not\subset X$. We then apply Lemma 8.2: Lemma 8.2(1) cannot occur by our assumption, and hence there exist polynomials $\sigma, \nu \in \mathbb{R}[t]$ such that at least one of them is non-constant, $\sigma(0) = 1$, $\nu(0) = 0$, and

$$(8.2) \qquad \overline{yU}\Phi(t) \subset X \quad \text{for all } t \in \mathbb{R} \text{ s.t. } \sigma(t) \neq 0$$

where $\Phi(t) = d(\sigma(t))v(\nu(t))$.

We claim that σ is non-constant. Suppose not and let

$$X_1 := \limsup_{t \to \infty} \overline{yU}\Phi(t) \subset X.$$

Since $\sigma(0) = 1$, $\Phi(t) \in V$ and hence $\overline{yU}\Phi(t) \subset yN$. Since yN is compact, we have $X_1 \neq \emptyset$. Applying Corollary 4.10 to $Y := \overline{yU}$, it follows that X_1 is invariant under a one-parameter subgroup $L < AV$. By Lemma 4.11, either $L = vAv^{-1}$ for some $v \in V$ or $L = V$. Since $X_1 \subset yN$ and yN is compact, we must have $L = V$. Together with the U-invariance of X_1, we have

$$X_1 = yN.$$

This contradicts $yN \not\subset X$, and the claim follows.

To finish the proof, suppose first that $y\Phi(t) \in \mathrm{RFM} \cdot U$ for all sufficiently large $t > 0$. Then by Proposition 5.5, $\limsup_{t\to\infty} yU\Phi(t)$ contains a $vAUv^{-1}$-orbit for some $v \in V$, and therefore (3) holds due to (8.2).

Otherwise, $y\Phi(t) \in \mathrm{BFM} \cdot V$ for some $t \in \mathbb{R}$ by Lemma 3.2. Since $\Phi(t) \in AV$, we have $y = zv^{-1}$ for some $z \in \mathrm{BFM}$ and $v \in V$. Since zH is a compact H-orbit, we have from Corollary 6.5 that

$$\overline{yU} = \overline{zU}v^{-1} = zHv^{-1} = zv^{-1}(vHv^{-1}).$$

Therefore,

$$zv^{-1}(vAUv^{-1}) \subset zv^{-1}(vHv^{-1}) = \overline{yU} \subset X,$$

and hence (3) holds, completing the proof. $\qquad\square$

We prove one more lemma.

Lemma 8.4. *Let $x \in \mathrm{RF}_+\mathsf{M}$ and $X := \overline{xU}$. Suppose that there exists $y \in X$ such that yN is compact. Then one of the following holds:*

(1) *xN is compact.*
(2) *there exists a non-constant polynomial $\sigma \in \mathbb{C}[t]$ with $\sigma(0) = 1$ so that for any $t \in \mathbb{R}$ satisfying $\sigma(t) \neq 0$, there exists $y_t \in yN$ such that*

$$y_t d(\sigma(t)) \in X.$$

Proof. Since $y \in X$, there exists a sequence $x_n \in xU$ such that $x_n \to y$ as $n \to \infty$. We may write $x_n = yg_n$ for some sequence $g_n \to e$ in G. After passing to a subsequence, we may assume that either $g_n \in \mathrm{N}_G(N)$ for all $n \geq 1$ or $g_n \notin \mathrm{N}_G(N)$ for all $n \geq 1$.

If $g_n \in \mathrm{N}_G(N)$ for all $n \geq 1$, then

$$x \in x_n U = yg_n U \subset yg_n N = yNg_n \quad \text{for all } n \geq 1.$$

Since yN is compact and $g_n \to e$, this implies that xN is compact, and (1) follows.

Now assume $g_n \notin \mathrm{N}_G(N)$ for all $n \geq 1$. Applying Lemma 4.8(2) with $S = N$, we obtain a non-constant polynomial $\sigma \in \mathbb{C}[t]$ with $\sigma(0) = 1$ satisfying

(8.3) $\qquad d(\sigma(t)) \in \limsup_{n\to\infty} Ng_n U \quad \text{for all } t \in \mathbb{R} \text{ s.t. } \sigma(t) \neq 0.$

Let $t \in \mathbb{R}$ be such that $\sigma(t) \neq 0$. By (8.3), there exist sequences $p_n \in N$ and $u_n \in U$ such that $p_n g_n u_n \to d(\sigma(t))$ as $n \to \infty$. Since yN is compact, after passing to a subsequence, yp_n^{-1} converges to some $y_t \in yN$. We then have

$$y_t d(\sigma(t)) = \lim_{n\to\infty} yp_n^{-1}(p_n g_n u_n) = \lim_{n\to\infty} x_n u_n \in X.$$

Therefore, (2) holds. $\qquad\square$

Remark 8.5. We remark that proofs of Lemma 8.2 and Lemma 8.4 work for a general Kleinian group $\Gamma < G$.

Proof of Theorem 8.1. Suppose that xN is not compact. By Corollary 8.3, it suffices to consider the case Corollary 8.3(2) that any compact N-orbit meeting X is contained in X. Assume that we are in such a case. By the hypothesis, there exists $y \in X$ such that yN is compact. Since xN is not compact, it follows from Lemma 8.4 that we can find a non-constant polynomial $\sigma \in \mathbb{C}[t]$ such that for all $t \in \mathbb{R}$ satisfying $\sigma(t) \neq 0$, we have for some $y_t \in yN$ that

$$y_t d(\sigma(t)) \in X.$$

Note that $d(\sigma(t)) \in \mathrm{N}_G(N)$ and hence

$$y_t d(\sigma(t))N = y_t N d(\sigma(t)) = yN d(\sigma(t))$$

is a compact N-orbit meeting X. Therefore, it follows from the hypothesis that

$$yN d(\sigma(t)) \subset X.$$

Since $\sigma \in \mathbb{C}[t]$ is non-constant, we have $|\sigma(t)| \to \infty$ as $t \to \infty$ in $\mathbb{R}$. Hence $X = \mathrm{RF}_+\mathsf{M}$ by Lemma 5.2. In particular, X contains a $vAUv^{-1}$-orbit for some $v \in V$. This finishes the proof. $\qquad\square$

9. A U-orbit closure without any compact U-orbit

Let $\Gamma < G$ be a geometrically finite Kleinian group with a round Sierpiński limit set and $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$. In this section, we consider an orbit closure $\overline{xU}$ without any compact U-orbit. The following is the main theorem of this section:

Theorem 9.1. *Let $x \in \mathrm{RFM}$ and $X := \overline{xU}$. Suppose that X does not contain any compact U-orbit. Then one of the following holds:*

(1) *there exists a compact N-orbit in X.*
(2) *there exists a $vAUv^{-1}$-orbit in X for some $v \in V$.*

To prove Theorem 9.1, we recall the notion of relatively minimal sets, introduced in [10]:

Definition 9.2. Let $W \subset \Gamma\backslash G$. A closed subset $Y \subset \Gamma\backslash G$ is called U-*minimal relative to* W if $Y \cap W \neq \emptyset$ and $\overline{yU} = Y$ for all $y \in Y \cap W$.

Note that a relatively U-minimal set is U-invariant. If W is compact, then any closed U-invariant set $Y \subset \Gamma\backslash G$ such that $Y \cap W \neq \emptyset$ contains a U-minimal set relative to W. This usually follows from Zorn's lemma.

Given two closed subsets in $\Gamma\backslash G$, we collect elements of G that deliver one subset to the other.

Definition 9.3. For any closed subsets Y_1, $Y_2 \subset \Gamma\backslash G$, We define

$$D(Y_1, Y_2) := \{g \in G : Y_1 g \cap Y_2 \neq \emptyset\}.$$

Note that for $x \in \Gamma\backslash G$ and a closed subset $Y \subset \Gamma\backslash G$,

$$\mathsf{T}_Y(x) = D(\{x\}, Y) \cap U.$$

Lemma 9.4. *Let $Y_1, Y_2 \subset \Gamma\backslash G$ be closed subsets and $S_1, S_2 < G$.*

(1) *If one of Y_i $(i = 1, 2)$ is compact, then $D(Y_1, Y_2)$ is closed.*
(2) *If $Y_iS_i = Y_i$ for $i = 1, 2$, then $D(Y_1, Y_2) = S_1D(Y_1, Y_2)S_2$.*

The proof of Lemma 9.4 is rather straightforward and will be omitted.

Lemma 9.5. *Let Y_1, $Y_2 \subset \Gamma\backslash G$ be closed U-invariant sets and $W \subset \Gamma\backslash G$ a compact subset. Suppose that Y_1 is U-minimal relative to W. Then*

$$D(Y_1 \cap W, Y_2) \cap \mathrm{N}_G(U) = \{g \in \mathrm{N}_G(U) : Y_1g \subset Y_2\}.$$

In particular, $D(Y_1 \cap W, Y_1) \cap \mathrm{N}_G(U)$ is a closed subsemigroup of $\mathrm{N}_G(U)$.

Proof. The hypothesis that Y_1 is U-minimal relative to W implies that $Y_1 \supset Y_1 \cap W \neq \emptyset$, and hence $\{g \in \mathrm{N}_G(U) : Y_1g \subset Y_2\} \subset D(Y_1 \cap W, Y_2) \cap \mathrm{N}_G(U)$.

Conversely, let $g \in D(Y_1 \cap W, Y_2) \cap \mathrm{N}_G(U)$. Then $y_1g = y_2$ for some $y_1 \in Y_1 \cap W$ and $y_2 \in Y_2$. Since Y_1 is U-minimal relative to W and $g \in \mathrm{N}_G(U)$, we have

$$Y_1g = \overline{y_1U}g = \overline{y_1gU} = \overline{y_2U} \subset Y_2.$$

Hence, the reverse inclusion follows. The last assertion is straightforward. $\square$

We will use the following lemma:

Lemma 9.6. [1, Lemma 8.2] *Let $Y \subset \Gamma\backslash G$ be a U-minimal set relative to a compact subset $W \subset \Gamma\backslash G$ and $y \in Y \cap W$. If $\mathsf{T}_{Y\cap W}(y)$ is unbounded, then there exists a sequence $u_n \to \infty$ in U such that $yu_n \to y$ as $n \to \infty$.*

An analogue of the following proposition was proved in [10, Theorem 9.4] when Γ is further assumed to be convex cocompact:

Lemma 9.7. *Let $Y \subset \Gamma\backslash G$ be a U-minimal set relative to a compact subset $W \subset \Gamma\backslash G$. Suppose that Y is not a compact U-orbit, and that $\mathsf{T}_W(y)$ is k-thick at ∞ for some $y \in Y \cap W$ and $k > 1$. Then there exists a one-parameter subsemigroup $L_+ < AV$ such that*

$$YL_+ \subset Y.$$

Proof. Since $Y \cap W$ is compact, $D(Y \cap W, Y)$ is closed by Lemma 9.4(1). We claim that there exists a non-trivial element in $D(Y \cap W, Y) \cap AV$ arbitrarily close to e. Since $AV < \mathrm{N}_G(U)$ is closed and

$$D(Y \cap W, Y) \cap \mathrm{N}_G(U) = \{g \in \mathrm{N}_G(U) : Yg \subset Y\} < \mathrm{N}_G(U)$$

is a closed subsemigroup by Lemma 9.5, the lemma follows from the claim.

Let $y \in Y \cap W$ be such that $\mathsf{T}_W(y)$ is k-thick at ∞. Since $yU \subset Y$, $\mathsf{T}_{Y\cap W}(y) = \mathsf{T}_W(y)$ is unbounded. By Lemma 9.6, there exists a sequence $u_n \to \infty$ in U such that $yu_n \to y$ as $n \to \infty$. We can write $yu_n = yg_n$ for some sequence $g_n \to e$ in G as $n \to \infty$. We in particular have

(9.1) $$g_n \in D(Y \cap W, Y).$$

We first observe that $g_n \notin U$ for all but finitely many $n \geq 1$. Since $\overline{yU} = Y$ and Y is not a compact U-orbit, yU is not compact. Hence, if $g_n \in U$, then it follows from $yu_n g_n^{-1} = y$ that $u_n = g_n$. Since $u_n \to \infty$ and $g_n \to e$ as $n \to \infty$, this is possible only for finitely many $n \geq 1$.

We now construct a non-trivial sequence $\ell_n \to e$ in $D(Y \cap W, Y) \cap AV$, which completes the proof as mentioned above. After passing to a subsequence, either $g_n \in AN$ for all $n \geq 1$ or $g_n \notin AN$ for all $n \geq 1$. First, consider the case that $g_n \in AN$ for all $n \geq 1$. Since $N = VU$, there exists a sequence $\widehat{u}_n \to e$ in U such that $g_n \widehat{u}_n \in AV$ for all $n \geq 1$, and $g_n \widehat{u}_n \to e$ in particular. By the above observation, $g_n \widehat{u}_n \neq e$ for all large enough $n \geq 1$. By (9.1) and Lemma 9.4(2), we have $g_n \widehat{u}_n \in D(Y \cap W, Y)$ for all $n \geq 1$. Therefore, we take $\ell_n = g_n \widehat{u}_n$ in this case.

Now assume that $g_n \notin AN$ for all $n \geq 1$. Let

$$T := \mathsf{T}_W(y)^{-1}.$$

For any $u \in T$ and $n \in \mathbb{N}$, we have

$$yu^{-1} \in Y \cap W \quad \text{and} \quad yu^{-1}(ug_n) = yg_n \in Y.$$

This implies $Tg_n \subset D(Y \cap W, Y)$ for all $n \geq 1$. By Lemma 9.4(2),

$$Tg_n U \subset D(Y \cap W, Y) \quad \text{for all } n \geq 1.$$

Since $\mathsf{T}_W(y)$ is k-thick at ∞, $T \subset U$ is so. By Lemma 4.6, $\limsup_{n \to \infty} Tg_n U$ contains a sequence $\ell_n \to e$ in $AV - \{e\}$ as $n \to \infty$. Since $D(Y \cap W, Y)$ is closed, $\limsup_{n \to \infty} Tg_n U \subset D(Y \cap W, Y)$. Therefore, $\ell_n \in D(Y \cap W, Y) \cap AV$ is the desired sequence, finishing the proof. $\qquad\square$

Remark 9.8. We remark that Lemma 9.6 and Lemma 9.7 hold for more general geometrically finite, acylindrical hyperbolic 3-manifolds, as the original statement of Lemma 9.6 in [1] was proved in such a setting. The proof of Lemma 9.7 works verbatim as long as Lemma 9.6 holds.

Proof of Theorem 9.1. Let $k_{\mathsf{M}} > 1$ and $R, \xi_{\mathsf{M}} > 0$ be constants as in Proposition 4.3, Proposition 4.4, and (3.4) respectively. Since $x \in \mathrm{RFM}$, there exists $\rho > 0$ such that $x \in W_{\rho+R}$. We simply write $W := W_{\rho+R}$ which is compact. Since $X = \overline{xU}$ is U-invariant and $X \cap W \neq \emptyset$, there exists a U-minimal set $Y \subset X$ relative to W.

Since Y cannot be a compact U-orbit and $\mathsf{T}_W(y)$ is $4k_{\mathsf{M}}$-thick at ∞ for any $y \in Y \cap W$ by Proposition 4.4(2), it follows from Lemma 9.7 that there exists a one-parameter subsemigroup $L_+ < AV$ such that $YL_+ \subset Y$.

We claim that one of the following holds:
  (a) Y contains a compact N-orbit.
  (b) there exist $y_0 \in Y$ and $v_0 \in V$ such that $y_0 v_0 \in \mathrm{BFM}$ and

$$y_0 v_0 H v_0^{-1} \subset Y.$$

  (c) there exists a sequence $\ell_n \to \infty$ in L_+ such that

$$\limsup_{n \to \infty} Y\ell_n \neq \emptyset.$$

Suppose first that there exists $z \in Y$ such that zN is compact. Since $zU \subset Y \subset X$ and X does not contain any compact U-orbit, zU is not compact. This implies

$$zN = \overline{zU} \subset Y,$$

from which (a) follows.

Now assume that

(9.2) $$zN \text{ is not compact for all } z \in Y.$$

Fix $y \in Y \cap W$ and let $g \in G$ be such that $y = [g]$. Let $L < AV$ be the one-parameter subgroup containing L_+. By Lemma 4.11, $L = vAv^{-1}$ for some $v \in V$ or $L = V$. Suppose first that $L = vAv^{-1}$ for some $v \in V$. There are two possible cases:

- Suppose that $yvU \cap \text{RFM} = \emptyset$. By Lemma 3.2, we have $yv \in \text{BFM} \cdot V$, and hence $yv_0 \in \text{BFM}$ for some $v_0 \in V$. Recalling that $\text{BFM} \subset \text{RFM}$ is a union of finitely many compact H-orbits,

  (9.3) $$yv_0 H = \overline{yv_0 U} = \overline{yU}v_0 = Yv_0$$

  since $yv_0 H$ is U-minimal by Corollary 6.5. Therefore, (b) follows in this case.

- Suppose that $yvU \cap \text{RFM} \neq \emptyset$. Then we have either $(gvU)^- \subset \Lambda$ or $(gvU)^-$ meets both Λ and $\widehat{\mathbb{C}} - \Lambda$. In any case, there exists $u \in U$ such that $(guv)^- = (gvu)^- \in \Lambda$ is conical. Hence, we can find a sequence $t_n \to \infty$ such that

  (9.4) $$\text{the sequence } yuva_{-t_n} \text{ converges.}$$

  By Lemma 5.1, $(guv)^+ = g^+$ is conical as well, since yN is not compact. We then have that for some sequence $s_n \to \infty$,

  (9.5) $$\text{the sequence } yuva_{s_n} \text{ converges.}$$

  Writing $A^+ := \{a_t \in A : t \geq 0\}$, there are two subcases:
  - if $L_+ = vA^+v^{-1}$, set $\ell_n = va_{s_n}v^{-1} \in L_+$. Then by (9.5), the sequence
    $$yu\ell_n = yu(va_{s_n}v^{-1})$$
    converges.
  - if $L_+ = v(A^+)^{-1}v^{-1}$, set $\ell_n = va_{-t_n}v^{-1} \in L_+$. Then by (9.4) the sequence
    $$yu\ell_n = yu(va_{-t_n}v^{-1})$$
    converges.

  Since $yu \in Y$, we have $\limsup_{n\to\infty} Y\ell_n \neq \emptyset$. Since $\ell_n \in L_+$ diverges, this shows (c).

Next, suppose $L = V$. There are two cases:

- if $Y \cap \mathrm{BFM} \cdot V \neq \emptyset$, then $y_0 v_0 \in \mathrm{BFM}$ for some $y_0 \in Y$ and $v_0 \in V$. Recalling that $\mathrm{BFM} \subset \mathrm{RFM}$ is a union of finitely many compact H-orbits,
$$y_0 v_0 H = \overline{y_0 v_0 U} = \overline{y_0 U} v_0 \subset Y v_0$$
since $y_0 v_0 H$ is U-minimal by Corollary 6.5 and $Y = \overline{yU}$ is U-invariant. Therefore, (b) follows in this case.
- if $Y \cap \mathrm{BFM} \cdot V = \emptyset$, then $Yv \subset \mathrm{RFM} \cdot U$ for all $v \in V$ by Lemma 3.2. Choose any sequence $\ell_n \to \infty$ in $L_+ < V$. Then by the U-invariance of Y, there exists a sequence $y_n \in Y$ such that $y_n \ell_n \in \mathrm{RFM}$ for all $n \geq 1$. By the hypothesis (9.2), $y_n \ell_n N = y_n N$ is not compact for all $n \geq 1$. We then have that
$$y_n \ell_n \in \mathrm{RFM} - F_{e^{-2R}\xi_{\mathsf{M}}}(N) \quad \text{for all } n \geq 1.$$
By Proposition 4.4(3),
$$y_n \ell_n U \cap W_R \neq \emptyset \quad \text{for all } n \geq 1$$
where $W_R = \mathrm{RFM} - \mathrm{int}(F_{e^{-2R}\xi_{\mathsf{M}}})$. Since $y_n \ell_n U \subset Y\ell_n$ and W_R is compact, it follows that $\limsup_{n\to\infty} Y\ell_n \neq \emptyset$, and therefore (c) holds.

To finish the proof, it suffices to consider the case (c) of the claim that for some sequence $\ell_n \to \infty$ in L_+, we have $\limsup_{n\to\infty} Y\ell_n \neq \emptyset$. Let $y_0 \in \limsup_{n\to\infty} Y\ell_n$ and take a sequence $y_n \in Y$ such that $y_n \ell_n \to y_0$ as $n \to \infty$. Then for any $\ell \in L$, we have $\ell_n \ell \in L_+$ for all large enough $n \geq 1$, and hence
$$y_n \ell_n \ell \in YL_+ \subset Y.$$
Taking the limit $n \to \infty$, this implies $y_0 \ell \in Y$. Since $\ell \in L$ is arbitrary, we have $y_0 L \subset Y$, and hence
$$y_0 LU \subset Y.$$
Again, $L = vAv^{-1}$ for some $v \in V$ or $L = V$ by Lemma 4.11. If $L = vAv^{-1}$ for some $v \in V$, then $y_0 vAUv^{-1} \subset Y$. If $L = V$, then $y_0 N \subset Y$. By Lemma 5.1, we have either $\overline{y_0 N} = \mathrm{RF}_+\mathsf{M}$ or $y_0 N$ is compact. Since $\overline{y_0 N} \subset X$, this completes the proof. $\qquad\square$

10. The classification

In this last section, we prove our classification of U-orbit closures. We restate Theorem 1.2 below:

Theorem 10.1. *Let* $\mathsf{M} = \Gamma\backslash\mathbb{H}^3$ *be a geometrically finite hyperbolic 3-manifold with a round Sierpiński limit set. Then for any* $x \in \mathrm{FM}$, *one of the following holds:*

1. xU *is closed.*
2. $\overline{xU} = xN$ *which is compact.*
3. $\overline{xU} = xvHv^{-1} \cap \mathrm{RF}_+\mathsf{M}$ *for some* $v \in V$.
4. $\overline{xU} = \mathrm{RF}_+\mathsf{M}$.

In the rest of the section, let M and Γ be as in Theorem 10.1. As observed in Theorem 9.1 and Theorem 8.1, it sometimes happen that a U-orbit closure contains a $vAUv^{-1}$-orbit for some $v \in V$. In this regard, we investigate AU-orbit closures further.

AU-orbit closures. Using the classification of H-orbit closures (Theorem 6.1), we obtain the following:

Lemma 10.2. *For any $x \in \mathrm{RF}_+\mathsf{M}$, there exist $y \in \mathrm{RF}_+\mathsf{M}$ and $v \in V$ such that $yv \in \mathrm{RFM}$, yvH is closed, and*

$$yvHv^{-1} \cap \mathrm{RF}_+\mathsf{M} \subset \overline{xAU}.$$

Proof. Suppose first that $x \in \mathrm{RFM} \cdot U$. We take $u \in U$ so that $xu \in \mathrm{RFM}$. Recall $K = \mathrm{PSU}(2)$ from (2.1) and let $K_H = H \cap K$. Since $H = AUK_H$ and K_H is compact, we have

$$\overline{xAU}K_H = \overline{xH} = \overline{xuH}.$$

By Theorem 6.1, either xuH is closed or $\overline{xuH} = \mathrm{RF}_+\mathsf{M} \cdot H$.

- If xuH is closed, then

$$\overline{xAU} \cap (xuH \cap \mathrm{RF}_+\mathsf{M}) \neq \emptyset$$

  since $K_H < H$. By the AU-minimality of $xuH \cap \mathrm{RF}_+\mathsf{M}$ (Corollary 6.6), this proves the claim with $y = xu \in \mathrm{RFM}$ and $v = e$.
- Otherwise, we have

$$\overline{xAU}K_H = \mathrm{RF}_+\mathsf{M} \cdot H.$$

  Let $y \in \mathrm{BFM}$. Then $yH \subset \mathrm{RFM}$ is closed. Hence

$$\overline{xAU} \cap yH \neq \emptyset,$$

  from which we can deduce the claim as above.

Now suppose that $x \notin \mathrm{RFM} \cdot U$. By Lemma 3.2, $x \in \mathrm{BFM} \cdot V$, and hence $xv \in \mathrm{BFM}$ for some $v \in V$. We then have $xvH \subset \mathrm{RFM}$ and xvH is compact. This implies that xvH is U-minimal by Corollary 6.5, and hence

$$\overline{xAU} \supset \overline{xU} = \overline{xvU}v^{-1} = xvHv^{-1}.$$

Setting $y = x$, this finishes the proof. $\qquad\square$

Corollary 10.3. *For any $x \in \mathrm{RF}_+\mathsf{M}$ and $v_0 \in V$, there exist $y \in \mathrm{RF}_+\mathsf{M}$ and $v \in V$ such that $yv \in \mathrm{RFM}$, yvH is closed, and*

$$yvHv^{-1} \cap \mathrm{RF}_+\mathsf{M} \subset \overline{xv_0AUv_0^{-1}}.$$

Proof. By Lemma 10.2, there exists $y' \in \mathrm{RF}_+\mathsf{M}$ and $v' \in V$ such that $y'v' \in \mathrm{RFM}$, $y'v'H$ is closed, and

$$y'v'Hv'^{-1} \cap \mathrm{RF}_+\mathsf{M} \subset \overline{xv_0AU}.$$

This implies that

$$y'v_0^{-1}(v_0v'Hv'^{-1}v_0^{-1}) \cap \mathrm{RF}_+\mathsf{M} \subset \overline{xv_0AUv_0^{-1}}.$$

We set $y = y'v_0^{-1} \in \mathrm{RF}_+\mathsf{M}$ and $v = v_0v' \in V$. Then $yv = y'v' \in \mathrm{RFM}$ and $yvH = y'v'H$ is closed. Moreover, the above inclusion is rewritten as follows:
$$yvHv^{-1} \cap \mathrm{RF}_+\mathsf{M} \subset \overline{xv_0AUv_0^{-1}}.$$
This completes the proof. $\square$

We now show the intermediate classification of U-orbit closures based on results from Section 8 and Section 9 as follows:

Proposition 10.4. *Let $x \in \mathrm{RF}_+\mathsf{M}$. Then one of the following holds:*

(1) *xU is closed.*
(2) *$\overline{xU} = xN$ which is compact.*
(3) *there exist $y \in \mathrm{RF}_+\mathsf{M}$ and $v \in V$ such that $yv \in \mathrm{RFM}$, yvH is closed, and*
$$yvHv^{-1} \cap \mathrm{RF}_+\mathsf{M} \subset \overline{xU}.$$

Proof. We first note that xN is compact or $\overline{xN} = \mathrm{RF}_+\mathsf{M}$ by Lemma 5.1. This implies that if $\overline{xU} = xN$, and hence xN is closed, then xN is compact. Therefore, it suffices to show that if $\overline{xU}$ is neither xU nor xN, then (3) holds.

Suppose that $\overline{xU}$ is neither xU nor xN. If $x \in \mathrm{BFM} \cdot V$, then $xv \in \mathrm{BFM} \subset \mathrm{RFM}$ for some $v \in V$, and hence xvH is a compact U-minimal set by Corollary 6.5. Therefore,
$$xvH = \overline{xvU} = \overline{xU}v,$$
from which (3) follows.

Now assume that $x \notin \mathrm{BFM} \cdot V$. By Lemma 3.2, $x \in \mathrm{RFM} \cdot U$ and hence we may assume that $x \in \mathrm{RFM}$ by replacing x with an element of xU. We claim that $\overline{xU}$ contains a $v_0AUv_0^{-1}$-orbit for some $v_0 \in V$. Once we show the claim, we apply Corollary 10.3 and (3) follows, finishing the proof.

To see the claim, first consider the case that $\overline{xU}$ does not contain any compact U-orbit. By Theorem 9.1, $\overline{xU}$ contains a $v_0AUv_0^{-1}$-orbit for some $v_0 \in V$ or a compact N-orbit. In the former case, we are done. If $\overline{xU}$ contains a compact N-orbit, it follows from Theorem 8.1 that $\overline{xU}$ contains a $v_0AUv_0^{-1}$-orbit for some $v_0 \in V$ or xN is compact. On the other hand, since $\overline{xU}$ is neither xU nor xN, xN cannot be compact. Indeed, if xN were compact, then xU is either compact or dense in xN, which is not the case here. Therefore, the claim follows.

Now suppose that $\overline{xU}$ contains a compact U-orbit, say yU. If $g \in G$ is such that $y = [g]$, then g^+ is a parabolic limit point, and hence yN is compact by Lemma 5.1. Hence $\overline{xU}$ meets a compact N-orbit and the claim follows from Theorem 8.1 as above. This finishes the proof. $\square$

Combined with the results from Section 7, we now complete the classification.

Proof of Theorem 10.1. Let $x \in \mathrm{FM}$. If $x \notin \mathrm{RF}_+\mathsf{M}$, then it is easy to see that xU is closed, noting that the Γ-action on $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ is a non-elementary convergence action with the limit set Λ.

Hence, we assume that $x \in \mathrm{RF}_+\mathsf{M}$. Suppose first that $x \in \mathrm{BFM} \cdot V$. Then $xv \in \mathrm{BFM}$ for some $v \in V$, and hence

$$\overline{xU} = \overline{xvU}v^{-1} = xvHv^{-1}$$

since $xvH \subset \mathrm{BFM} \subset \mathrm{RFM}$ is a compact U-minimal set by Corollary 6.5. Therefore, (3) follows in this case.

It remains to consider the case $x \notin \mathrm{BFM} \cdot V$. By Lemma 3.2, we have $x \in \mathrm{RFM} \cdot U$. By Proposition 10.4, it suffices to consider the case that there exist $y \in \mathrm{RF}_+\mathsf{M}$ and $v \in V$ such that $yv \in \mathrm{RFM}$, yvH is closed, and

$$yvHv^{-1} \cap \mathrm{RF}_+\mathsf{M} \subset \overline{xU}.$$

Since $x \notin \mathrm{BFM} \cdot V$, $xv \notin \mathrm{BFM} \cdot V$ and hence $xv \in \mathrm{RFM} \cdot U$ by Lemma 3.2. Replacing x with an element of xU, we may assume that $xv \in \mathrm{RFM}$.

If $yv \in \mathrm{BFM}$, then

$$yvH = yvH \cap \mathrm{RF}_+\mathsf{M} \subset \overline{xvU}.$$

By Proposition 7.4, this implies that either

$$\overline{xU}v = \overline{xvU} = yvH \quad \text{or} \quad \overline{xU}v = \mathrm{RF}_+\mathsf{M}.$$

In any case, (3) or (4) follows.

If $yv \notin \mathrm{BFM}$, it then follows from Proposition 7.7 that either

$$\overline{xU}v = yvH \cap \mathrm{RF}_+\mathsf{M} \quad \text{or} \quad \overline{xU}v = \mathrm{RF}_+\mathsf{M}.$$

Again, (3) or (4) follows in either case. This completes the proof. $\qquad\square$

References

[1] Y. Benoist and H. Oh. Geodesic planes in geometrically finite acylindrical 3-manifolds. *Ergodic Theory Dynam. Systems*, 42(2):514–553, 2022.

[2] F. Dal'bo. Topologie du feuilletage fortement stable. *Ann. Inst. Fourier (Grenoble)*, 50(3):981–993, 2000.

[3] S. Dani and G. Margulis. Values of quadratic forms at primitive integral points. *Invent. Math.*, (2):405–424, 1989.

[4] D. Ferte. Flot horosphérique des repères sur les variétés hyperboliques de dimension 3 et spectre des groupes kleiniens. *Bull. Braz. Math. Soc. (N.S.)*, 33(1):99–123, 2002.

[5] J. Jeoung and S. Lim. $\mathrm{PGL}_2(\mathbb{Q}_p)$-orbit closures on a p-adic homogeneous space of infinite volume. *preprint arXiv:2311.11118*, 2023.

[6] O. Khalil. Geodesic planes in geometrically finite manifolds. *Discrete Contin. Dyn. Syst.*, 39(2):881–903, 2019.

[7] M. Lee and H. Oh. Orbit closures of unipotent flows for hyperbolic manifolds with Fuchsian ends. *Geom. Topol.*, 28(7):3373–3473, 2024.

[8] C. McMullen. Iteration on Teichmüller space. *Invent. Math.*, 99(2):425–454, 1990.

[9] C. McMullen, A. Mohammadi, and H. Oh. Horocycles in hyperbolic 3-manifolds. *Geom. Funct. Anal.*, 26(3):961–973, 2016.

[10] C. McMullen, A. Mohammadi, and H. Oh. Geodesic planes in hyperbolic 3-manifolds. *Invent. Math.*, 209(2):425–461, 2017.

[11] C. McMullen, A. Mohammadi, and H. Oh. Geodesic planes in the convex core of an acylindrical 3-manifold. *Duke Math. J.*, 171(5):1029–1060, 2022.

[12] H. Oh and N. Shah. Equidistribution and counting for orbits of geometrically finite hyperbolic groups. *J. Amer. Math. Soc.*, 26(2):511–562, 2013.

[13] M. Ratner. Raghunathan's topological conjecture and distributions of unipotent flows. *Duke Math. J.*, 63(1):235–280, 1991.

[14] N. Shah. *Unipotent flows on homogeneous spaces of* $\mathrm{SL}(2,\mathbb{C})$. 1992. Thesis (M.Sc.)– Tata Institute of Fundamental Research.

[15] W. Thurston. Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds. *Ann. of Math. (2)*, 124(2):203–246, 1986.

[16] T. Torkaman and Y. Zhang. Geodesic planes in a geometrically finite end and the halo of a measured lamination. *Adv. Math.*, 456:Paper No. 109902, 67, 2024.

[17] D. Winter. Mixing of frame flow for rank one locally symmetric spaces and measure classification. *Israel J. Math.*, 210(1):467–507, 2015.

[18] Y. Zhang. Existence of an exotic plane in an acylindrical 3-manifold. *Math. Res. Lett.*, 30(2):611–631, 2023.

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06511, USA
Email address: dongryul.kim@yale.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637, USA
Email address: minju1@uchicago.edu