PROPERLY DISCONTINUOUS ACTIONS, GROWTH INDICATORS, AND CONFORMAL MEASURES FOR TRANSVERSE SUBGROUPS

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Abstract. Let G be a connected semisimple real algebraic group. The class of transverse subgroups of G includes all discrete subgroups of rank one Lie groups and any subgroups of Anosov or relative Anosov subgroups. Given a transverse subgroup Γ, we show that the Γ-action on the Weyl chamber flow space determined by its limit set is properly discontinuous. This allows us to consider the quotient space and define Bowen-Margulis-Sullivan measures. We then establish the ergodic dichotomy for the Weyl chamber flow, in the original spirit of Hopf-Tsuji-Sullivan. We also introduce the notion of growth indicators and discuss their properties and roles in the study of conformal measures, extending the work of Quint. We discuss several applications as well.

Contents

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1. Introduction

Patterson-Sullivan theory on conformal measures of a discrete subgroup of a rank one simple real algebraic group G has played a pivotal role in the study of dynamics on rank one homogeneous spaces. One of the basic results due to Sullivan in 1979 is the relation between the support of a conformal measure and its dimension, which we recall for $G = \mathrm{SO}^\circ(n,1)$, the identity component of the special orthogonal group $\mathrm{SO}(n,1)$. The group $\mathrm{SO}^\circ(n,1)$ is the group of orientation-preserving isometries of the real hyperbolic space $(\mathbb{H}^n, d)$. The geometric boundary of $\mathbb{H}^n$ can be identified with the sphere $\mathbb{S}^{n-1}$. For a discrete subgroup $\Gamma < G$, denote by $\Lambda^{\mathsf{con}} \subset \mathbb{S}^{n-1}$ the conical set of Γ, which consists of the endpoints of all geodesic rays in $\mathbb{H}^n$ which accumulate modulo Γ. Let δ_Γ denote the critical exponent of Γ, which is the abscissa of convergence of the Poincaré series $s \mapsto \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}$, $o \in \mathbb{H}^n$.

For a given Γ-conformal measure ν, we denote by m_ν the Bowen-Margulis-Sullivan measure on the unit tangent bundle $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$, which is a locally finite measure invariant under the geodesic flow. The following theorem is often referred to as the Hopf-Tsuji-Sullivan dichotomy (see [43], [19], [42], [1], [39, Theorem 1.7]).

Theorem 1.1 (Sullivan, [42, Corollaries 4, 20, Theorem 21], see also [1], [12], [39])**.** *Let $\Gamma < \mathrm{SO}^\circ(n,1)$, $n \geq 2$, be a non-elementary discrete subgroup. Suppose that there exists a Γ-conformal measure ν on $\mathbb{S}^{n-1}$ of dimension $s \geq 0$.*

(1) *We have*
$$s \geq \delta_\Gamma.$$
(2) *The following are equivalent:*
   (a) $\sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} = \infty$ *(resp. $\sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} < \infty$);*
   (b) $\nu(\Lambda^{\mathsf{con}}) = 1$ *(resp. $\nu(\Lambda^{\mathsf{con}}) = 0$);*
   (c) *the geodesic flow on $(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n), \mathsf{m}_\nu)$ is completely conservative and ergodic.*
   (resp. the geodesic flow on $(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n), \mathsf{m}_\nu)$ is completely dissipative and non-ergodic.)

In the former case, $s = \delta_\Gamma$ and ν is the unique Γ-conformal measure of dimension δ_Γ.

The main aim of this paper is to establish an analogous result for a class of discrete subgroups of a general connected semisimple real algebraic group G, called θ-transverse subgroups. The class of θ-transverse subgroups includes all discrete subgroups of rank one Lie groups, θ-Anosov subgroups and their relative versions. This class is regarded as a generalization of all rank one discrete subgroups while Anosov subgroups are regarded as higher rank analogues of convex cocompact subgroups.

We need to introduce some notations to state our results precisely. Let $P < G$ be a minimal parabolic subgroup with a fixed Langlands decomposition $P = MAN$ where A is a maximal real split torus of G, M is the maximal compact subgroup of P commuting with A and N is the unipotent radical of P. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebra of G and A. Fix a positive closed Weyl chamber $\mathfrak{a}^+ < \mathfrak{a}$ so that $\log N$ consists of positive root subspaces and set $A^+ = \exp \mathfrak{a}^+$. We fix a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. We denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection defined by the condition $g \in K \exp \mu(g) K$ for $g \in G$. Let Π denote the set of all simple roots for $(\mathfrak{g}, \mathfrak{a}^+)$. As usual, the Weyl group is the quotient of the normalizer of A in K by the centralizer of A in K. Let $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ denote the opposition involution, that is, $\mathrm{i}(u) = -\mathrm{Ad}_{w_0}(u)$ for all $u \in \mathfrak{a}$ where w_0 is the longest Weyl element. It induces an involution on Π which we denote by the same notation i. Throughout the introduction, we fix a non-empty subset

$$\theta \subset \Pi.$$

Let $\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha$ and let $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ be the unique projection, invariant under all Weyl elements fixing $\mathfrak{a}_\theta$ pointwise. Let P_θ be the standard parabolic subgroup corresponding to θ (our convention is that $P = P_\Pi$) and consider the θ-boundary:

$$\mathcal{F}_\theta = G/P_\theta.$$

We say that $\xi \in \mathcal{F}_\theta$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ are in general position if the pair (ξ, η) belongs to the unique open G-orbit in $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)}$ under the diagonal action of G.

Let $\Gamma < G$ be a discrete subgroup. The following properties of Γ are natural to consider in studying analogues of Theorem 1.1 for Γ-conformal measures on the θ-boundary $\mathcal{F}_\theta$. Let $\Lambda_\theta = \Lambda_\theta(\Gamma)$ denote the θ-limit set of Γ in $\mathcal{F}_\theta$ (Definition 5.1).

Definition 1.2. A discrete subgroup Γ is said to be *θ-transverse* if

- Γ is *θ-regular*, i.e., $\liminf_{\gamma \in \Gamma} \alpha(\mu(\gamma)) = \infty$ for all $\alpha \in \theta$; and
- Γ is *θ-antipodal*, i.e., if any two distinct $\xi, \eta \in \Lambda_{\theta \cup \mathrm{i}(\theta)}$ are in general position.

A θ-transverse subgroup Γ is called *non-elementary* if $\#\Lambda_\theta \geq 3$.

Note that the θ-transverse property is hereditary: a subgroup of a θ-transverse subgroup is also θ-transverse.

We assume that Γ is θ-transverse in the rest of the introduction. We define the θ-growth indicator $\psi_\Gamma^\theta : \mathfrak{a}_\theta \to [-\infty, \infty]$ as follows: fixing any norm $\|\cdot\|$ on $\mathfrak{a}_\theta$, if $u \in \mathfrak{a}_\theta$ is non-zero,

$$\psi_\Gamma^\theta(u) = \|u\| \inf_{u \in \mathcal{C}} \tau_{\mathcal{C}}^\theta \tag{1.1}$$

where $\tau_{\mathcal{C}}^\theta$ is the abscissa of convergence of the series $\sum_{\gamma \in \Gamma, \mu_\theta(\gamma) \in \mathcal{C}} e^{-s\|\mu_\theta(\gamma)\|}$ and $\mathcal{C} \subset \mathfrak{a}_\theta$ ranges over all open cones containing u. Set $\psi_\Gamma^\theta(0) = 0$. This definition is independent of the choice of a norm on $\mathfrak{a}_\theta$. For $\theta = \Pi$, ψ_Γ^Π

coincides with Quint's growth indicator ψ_Γ [35]. For a general $\theta \subset \Pi$, we have:

$$\psi_\Gamma^\theta \circ p_\theta \geq \psi_\Gamma. \tag{1.2}$$

(Lemma 3.13, see also Lemma 3.14 for a precise relation for G simple). We show that $\psi_\Gamma^\theta < \infty$, and ψ_Γ^θ is a homogeneous, upper semi-continuous and concave function. It also follows from (1.2) that

$$\{\psi_\Gamma^\theta \geq 0\} = \mathcal{L}_\theta \quad \text{and} \quad \psi_\Gamma^\theta > 0 \quad \text{on int}\, \mathcal{L}_\theta \tag{1.3}$$

where $\mathcal{L}_\theta = \mathcal{L}_\theta(\Gamma)$ is the θ-limit cone of Γ (Theorem 3.3).

Denote by $\mathfrak{a}_\theta^* = \operatorname{Hom}(\mathfrak{a}_\theta, \mathbb{R})$ the space of all linear forms on $\mathfrak{a}_\theta$. For $\psi \in \mathfrak{a}_\theta^*$, a Borel probability measure ν on $\mathcal{F}_\theta$ is called a (Γ, ψ)-conformal measure if

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi^\theta(e,\gamma))} \quad \text{for all } \gamma \in \Gamma \text{ and } \xi \in \mathcal{F}_\theta$$

where $\gamma_*\nu(D) = \nu(\gamma^{-1}D)$ for any Borel subset $D \subset \mathcal{F}_\theta$ and β_ξ^θ denotes the $\mathfrak{a}_\theta$-valued Busemann map defined in (5.6). We find it convenient to call the linear form ψ the *dimension* of ν.

For a collection $\{E_n : n \in \mathbb{N}\}$ of subsets of a given metric space $\mathcal{X}$, its topological limsup is the set of accumulation points of all sequences $\{x_n \in E_n : n \in \mathbb{N}\}$, and is denoted by $\limsup_n E_n$. We define the θ-conical set of Γ as

$$\Lambda_\theta^{\mathsf{con}} = \left\{ gP_\theta \in \mathcal{F}_\theta : \limsup_{\gamma \in \Gamma} \gamma g M_\theta A^+ \neq \emptyset \right\}, \tag{1.4}$$

where $M_\theta = K \cap P_\theta$ (see Lemma 5.4 for an equivalent definition). If Γ is θ-regular, then $\Lambda_\theta^{\mathsf{con}} \subset \Lambda_\theta$ (Proposition 5.6).

Definition 1.3. We say $\psi \in \mathfrak{a}_\theta^*$ is (Γ, θ)-proper if $\psi \circ \mu_\theta : \Gamma \to [-\varepsilon, \infty)$ is a proper map for some $\varepsilon > 0$.

For example, a linear form $\psi \in \mathfrak{a}_\theta^*$ which is positive on $\mathcal{L}_\theta - \{0\}$ is (Γ, θ)-proper. For a (Γ, θ)-proper form ψ, the critical exponent $0 < \delta_\psi = \delta_\psi(\Gamma) \leq \infty$ of the ψ-Poincaré series $\mathcal{P}_\psi(s) = \sum_{\gamma \in \Gamma} e^{-s\psi(\mu_\theta(\gamma))}$ is well-defined and we have

$$\delta_\psi = \limsup_{t \to \infty} \frac{1}{t} \# \log\{\gamma \in \Gamma : \psi(\mu_\theta(\gamma)) < t\}$$

(see Lemma 4.2).

A linear form $\psi \in \mathfrak{a}_\theta^*$ is said to be (Γ, θ)-critical if ψ is tangent to the θ-growth indicator ψ_Γ^θ, i.e., $\psi \geq \psi_\Gamma^\theta$ and $\psi(u) = \psi_\Gamma^\theta(u)$ for some $u \in \mathfrak{a}_\theta^+ - \{0\}$.

Main theorems. Our main theorems extend Theorem 1.1 to higher rank.

Theorem 1.4. *Let $\Gamma < G$ be a Zariski dense θ-transverse subgroup. Suppose that there exists a (Γ, ψ)-conformal measure ν on $\mathcal{F}_\theta$ for $\psi \in \mathfrak{a}_\theta^*$.*

(1) *If ψ is (Γ, θ)-proper, then*

$$\psi \geq \psi_\Gamma^\theta. \tag{1.5}$$

(2) *The following are equivalent:*

(a) $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$ *(resp.* $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} < \infty$*);*
(b) $\nu(\Lambda_\theta^{\mathsf{con}}) = 1$ *(resp.* $\nu(\Lambda_\theta^{\mathsf{con}}) = 0$*).*

In the former case, any (Γ, θ)*-proper* ψ *is necessarily* (Γ, θ)*-critical and* ν *is the unique* (Γ, ψ)*-conformal measure on* $\mathcal{F}_\theta$*.*

When $\theta = \Pi$, Theorem 1.4(1) for a Zariski dense discrete subgroup was proved by Quint. For a general θ, only a weaker bound as (8.7) was known by [36, Theorem 8.1]. It implies:

Theorem 1.5. *Let* $\Gamma < G$ *be a Zariski dense* θ*-transverse subgroup. If there exists a* (Γ, ψ)*-conformal measure on* $\mathcal{F}_\theta$ *for a* (Γ, θ)*-proper* $\psi \in \mathfrak{a}_\theta^*$*, then*

$$\delta_\psi \le 1.$$

Remark 1.6.
(1) Canary-Zhang-Zimmer [9, Theorem 1.4] proved the equivalence of (a) and (b) in Theorem 1.4(2) for θ symmetric, that is, $\theta = \mathrm{i}(\theta)$, and for conformal measures supported on Λ_θ. We mention that transverse subgroups are sometimes called RA-subgroups (cf. [14]).
(2) For θ symmetric and conformal measures supported on Λ_θ, Theorem 1.5 was shown in [9, Theorem 1.4]. For some special class of θ-*Anosov* subgroups and for conformal measures *supported on* Λ_θ, Theorem 1.5 was also proved in [34, Theorem C] and [41, Theorem A].

As in the original Hopf-Tsuji-Sullivan dichotomy (Theorem 1.1), Theorem 1.4 can be extended to the dichotomy on the ergodicity of the Weyl chamber flow. Recalling the Hopf parametrization $\Gamma\backslash(\mathcal{F}_\Pi^{(2)} \times \mathfrak{a}) \simeq \Gamma\backslash G/M$, a natural space to consider is the quotient space $\Gamma\backslash(\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta)$ where $\mathcal{F}_\theta^{(2)} = \{(\xi, \eta) \in \mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)} : \xi, \eta \text{ are in general position}\}$ and Γ acts on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ from the left by

$$\gamma.(\xi, \eta, u) = (\gamma\xi, \gamma\eta, u + \beta_\xi^\theta(\gamma^{-1}, e)) \tag{1.6}$$

for all $\gamma \in \Gamma$ and $(\xi, \eta, u) \in \mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$. However the Γ-action on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ is not properly discontinuous in general; so the quotient space $\Gamma\backslash(\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta)$ is not locally compact.

On the other hand, the restriction of the Γ-action on the subspace $\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ turns out to be properly discontinuous where $\Lambda_\theta^{(2)} = \mathcal{F}_\theta^{(2)} \cap (\Lambda_\theta \times \Lambda_{\mathrm{i}(\theta)})$ (Theorem 9.1):

Theorem 1.7 (Properly discontinuous action)**.** *Let* $\Gamma < G$ *be a non-elementary* θ*-transverse subgroup. Then the* Γ*-action on* $\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ *given by* (1.6) *is properly discontinuous, and hence the quotient space*

$$\Omega_\theta := \Gamma\backslash\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$$

is a locally compact Hausdorff space on which $\mathfrak{a}_\theta$ *acts by translations from the right.*

Indeed, we prove a stronger property of the action: for a (Γ,θ)-proper $\varphi\in\mathfrak{a}_\theta^*$, we have a projection $\Lambda_\theta^{(2)}\times\mathfrak{a}_\theta\to\Lambda_\theta^{(2)}\times\mathbb{R}$ given by $(\xi,\eta,u)\mapsto(\xi,\eta,\varphi(u))$. The action (1.6) descends to the action

$$\gamma.(\xi,\eta,s)=(\gamma\xi,\gamma\eta,s+\varphi(\beta_\xi^\theta(\gamma^{-1},e))) \tag{1.7}$$

for all $\gamma\in\Gamma$ and $(\xi,\eta,s)\in\Lambda_\theta^{(2)}\times\mathbb{R}$. We show that the action (1.7) is properly discontinuous, and prove the following (Theorem 9.2):

Theorem 1.8. *Let $\Gamma<G$ be a non-elementary θ-transverse subgroup. For any (Γ,θ)-proper $\varphi\in\mathfrak{a}_\theta^*$, $\Omega_\varphi:=\Gamma\backslash\Lambda_\theta^{(2)}\times\mathbb{R}$ is a locally compact Hausdorff space. Moreover, Ω_φ is compact if and only if Γ is θ-Anosov.*

Furthermore, we have a trivial $\ker\varphi$-bundle $\Omega_\theta\to\Omega_\varphi$ so that Ω_θ is homeomorphic to $\Omega_\varphi\times\ker\varphi$ (10.4).

For $\psi\in\mathfrak{a}_\theta^*$, we denote by $\mathcal{M}_\psi^\theta$ the space of all (Γ,ψ)-conformal measures *supported on* Λ_θ. For a pair $(\nu,\nu_{\mathrm{i}})\in\mathcal{M}_\psi^\theta\times\mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$, we denote by $\mathsf{m}_{\nu,\nu_{\mathrm{i}}}$ the associated Bowen-Margulis-Sullivan measure on Ω_θ (see (10.1) for its definition).

We expand Theorem 1.4 to the dichotomy on conservativity and ergodicity of the $\mathfrak{a}_\theta$-action on the space $(\Omega_\theta,\mathsf{m}_{\nu,\nu_{\mathrm{i}}})$. See Theorem 11.2 for a more elaborate statement.

Theorem 1.9. *Let $\Gamma<G$ be a non-elementary θ-transverse subgroup. Let $\psi\in\mathfrak{a}_\theta^*$ be (Γ,θ)-proper such that $\mathcal{M}_\psi^\theta\neq\emptyset$. In each of the following complementary cases, the claims $(1)-(4)$ are equivalent to each other.*

The first case:

(1) $\sum_{\gamma\in\Gamma}e^{-\psi(\mu_\theta(\gamma))}=\infty$;
(2) *For any $\nu\in\mathcal{M}_\psi^\theta$, $\nu(\Lambda_\theta^{\mathsf{con}})=1$;*
(3) *For any $(\nu,\nu_{\mathrm{i}})\in\mathcal{M}_\psi^\theta\times\mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$, the Γ-action on $(\Lambda_\theta^{(2)},\nu\times\nu_{\mathrm{i}})$ is completely conservative and ergodic;*
(4) *For any $(\nu,\nu_{\mathrm{i}})\in\mathcal{M}_\psi^\theta\times\mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$, the $\mathfrak{a}_\theta$-action on $(\Omega_\theta,\mathsf{m}_{\nu,\nu_{\mathrm{i}}})$ is completely conservative and ergodic.*

The second case:

(1) $\sum_{\gamma\in\Gamma}e^{-\psi(\mu_\theta(\gamma))}<\infty$;
(2) *For any $\nu\in\mathcal{M}_\psi^\theta$, $\nu(\Lambda_\theta^{\mathsf{con}})=0$;*
(3) *For any $(\nu,\nu_{\mathrm{i}})\in\mathcal{M}_\psi^\theta\times\mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$, the Γ-action on $(\Lambda_\theta^{(2)},\nu\times\nu_{\mathrm{i}})$ is completely dissipative and non-ergodic;*
(4) *For any $(\nu,\nu_{\mathrm{i}})\in\mathcal{M}_\psi^\theta\times\mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$, the $\mathfrak{a}_\theta$-action on $(\Omega_\theta,\mathsf{m}_{\nu,\nu_{\mathrm{i}}})$ is completely dissipative and non-ergodic.*

When θ is symmetric, the equivalences (1)-(3) in both cases were proved in [9, Theorem 1.4] by a different approach.

Disjoint dimensions phenomenon. Let

$$\mathcal{D}_\Gamma^\theta = \{\psi \in \mathfrak{a}_\theta^* : (\Gamma, \theta)\text{-proper}, \delta_\psi(\Gamma) = 1 \text{ and } \mathcal{P}_\psi(1) = \infty\}.$$

This is in fact same as

$$\{\psi \in \mathfrak{a}_\theta^* : (\Gamma, \theta)\text{-proper}, \exists \text{ a } (\Gamma, \psi)\text{-conformal measure}, \mathcal{P}_\psi(1) = \infty\}$$

when Γ is a θ-transverse subgroup (see Lemma 12.4).

Inspired by the entropy drop phenomenon proved by Canary-Zhang-Zimmer [9, Theorem 4.1] for $\theta = \mathrm{i}(\theta)$, we deduce from Theorem 1.4 the following disjointness of dimensions (Theorem 12.5), which turns out to be equivalent to the entropy drop phenomenon (Corollary 12.6):

Corollary 1.10 (Disjoint dimensions)**.** *Let $\Gamma < G$ be a non-elementary θ-transverse subgroup. For any subgroup $\Gamma_0 < \Gamma$ with $\Lambda_\theta(\Gamma_0) \neq \Lambda_\theta(\Gamma)$, we have*

$$\mathcal{D}_\Gamma^\theta \cap \mathcal{D}_{\Gamma_0}^\theta = \emptyset.$$

In the rank one case, this corollary says that if $\Lambda(\Gamma_0) \neq \Lambda(\Gamma)$ and $\Gamma_0 < \Gamma$ are of divergence type, that is, their Poincaré series diverge at the critical exponents, then $\delta_{\Gamma_0} < \delta_\Gamma$. We refer to [9] for a more detailed background on this phenomenon.

θ-Anosov subgroups. A finitely generated subgroup $\Gamma < G$ is a θ-Anosov subgroup if there exists $C > 0$ such that for all $\gamma \in \Gamma$,

$$\min_{\alpha \in \theta} \alpha(\mu(\gamma)) \geq C|\gamma| - C^{-1} \tag{1.8}$$

where $|\gamma|$ denotes the word length of γ with respect to a fixed finite generating set of Γ ([28], [17], [21], [22], [23]). All θ-Anosov subgroups are θ-transverse and $\Lambda_\theta = \Lambda_\theta^{\mathsf{con}}$ ([18], [22]). We deduce the following from Theorem 1.4:

Theorem 1.11. *Let $\Gamma < G$ be a Zariski dense θ-Anosov subgroup. Suppose that there exists a (Γ, ψ)-conformal measure ν on $\mathcal{F}_\theta$ for $\psi \in \mathfrak{a}_\theta^*$. We have:*

(1) *The linear form ψ is (Γ, θ)-proper and $\psi \geq \psi_\Gamma^\theta$.*
(2) *The following are equivalent to each other:*
    (a) $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$ *(resp. $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} < \infty$);*
    (b) $\nu(\Lambda_\theta) = 1$ *(resp. $\nu(\Lambda_\theta) = 0$);*
    (c) *ψ is (Γ, θ)-critical (resp. ψ is not (Γ, θ)-critical).*
(3) *For each (Γ, θ)-critical $\psi \in \mathfrak{a}_\theta^*$, there exists a unique (Γ, ψ)-conformal measure, say, ν_ψ, on $\mathcal{F}_\theta$, which is necessarily supported on Λ_θ. Moreover the $\mathfrak{a}_\theta$-action on $(\Omega_\theta, \mathsf{m}_{\nu_\psi, \nu_{\psi \circ \mathrm{i}}})$ is completely conservative and ergodic.*

The equivalence $(a) \Leftrightarrow (b)$ in (2) answers a question asked by Sambarino [41, Remark 5.10].

Analogue of Ahlfors measure conjecture for θ-Anosov groups. We denote by Leb_θ the Lebesgue measure on $\mathcal{F}_\theta$, which is the unique K-invariant probability measure on $\mathcal{F}_\theta$. The following corollary is motivated by the Ahlfors measure conjecture [2].

Corollary 1.12. *If $\Gamma < G$ is Zariski dense θ-Anosov, then*

$$\text{either } \Lambda_\theta = \mathcal{F}_\theta \quad \text{or} \quad \mathrm{Leb}_\theta(\Lambda_\theta) = 0.$$

Moreover, in the former case, θ is the simple root of a rank one factor, say G_0, of G and Γ projects to a cocompact lattice of G_0.

See Theorem 12.1 for a more general version stated for a θ-transverse subgroup.

Critical forms and conformal measures. We set

$$\mathcal{T}_\Gamma^\theta := \{\psi \in \mathfrak{a}_\theta^* : \psi \text{ is } (\Gamma, \theta)\text{-critical}\}.$$

Note that $\mathcal{D}_\Gamma^\theta \subset \mathcal{T}_\Gamma^\theta$ (Corollary 4.6). For θ-Anosov subgroups, we further have $\mathcal{T}_\Gamma^\theta = \mathcal{D}_\Gamma^\theta$, which is again same as the set of all $\psi \in \mathfrak{a}_\theta^*$ for which there exists a (Γ, ψ)-conformal measure supported on Λ_θ (Lemma 13.4). A Γ-conformal measure is said to be of *critical* dimension if the associated linear form belongs to $\mathcal{T}_\Gamma^\theta$. Using Sambarino's parametrization of the space of all conformal measures on Λ_θ as $\{\delta_\psi = 1\}$ [41, Theorem A], we deduce:

Corollary 1.13. *For any Zariski dense θ-Anosov subgroup $\Gamma < G$, we have a one-to-one correspondence among*

1. *the set $\mathcal{T}_\Gamma^\theta$ of all (Γ, θ)-critical forms on $\mathfrak{a}_\theta$;*
2. *the set of all unit vectors in $\operatorname{int} \mathcal{L}_\theta$;*
3. *the set of all Γ-conformal measures supported on Λ_θ;*
4. *the set of all Γ-conformal measures on $\mathcal{F}_\theta$ of critical dimensions.*

More precisely, for any $\psi \in \mathcal{T}_\Gamma^\theta$, there exists a unique unit vector $u_\psi \in \mathfrak{a}_\theta^+$ such that $\psi(u_\psi) = \psi_\Gamma^\theta(u_\psi)$; moreover $u_\psi \in \operatorname{int} \mathcal{L}_\theta$. There also exists a unique (Γ, ψ)-conformal measure ν_ψ on $\mathcal{F}_\theta$, which is necessarily supported on Λ_θ. Moreover every Γ-conformal measure supported on Λ_θ arises in this way.

Corollary 1.14 (Disjoint critical dimensions)**.** *Let $\Gamma < G$ be a Zariski dense θ-Anosov subgroup. For any subgroup $\Gamma_0 < \Gamma$ such that $\Lambda_\theta(\Gamma_0) \neq \Lambda_\theta(\Gamma)$, we have*

$$\mathcal{T}_\Gamma^\theta \cap \mathcal{T}_{\Gamma_0}^\theta = \emptyset \quad \text{and} \quad \psi_{\Gamma_0}^\theta < \psi_\Gamma^\theta \text{ on } \operatorname{int} \mathcal{L}_\theta(\Gamma).$$

Indeed, the above two conclusions are equivalent to each other by the vertical tangency and concavity of ψ_Γ^θ (see Corollary 13.3 for the proof).

Remark 1.15. Related dichotomy properties for conformal measures were studied in [14], [6], [30], [15], [41], [9], etc. In particular, when Γ is Π-Anosov, Theorem 1.11, Corollaries 1.12 and 1.13 were proved by Lee-Oh [30, Theorems 1.3, 1.4]. The papers [14], [41], and [9] study conformal measures *supported on the limit set* Λ_θ and the papers [6] and [15] study the role of *directional* conical sets in the ergodic behavior of conformal measures.

Our focus on this paper is to address general conformal measures without restriction on their supports following [30] and to study the relationship between the dimensions of conformal measures and θ-growth indicators so as to establish an analogue of Sullivan's theorem (Theorem 1.1) and the analogue of the Ahlfors measure conjecture. We also emphasize that the θ-growth indicator is first introduced in our paper. Notably, Theorem 1.7 provides a new locally compact Hausdorff space $\Omega_\theta := \Gamma\backslash\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ which is a non-wandering set for the Weyl chamber flow A_θ. This allows us to define Bowen-Margulis-Sullivan measures as in the rank one setting. Hence the dynamical properties of the Weyl chamber flow can be studied also in higher rank, fully recovering the original work of Hopf-Tsuji-Sullivan.

Finally, we mention that there is a plethora of examples of θ-transverse subgroups which are not θ-Anosov. First of all, any subgroup of θ-Anosov subgroups are θ-transverse. For instance, a co-abelian subgroup of a θ-Anosov subgroup of infinite index is θ-transverse but not θ-Anosov. The images of cusped Hitchin representations of geometrically finite Fuchsian groups by [7] are also θ-transverse but not θ-Anosov. Another important examples are self-joinings of geometrically finite subgroups of rank one Lie groups, that is, $\Gamma = (\prod_{i=1}^k \rho_i)(\Delta) = \{(\rho_i(g))_i : g \in \Delta\}$ where Δ is a geometrically finite subgroup of a rank one simple real algebraic group G_0 and $\rho_i : \Delta \to G_i$ is a type-preserving isomorphism onto its image $\rho_i(\Delta)$ which is a geometrically finite subgroup of a rank one simple real algebraic group G_i for each $1 \le i \le k$. It follows from [44, Theorem 3.3] and [13, Theorem A.4] (see also [46, Theorem 0.1]) that there exists a ρ_i-equivariant homeomorphism between the limit set of Δ and the limit set of $\rho_i(\Delta)$ for each $1 \le i \le k$. This implies that Γ is Π-transverse.

Organization.

- In section 2, we introduce the notion of convergence of elements of G to those of $\mathcal{F}_\theta$ and present some basic lemmas which will be used in the proof of our main theorems.
- In section 3, we define the θ-growth indicator ψ_Γ^θ for a θ-discrete subgroup $\Gamma < G$. Properties of the θ-growth indicator and its relationship with Quint's growth indicator [35] are also discussed.
- In section 4, we introduce (Γ, θ)-proper linear forms and (Γ, θ)-critical linear forms and discuss properties of their critical exponents.
- In section 5, we define the θ-limit set and the θ-conical set of Γ. For θ-regular subgroups, we show that the θ-conical set is a subset of the θ-limit set and construct conformal measures supported on the θ-limit set for each $\psi \in \mathcal{D}_\Gamma^\theta$.
- In section 6, we prove that for θ-transverse subgroups, θ-shadows with bounded width have bounded multiplicity, which is one of the key technical ingredients of our main results.

- In section 7, we show that if Γ is a θ-transverse subgroup, the dimension of a Γ-conformal measure is at least ψ_Γ^θ (Theorem 7.1).
- In section 8, we prove the zero-one law for the ν-size of the conical set depending on whether or not the associated Poincaré series converges at its dimension (Theorem 8.2).
- In section 9, we prove that a θ-transverse subgroup Γ acts properly discontinuously on $\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ and define Bowen-Margulis-Sullivan measures on the space $\Omega_\theta = \Gamma \backslash \Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$. For any (Γ, θ)-proper form φ, we also show that the φ-twisted Γ-action on $\Lambda_\theta^{(2)} \times \mathbb{R}$ is properly discontinuous and gives rise to a trivial vector bundle $\Omega_\theta \to \Omega_\varphi = \Gamma \backslash \Lambda_\theta^{(2)} \times \mathbb{R}$.
- In section 10, we give the definition of Bowen-Margulis-Sullivan measures.
- In section 11, we expand the equivalence between dichotomies to conservativity and ergodicity of the $\mathfrak{a}_\theta$-action on Ω_θ, proving Theorem 1.9. We also explain how to deduce Theorem 1.4 from Theorems 7.1 and 8.2.
- In section 12, we discuss several consequences of Theorem 8.2, including disjoint dimension phenomenon.
- Finally, in section 13 we discuss how our theorems are applied for θ-Anosov groups. We also prove Corollary 1.12.

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2. Convergence in $G \cup \mathcal{F}_\theta$.

In the whole paper, let G be a connected semisimple real algebraic group. Let $P < G$ be a minimal parabolic subgroup with a fixed Langlands decomposition $P = MAN$ where A is a maximal real split torus of G, M is the maximal compact subgroup of P commuting with A and N is the unipotent radical of P. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebras of G and A. Fix a positive closed Weyl chamber $\mathfrak{a}^+ < \mathfrak{a}$ so that $\log N$ consists of positive root subspaces and set $A^+ = \exp \mathfrak{a}^+$. We fix a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. We denote by

$$\mu : G \to \mathfrak{a}^+$$

the Cartan projection defined by the condition $g \in K \exp \mu(g) K$ for $g \in G$. Let $X = G/K$ be the associated Riemannian symmetric space, and set $o = [K] \in X$. Fix a K-invariant norm $\|\cdot\|$ on $\mathfrak{g}$ induced from the Killing form on $\mathfrak{g}$ and let d denote the Riemannian metric on X induced by $\|\cdot\|$.

Lemma 2.1. [3, Lemma 4.6] *For any compact subset $Q \subset G$, there exists $C = C(Q) > 0$ such that for all $g \in G$,*

$$\sup_{q_1, q_2 \in Q} \|\mu(q_1 g q_2) - \mu(g)\| \leq C.$$

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$ denote the set of all roots, $\Phi^+ \subset \Phi$ the set of all positive roots, and $\Pi \subset \Phi^+$ the set of all simple roots. We denote by $N_K(A)$ and $C_K(A)$ the normalizer and centralizer of A in K respectively. Consider the Weyl group $\mathcal{W} = N_K(A)/C_K(A)$. Fix an element

$$w_0 \in N_K(A)$$

representing the longest Weyl element so that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$ and $w_0^{-1} = w_0$. Hence the map

$$\mathrm{i} = -\mathrm{Ad}_{w_0} : \mathfrak{a} \to \mathfrak{a}$$

defines an involution of $\mathfrak{a}$ preserving $\mathfrak{a}^+$; this is called the opposition involution. It induces a map $\Phi \to \Phi$ preserving Π, for which we use the same notation i, such that $\mathrm{i}(\alpha) \circ \mathrm{Ad}_{w_0} = -\alpha$ for all $\alpha \in \Phi$. We have

$$\mu(g^{-1}) = \mathrm{i}(\mu(g)) \quad \text{ for all } g \in G . \tag{2.1}$$

In the rest of the paper, we fix a non-empty subset $\theta \subset \Pi$. Let P_θ denote a standard parabolic subgroup of G corresponding to θ; that is, P_θ is generated by MA and all root subgroups U_α, $\alpha \in \Phi^+ \cup [\Pi - \theta]$ where $[\Pi - \theta]$ denotes the set of all roots in Φ which are $\mathbb{Z}$-linear combinations of $\Pi - \theta$. Hence $P_\Pi = P$. The subgroup P_θ is equal to its own normalizer; for $g \in G$, $gP_\theta g^{-1} = P_\theta$ if and only if $g \in P_\theta$. Let

$$\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha, \quad \mathfrak{a}_\theta^+ = \mathfrak{a}_\theta \cap \mathfrak{a}^+,$$
$$A_\theta = \exp \mathfrak{a}_\theta, \text{ and } \quad A_\theta^+ = \exp \mathfrak{a}_\theta^+.$$

Let

$$p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$$

denote the projection invariant under $w \in \mathcal{W}$ fixing $\mathfrak{a}_\theta$ pointwise.

Let L_θ denote the centralizer of A_θ; it is a Levi subgroup of P_θ and $P_\theta = L_\theta N_\theta$ where $N_\theta = R_u(P_\theta)$ is the unipotent radical of P_θ. We set $M_\theta = K \cap P_\theta \subset L_\theta$. We may then write $L_\theta = A_\theta S_\theta$ where S_θ is an almost direct product of a connected semisimple real algebraic subgroup and a compact subgroup. Then $B_\theta = S_\theta \cap A$ is a maximal $\mathbb{R}$-split torus of S_θ and $\Pi - \theta$ is the set of simple roots for $(\mathrm{Lie}\, S_\theta, \mathrm{Lie}\, B_\theta)$. Letting

$$B_\theta^+ = \{b \in B_\theta : \alpha(\log b) \geq 0 \text{ for all } \alpha \in \Pi - \theta\},$$

we have the Cartan decomposition of S_θ:

$$S_\theta = M_\theta B_\theta^+ M_\theta.$$

Any $u \in \mathfrak{a}$ can be written as $u = u_1 + u_2$ for unique $u_1 \in \mathfrak{a}_\theta$ and $u_2 \in \mathrm{Lie}\, B_\theta$, and we have $p_\theta(u) = u_1$. In particular, we have

$$A = A_\theta B_\theta \quad \text{and} \quad A^+ \subset A_\theta^+ B_\theta^+.$$

We denote by $\mathfrak{a}_\theta^* = \mathrm{Hom}(\mathfrak{a}_\theta, \mathbb{R})$ the dual space of $\mathfrak{a}_\theta$. It can be identified with the subspace of $\mathfrak{a}^*$ which is p_θ-invariant: $\mathfrak{a}_\theta^* = \{\psi \in \mathfrak{a}^* : \psi \circ p_\theta = \psi\}$; so for $\theta_1 \subset \theta_2$, we have $\mathfrak{a}_{\theta_1}^* \subset \mathfrak{a}_{\theta_2}^*$.

The θ-boundary $\mathcal{F}_\theta$ and convergence to $\mathcal{F}_\theta$. We set

$$\mathcal{F}_\theta = G/P_\theta \quad \text{and} \quad \mathcal{F} = G/P.$$

Let

$$\pi_\theta : \mathcal{F} \to \mathcal{F}_\theta$$

denote the canonical projection map given by $gP \mapsto gP_\theta$, $g \in G$. We set

$$\xi_\theta = [P_\theta] \in \mathcal{F}_\theta. \tag{2.2}$$

By the Iwasawa decomposition $G = KP = KAN$, the subgroup K acts transitively on $\mathcal{F}_\theta$, and hence

$$\mathcal{F}_\theta \simeq K/M_\theta.$$

We consider the following notion of convergence of a sequence in G to an element of $\mathcal{F}_\theta$.

Definition 2.2. For a sequence $g_i \in G$ and $\xi \in \mathcal{F}_\theta$, we write $\lim_{i\to\infty} g_i = \lim g_i o = \xi$ and say g_i (or $g_i o \in X$) converges to ξ if

- $\min_{\alpha\in\theta} \alpha(\mu(g_i)) \to \infty$; and
- $\lim_{i\to\infty} \kappa_{g_i}\xi_\theta = \xi$ in $\mathcal{F}_\theta$ for some $\kappa_{g_i} \in K$ such that $g_i \in \kappa_{g_i} A^+ K$.

Points in general position. Let P_θ^+ be the standard parabolic subgroup of G opposite to P_θ such that $P_\theta \cap P_\theta^+ = L_\theta$. Set $P^+ := P_\Pi^+$. We have $P_\theta^+ = w_0 P_{\mathrm{i}(\theta)} w_0^{-1}$ and hence

$$\mathcal{F}_{\mathrm{i}(\theta)} = G/P_\theta^+.$$

In particular, if θ is symmetric in the sense that $\theta = \mathrm{i}(\theta)$, then $\mathcal{F}_\theta = G/P_\theta^+$. Let N_θ^+ denote the unipotent radical of P_θ^+. The set $N_\theta^+ P_\theta$ is a Zariski open and dense subset of G. In particular, $N_\theta^+\xi_\theta \cap hN_\theta^+\xi_\theta \neq \emptyset$ for any $h \in G$. The G-orbit of (P_θ, P_θ^+) is the unique open G-orbit in $G/P_\theta \times G/P_\theta^+$ under the diagonal G-action. Since $P = MAN$ and $P^+ = MAN^+$, $a \in A^+$ centralizes MA, and its conjugation action on N (resp. N^+) contracts (resp. expands), the following is immediate:

Lemma 2.3. *Let $Q \subset P$ and $Q^+ \subset P^+$ be bounded subsets. For any sequence $a_i \in A^+$, both sequences $a_i^{-1} Q a_i$ and $a_i Q^+ a_i^{-1}$ are uniformly bounded.*

Definition 2.4. Two elements $\xi \in \mathcal{F}_\theta$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ are said to be in general position if $(\xi, \eta) \in G.(P_\theta, w_0 P_{\mathrm{i}(\theta)}) = G.(P_\theta, P_\theta^+)$, i.e., $\xi = gP_\theta$ and $\eta = gw_0 P_{\mathrm{i}(\theta)}$ for some $g \in G$.

We set

$$\mathcal{F}_\theta^{(2)} = \{(\xi, \eta) \in \mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)} : \xi,\, \eta \text{ are in general position}\}, \tag{2.3}$$

which is the unique open G-orbit in $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)}$. It follows from the identity $P_\theta^+ = N_\theta^+(P_\theta \cap P_\theta^+)$ that

$$(gP_\theta, P_\theta^+) \in \mathcal{F}_\theta^{(2)} \quad \text{if and only if} \quad g \in N_\theta^+ P_\theta. \tag{2.4}$$

Basic lemmas. We generalize [29, Lemmas 2.9-11] from $\theta = \Pi$ to a general θ as follows. For subsets $S_i \subset G$, we often write $g = g_1 g_2 g_3 \in S_1 S_2 S_3$ to mean that $g_i \in S_i$ for each i, in addition to $g = g_1 g_2 g_3$.

Lemma 2.5. *Consider a sequence $g_i = k_i a_i h_i^{-1}$ where $k_i \in K$, $a_i \in A^+$, and $h_i \in G$. Suppose that $k_i \to k_0 \in K$, $h_i \to h_0 \in G$, and $\min_{\alpha\in\theta} \alpha(\log a_i) \to \infty$, as $i \to \infty$. Then for any $\xi \in h_0 N_\theta^+ \xi_\theta$ (i.e., ξ is in general position with $h_0 P_\theta^+$), we have*

$$\lim_{i\to\infty} g_i \xi = k_0 \xi_\theta.$$

Proof. Since $h_i^{-1}\xi$ converges to the element $h_0^{-1}\xi \in N_\theta^+\xi_\theta$ by the hypothesis and $N_\theta^+\xi_\theta \subset \mathcal{F}_\theta$ is open, we have $h_i^{-1}\xi \in N_\theta^+\xi_\theta$ for all large i. Hence we can write $h_i^{-1}\xi = n_i \xi_\theta$ with $n_i \in N_\theta^+$ uniformly bounded. Since $\min_{\alpha\in\theta}\alpha(\log a_i) \to \infty$ and $n_i \in N_\theta^+$ is uniformly bounded, we have $a_i n_i a_i^{-1} \to e$ as $i \to \infty$. Therefore the sequence $a_i h_i^{-1}\xi = a_i n_i a_i^{-1}\xi_\theta$ converges to ξ_θ. Hence we have

$$\lim_{i\to\infty} g_i\xi = \lim_{i\to\infty} k_i(a_i h_i^{-1}\xi) = k_0\xi_\theta.$$

□

Corollary 2.6. *If $w \in N_K(A)$ is such that $mw \in N_\theta^+ P_\theta$ for some $m \in M_\theta$, then $w \in M_\theta$. In particular, if wP_θ and P_θ^+ are in general position, then $w \in M_\theta$.*

Proof. Choose any sequence $a_i \in A_\theta^+$ such that $\min_{\alpha\in\theta}\alpha(\log a_i) \to \infty$. Since $mw\xi_\theta \in N_\theta^+\xi_\theta$, we deduce from Lemma 2.5 that $a_i m w \xi_\theta$ converges to ξ_θ as $i \to \infty$. On the other hand, since $w \in N_K(A)$, $A \subset P_\theta$ and $m \in M_\theta$, we have $a_i m w\xi_\theta = mw(w^{-1}a_i w)\xi_\theta = mw\xi_\theta$ for all i. Hence $mw\xi_\theta = \xi_\theta$. Since $m \in M_\theta$, this implies $w\xi_\theta = \xi_\theta$ and hence $w \in P_\theta \cap K = M_\theta$. □

It turns out that the convergence of $g_i \to \xi$ is equivalent to $g_i p \to \xi$ for any $p \in X$. More generally, we have

Lemma 2.7. *If a sequence $g_i \in G$ converges to $\xi \in \mathcal{F}_\theta$ and $p_i \in X$ is a bounded sequence, then*

$$\lim_{i\to\infty} g_i p_i = \xi.$$

Proof. Let $g_i' \in G$ be such that $g_i' o = p_i$; then g_i' is bounded. Since $\lim g_i = \xi$, we may write $g_i = k_i a_i \ell_i^{-1}$ with $k_i, \ell_i \in K$ and $a_i \in A^+$ where $\min_{\alpha\in\theta}\alpha(\log a_i) \to \infty$, and $k_i\xi_\theta \to \xi$ as $i \to \infty$. Write $g_i g_i' = k_i' a_i' (\ell_i')^{-1} \in KA^+K$. Since g_i' is bounded, $\lim_{i\to\infty}\min_{\alpha\in\theta}\alpha(\log a_i') = \infty$, by Lemma 2.1. Let $q \in K$ be a limit of the sequence $q_i := k_i^{-1}k_i'$. By passing to a subsequence, we may assume that $q_i \to q$. Since $d(o, p_i) = d(g_i o, g_i p_i) = d(o, a_i^{-1} q_i a_i' o)$, the sequence $h_i^{-1} := a_i^{-1} q_i a_i'$ is bounded. Passing to a subsequence, we may assume that h_i converges to some $h_0 \in G$. Choose any $\eta \in N_\theta^+\xi_\theta \cap h_0 N_\theta^+\xi_\theta$. By Lemma 2.5, we have

$$\lim_{i\to\infty} a_i h_i^{-1}\eta = \xi_\theta \quad \text{and} \quad \lim_{i\to\infty} q_i a_i'\eta = q\xi_\theta.$$

Since $a_i h_i^{-1} = q_i a_i'$, it follows that $q\xi_\theta = \xi_\theta$; so $q \in K \cap P_\theta$. Hence $\xi = \lim k_i \xi_\theta = \lim k_i' \xi_\theta$. It follows that $\lim g_i p_i = \xi$. $\square$

Lemma 2.8. *If a sequence $g_i \in G$ converges to g and a sequence $a_i \in A^+$ satisfies $\min_{\alpha\in\theta} \alpha(\log a_i) \to \infty$ as $i \to \infty$, then for any $p \in X$, we have*

$$\lim_{i\to\infty} g_i a_i p = g\xi_\theta.$$

Proof. By Lemma 2.7, it suffices to consider the case when $p = o$. Write $g_i a_i = k_i b_i \ell_i^{-1}$ with $k_i, \ell_i \in K$ and $b_i \in A^+$. Since the sequence g_i is bounded, $\lim_{i\to\infty} \min_{\alpha\in\theta} \alpha(\log b_i) = \infty$ by Lemma 2.1. Let k_0 be a limit of the sequence k_i; without loss of generality, we may assume that k_i converges to k_0 as $i \to \infty$. Then $\lim_{i\to\infty} g_i a_i o = k_0 \xi_\theta$. We may also assume that ℓ_i converges to some $\ell_0 \in K$. Choose $\xi \in \ell_0 N_\theta^+ \xi_\theta \cap N_\theta^+ \xi_\theta$. Then by Lemma 2.5, as $i \to \infty$, $g_i a_i \xi \to k_0 \xi_\theta$ and $a_i \xi \to \xi_\theta$. Since g_i converges to g, this implies that $k_0 \xi_\theta = g\xi_\theta$. This finishes the proof. $\square$

3. Growth indicators

Let $\Gamma < G$ be a discrete subgroup. We set

$$\mu_\theta := p_\theta \circ \mu : G \to \mathfrak{a}_\theta^+. \tag{3.1}$$

Definition 3.1. We say that Γ is θ-discrete if the restriction $\mu_\theta|_\Gamma : \Gamma \to \mathfrak{a}_\theta^+$ is proper.

The θ-discreteness of Γ implies that $\mu_\theta(\Gamma)$ is a closed discrete subset of $\mathfrak{a}_\theta^+$. Indeed, Γ is θ-discrete if and only if the counting measure on $\mu_\theta(\Gamma)$ weighted with multiplicity is a Radon measure on $\mathfrak{a}_\theta^+$.

Definition 3.2 (θ-growth indicator)**.** For a θ-discrete subgroup $\Gamma < G$, we define the θ-growth indicator $\psi_\Gamma^\theta : \mathfrak{a}_\theta \to [-\infty, \infty]$ as follows: if $u \in \mathfrak{a}_\theta$ is non-zero,

$$\psi_\Gamma^\theta(u) = \|u\| \inf_{u\in\mathcal{C}} \tau_{\mathcal{C}}^\theta \tag{3.2}$$

where $\mathcal{C} \subset \mathfrak{a}_\theta$ ranges over all open cones containing u, and $\psi_\Gamma^\theta(0) = 0$. Here $-\infty \le \tau_{\mathcal{C}}^\theta \le \infty$ denotes the abscissa of convergence of the series $\mathcal{P}_{\mathcal{C}}^\theta(s) = \sum_{\gamma\in\Gamma, \mu_\theta(\gamma)\in\mathcal{C}} e^{-s\|\mu_\theta(\gamma)\|}$, that is,

$$\tau_{\mathcal{C}}^\theta = \sup\{s \in \mathbb{R} : \mathcal{P}_{\mathcal{C}}^\theta(s) = \infty\} = \inf\{s \in \mathbb{R} : \mathcal{P}_{\mathcal{C}}^\theta(s) < \infty\}.$$

This definition is independent of the choice of a norm on $\mathfrak{a}_\theta$. For $\theta = \Pi$, we set

$$\psi_\Gamma := \psi_\Gamma^\Pi.$$

The main goal of this section is to establish the following properties of ψ_Γ^θ for a general $\theta \subset \Pi$: for $\theta = \Pi$, this theorem is due to Quint [35, Theorem 1.1.1].

Theorem 3.3. *Let $\Gamma < G$ be a Zariski dense θ-discrete subgroup.*

(1) $\psi_\Gamma^\theta < \infty$.
(2) ψ_Γ^θ *is a homogeneous, upper semi-continuous and concave function.*
(3) $\mathcal{L}_\theta = \{\psi_\Gamma^\theta \geq 0\}$, $\psi_\Gamma^\theta = -\infty$ *outside* $\mathcal{L}_\theta$ *and* $\psi_\Gamma^\theta > 0$ *on* $\operatorname{int}\mathcal{L}_\theta$.

Here, $\mathcal{L}_\theta \subset \mathfrak{a}_\theta^+$ denotes the θ-limit cone of Γ, which is the asymptotic cone of $\mu_\theta(\Gamma)$:

$$\mathcal{L}_\theta = \{\lim t_i \mu_\theta(\gamma_i) : \gamma_i \in \Gamma, t_i \to 0\}. \tag{3.3}$$

We set $\mathcal{L} = \mathcal{L}_\Pi$, which is the usual limit cone. By [3, Sections 1.2, 4.6], if Γ is Zariski dense, then $\mathcal{L}$ is a convex cone with non-empty interior and $\mu(\Gamma)$ is within a bounded distance from $\mathcal{L}$. We have

$$\mathcal{L} = \{\psi_\Gamma \geq 0\}, \quad \text{and} \quad \psi_\Gamma > 0 \text{ on } \operatorname{int}\mathcal{L} \tag{3.4}$$

and $\psi_\Gamma = -\infty$ outside $\mathcal{L}$ [35, Theorem 1.1.1]. Noting that $\mathcal{L}_\theta = p_\theta(\mathcal{L})$, we get:

Lemma 3.4. *Let* Γ *be a Zariski dense* θ*-discrete subgroup. The* θ*-limit cone* $\mathcal{L}_\theta$ *is a convex cone in* $\mathfrak{a}_\theta^+$ *with non-empty interior and* $\mu_\theta(\Gamma)$ *is within a bounded distance from* $\mathcal{L}_\theta$.

$\psi_\Gamma^\theta < \infty$ **and** θ**-critical exponent.** In this subsection, we show Theorem 3.3(1), that is, for a Zariski dense θ-discrete $\Gamma < G$, ψ_Γ^θ does not take $+\infty$-value. This will be achieved by proving $\delta_\Gamma^\theta < \infty$ (Proposition 3.7) where

$$-\infty \leq \delta_\Gamma^\theta \leq \infty$$

denotes the abscissa of convergence of the series $s \mapsto \sum_{\gamma\in\Gamma} e^{-s\|\mu_\theta(\gamma)\|}$. For $\theta = \Pi$, we have $0 < \delta_\Gamma = \delta_\Gamma^\Pi < \infty$ [35, Theorem 4.2.2]. Since $\|\mu_\theta(g)\| \leq \|\mu(g)\|$ for all $g \in G$ and hence $\sum_{\gamma\in\Gamma} e^{-s\|\mu(\gamma)\|} \leq \sum_{\gamma\in\Gamma} e^{-s\|\mu_\theta(\gamma)\|}$ for all $s \geq 0$, we have

$$0 < \delta_\Gamma \leq \delta_\Gamma^\theta. \tag{3.5}$$

Lemma 3.5. *If* Γ *is Zariski dense and* θ*-discrete, then*

$$\delta_\Gamma^\theta = \limsup_{t\to\infty} \frac{1}{t} \# \log\{\gamma \in \Gamma : \|\mu_\theta(\gamma)\| < t\} \in (0, \infty].$$

Proof. For $x \in \mathfrak{a}_\theta$, we denote by D_x the Dirac mass at x. Since $\sum_{\gamma\in\Gamma} D_{\mu_\theta(\gamma)}$ is a Radon measure on $\mathfrak{a}_\theta^+$ and $\delta_\Gamma^\theta > 0$ by (3.5), it follows from [35, Lemma 3.1.1]. $\square$

For a general discrete subgroup $\Gamma < G$, δ_Γ^θ may be infinite (e.g., $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_i is an infinite discrete subgroup of G_i for both $i = 1, 2$ and θ consists of simple roots of G_1). Since $\tau_{\mathcal{C}}^\theta \leq \delta_\Gamma^\theta$ for all cones $\mathcal{C}$ in $\mathfrak{a}_\theta$, we have

$$\sup_{u\in\mathfrak{a}_\theta, \|u\|=1} \psi_\Gamma^\theta(u) \leq \delta_\Gamma^\theta.$$

Hence Theorem 3.3(1) follows once we show the that $\delta_\Gamma^\theta < \infty$ for any θ-discrete subgroup $\Gamma < G$ as in Proposition 3.7.

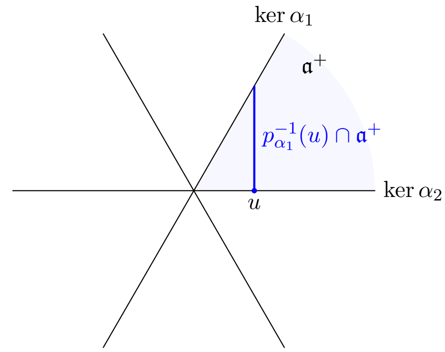

FIGURE 1. $G = \mathrm{PSL}_3(\mathbb{R})$ and $\theta = \{\alpha_1\}$.

Lemma 3.6. *If $p_\theta|_{\mathfrak{a}^+}$ is a proper map (e.g., G is simple), then*

$$\delta_\Gamma^\theta < \infty$$

for any discrete subgroup $\Gamma < G$. In particular, if G is simple, any discrete subgroup $\Gamma < G$ is θ-discrete.

Proof. First, observe that if G is simple, then the angle between any two walls of $\mathfrak{a}^+$ is strictly smaller than $\pi/2$ and hence $p_\theta|_{\mathfrak{a}^+}$ is a proper map (see Figure 1). Now, if $p_\theta|_{\mathfrak{a}^+}$ is a proper map, then for some constant $C > 1$, we have

$$C^{-1}\|u\| \le \|p_\theta(u)\| \le C\|u\|$$

for all $u \in \mathfrak{a}^+$. Hence $\delta_\Gamma < \infty$ implies that

$$\delta_\Gamma^\theta < \infty.$$

$\square$

It follows from the definition of δ_Γ^θ that the finiteness of δ_Γ^θ implies the θ-discreteness of Γ. Indeed, the converse holds as well from which Theorem 3.3(1) follows.

Proposition 3.7. *We have*

$$\Gamma \text{ is } \theta\text{-discrete} \quad \text{if and only if} \quad \delta_\Gamma^\theta < \infty.$$

Proof. It suffices to show that the θ-discreteness of Γ implies $\delta_\Gamma^\theta < \infty$. Write $G = G_1 G_2$ as an almost direct product of semisimple real algebraic groups where G_1 is the smallest group such that θ is contained in the set of simple roots for $(\mathfrak{g}_1, \mathfrak{a}_1^+ = \mathfrak{a}^+ \cap \mathfrak{g}_1)$. Then $\mu_\theta(\Gamma) \subset \mathfrak{a}_\theta^+ \subset \mathfrak{a}_1^+$. Since the kernel of $p_\theta|_{\mu(\Gamma)}$ contains $\mu(\Gamma \cap (\{e\} \times G_2))$, the properness hypothesis implies that $\Gamma \cap (\{e\} \times G_2)$ is finite. By passing to a subgroup of finite index, we may assume that $\Gamma \cap (\{e\} \times G_2)$ is trivial. The properness of $\mu_\theta|_\Gamma$ also implies that the projection of Γ to G_1 is a discrete subgroup, which we denote by Γ_1. Since there exists a unique element, say, $\sigma(\gamma_1) \in G_2$ such that $(\gamma_1, \sigma(\gamma_1)) \in \Gamma$ for each $\gamma_1 \in \Gamma_1$, we get a faithful representation $\sigma : \Gamma_1 \to G_2$, and Γ is of the form $\{(\gamma_1, \sigma(\gamma_1)) : \gamma \in \Gamma_1\}$. Since $\mu_\theta(\gamma) = \mu_\theta(\gamma_1)$ for $\gamma = (\gamma_1, \sigma(\gamma_1)) \in \Gamma$, we have

$$\delta_\Gamma^\theta = \delta_{\Gamma_1}^\theta.$$

Hence we may assume without loss of generality that θ contains at least one root of each simple factor of G. Since the restriction $p_\theta : \mathfrak{a}^+ \cap \operatorname{Lie} G_0 \to \mathfrak{a}_\theta \cap \operatorname{Lie} G_0$ is a proper map for each simple factor G_0 of G as mentioned before, it follows that p_θ is a proper map. Hence the claim $\delta_\Gamma^\theta < \infty$ follows by Lemma 3.6. □

Remark 3.8. We remark that the θ-discreteness of Γ does not necessarily imply that the map $p_\theta|_{\mathcal{L}}$ is a proper map. For example, let Γ_0 be a Zariski dense and convex cocompact subgroup of $\mathrm{SO}^\circ(k,1)$, $k \geq 2$, and let $\sigma : \Gamma_0 \to \mathrm{SO}^\circ(k,1)$ be a discrete faithful representation such that $\sigma(\Gamma_0)$ is Zariski dense but not convex cocompact. Consider $\Gamma = \{(g, \sigma(g)) : g \in \Gamma_0\}$ and $G = \mathrm{SO}^\circ(k,1) \times \mathrm{SO}^\circ(k,1)$. We may identify $\mathfrak{a} = \{(x_1, x_2) \in \mathbb{R}^2\}$ and $\mathfrak{a}^+ = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Then the limit cone of Γ is a convex cone of $\mathfrak{a}^+$ containing the x_1-axis; otherwise, σ must be convex cocompact. Hence for $\theta = \{\alpha_2\}$ where $\alpha_2(x_1, x_2) = x_2$, the fiber $p_\theta^{-1}(0)$ is the whole x_1-axis, and hence $p_\theta|_{\mathcal{L}}$ is not proper. On the other hand, the discreteness of $\sigma(\Gamma_0)$ is same as θ-discreteness of Γ.

Concavity of ψ_Γ^θ. The growth indicator ψ_Γ^θ is clearly a homogeneous and upper semi-continuous function [35, Lemma 3.1.7]. It is also a concave function, but its proof requires the following lemma, which is proved in [35, Proposition 2.3.1] for $\theta = \Pi$.

Lemma 3.9. *Suppose that Γ is Zariski dense and θ-discrete. Then there exists a map $\pi : \Gamma \times \Gamma \to \Gamma$ satisfying the following:*

1. *there exists $\kappa \geq 0$ such that for every $\gamma_1, \gamma_2 \in \Gamma$,*
$$\|\mu_\theta(\pi(\gamma_1, \gamma_2)) - \mu_\theta(\gamma_1) - \mu_\theta(\gamma_2)\| < \kappa; \text{ and}$$
2. *for every $R \geq 0$, there exists a finite subset H of Γ such that for $\gamma_1, \gamma_1', \gamma_2, \gamma_2' \in \Gamma$ with $\|\mu_\theta(\gamma_i) - \mu_\theta(\gamma_i')\| \leq R$ for $i = 1, 2$,*
$$\pi(\gamma_1, \gamma_2) = \pi(\gamma_1', \gamma_2') \Rightarrow \gamma_1' \in \gamma_1 H \text{ and } \gamma_2' \in H\gamma_2.$$

Proof. Since p_θ is norm-decreasing, (1) follows from [35, Proposition 2.3.1(1)]. By the proof of [35, Proposition 2.3.1(2)], the claim (2) holds if we set H to be the subset consisting of all elements $\gamma \in \Gamma$ such that $\mu_\theta(\gamma) < R'$ for some $R' > 0$ depending only on R. Since Γ is θ-discrete, this subset H is finite, as desired. □

Proposition 3.10. *If Γ is Zariski dense and θ-discrete, then ψ_Γ^θ is concave, and hence there exists a unique unit vector $u_\Gamma^\theta \in \mathfrak{a}_\theta^+$ such that*

$$\psi_\Gamma^\theta(u_\Gamma^\theta) = \max_{\|u\|=1, u \in \mathfrak{a}_\theta^+} \psi_\Gamma^\theta(u) = \delta_\Gamma^\theta.$$

Proof. By Lemma 3.9, the counting measure $\sum_{\gamma\in\Gamma} D_{\mu_\theta(\gamma)}$ is of concave growth (see [35, Section 3.2] for details). It follows from [35, Theorem 3.2.1] that ψ_Γ^θ is concave. By [35, Corollary 3.1.4, Corollary 3.3.5], the second claim follows. ☐

Definition 3.11. A linear form $\psi \in \mathfrak{a}_\theta^*$ is said to be tangent to ψ_Γ^θ (at $u \in \mathfrak{a}_\theta^+ - \{0\}$) if $\psi \geq \psi_\Gamma^\theta$ on $\mathfrak{a}_\theta^+$ and $\psi(u) = \psi_\Gamma^\theta(u)$.

By the supporting hyperplane theorem, we have the following corollary:

Corollary 3.12. *Let $\Gamma < G$ be Zariski dense and θ-discrete. For any $u \in \operatorname{int}\mathcal{L}_\theta$, there exists a linear form $\psi \in \mathfrak{a}_\theta^*$ tangent to ψ_Γ^θ at u.*

Positivity of ψ_Γ^θ. By Lemma 3.4, we have $\psi_\Gamma^\theta = -\infty$ outside $\mathcal{L}_\theta$. If $\Theta \supset \theta$, then any θ-discrete Γ is Θ-discrete as well. The following lemma shows how ψ_Γ^θ is related to ψ_Γ^Θ from which Theorem 3.3(3) follows:

Lemma 3.13. *For $\Theta \supset \theta$, let $p_\theta = p_\theta|_{\mathfrak{a}_\Theta} : \mathfrak{a}_\Theta \to \mathfrak{a}_\theta$ by abuse of notation. For any Zariski dense θ-discrete $\Gamma < G$, we have*

$$\psi_\Gamma^\theta \circ p_\theta \geq \psi_\Gamma^\Theta \quad \text{on } \mathfrak{a}_\Theta. \tag{3.6}$$

In particular,

$$\psi_\Gamma^\theta \geq 0 \text{ on } \mathcal{L}_\theta \quad \text{and} \quad \psi_\Gamma^\theta > 0 \text{ on } \operatorname{int}\mathcal{L}_\theta. \tag{3.7}$$

Proof. Note that (3.7) follows from (3.4) and (3.6). By homogeneity, it suffices to prove (3.6) for all $v \in p_\theta^{-1}(u) \cap \mathfrak{a}_\Theta$, where $u \in \mathcal{L}_\theta$ is an arbitrary unit vector. Let $v \in p_\theta^{-1}(u) \cap \mathfrak{a}_\Theta$. Let $\mathcal{C} \subset \mathfrak{a}_\theta$ be an open cone containing u. For each $\varepsilon > 0$, set

$$\mathcal{C}(v,\varepsilon) := \left\{ w \in \mathfrak{a}_\Theta : p_\theta(w) \neq 0 \text{ and } \left\| \tfrac{w}{\|p_\theta(w)\|} - v \right\| < \varepsilon \right\}. \tag{3.8}$$

Since $\|p_\theta(v)\| = \|u\| = 1$, $\mathcal{C}(v,\varepsilon)$ is an open cone containing v. In the following, let $\varepsilon > 0$ be small enough so that $\mathcal{C}(v,\varepsilon) \subset p_\theta^{-1}(\mathcal{C})$.

Then for all $s \in \mathbb{R}$, we have

$$\begin{aligned}\sum_{\gamma\in\Gamma, \mu_\Theta(\gamma)\in\mathcal{C}(v,\varepsilon)} e^{-s\|\mu_\Theta(\gamma)\|} &\leq \sum_{\gamma\in\Gamma, \mu_\Theta(\gamma)\in\mathcal{C}(v,\varepsilon)} e^{-(s\|v\| - |\varepsilon s|)\|\mu_\theta(\gamma)\|} \\ &\leq \sum_{\gamma\in\Gamma, \mu_\theta(\gamma)\in\mathcal{C}} e^{-(s\|v\| - |\varepsilon s|)\|\mu_\theta(\gamma)\|}.\end{aligned}$$

Hence we have

$$\tau_{\mathcal{C}(v,\varepsilon)}^\Theta \leq (\|v\| - \varepsilon)^{-1}\, \tau_{\mathcal{C}}^\theta.$$

Therefore we have

$$\psi_\Gamma^\Theta(v) \leq \|v\| \tau_{\mathcal{C}(v,\varepsilon)}^\Theta \leq \|v\| \, (\|v\| - \varepsilon)^{-1}\, \tau_{\mathcal{C}}^\theta.$$

Taking $\varepsilon \to 0$ yields that

$$\psi_\Gamma^\Theta(v) \leq \tau_{\mathcal{C}}^\theta.$$

Since $\mathcal{C} \subset \mathfrak{a}_\Theta$ is an arbitrary open cone in $\mathfrak{a}_\theta$ containing u, it follows that

$$\psi_\Gamma^\Theta(v) \le \psi_\Gamma^\theta(u),$$

and hence (3.6) is proved. Last claim follows the from (3.4) and (3.6) applied to $\Theta = \Pi$. $\square$

Comparison between ψ_Γ^θ and ψ_Γ^Θ. Note that for a discrete subgroup $\Gamma < G$, the properness of $p_\theta|_{\mathcal{L}_\theta}$ implies the θ-discreteness of Γ as $\mu(\Gamma)$ is within a bounded distance from $\mathcal{L}$. The following lemma is to appear in [16] in a more general context.

Lemma 3.14. *Let $\Gamma < G$ be a Zariski dense discrete subgroup. If $p_\theta|_{\mathcal{L}}$ is a proper map (e.g., G is simple), then for any $\Theta \supset \theta$ and for any $u \in \mathfrak{a}_\theta$,*

$$\psi_\Gamma^\theta(u) = \max_{v \in p_\theta^{-1}(u)} \psi_\Gamma^\Theta(v) \tag{3.9}$$

where $p_\theta = p_\theta|_{\mathfrak{a}_\Theta}$ by abuse of notation.

Proof. Suppose that $p_\theta|_{\mathcal{L}} : \mathcal{L} \to \mathfrak{a}_\theta$ is a proper map. By Lemma 3.13, it suffices to consider a unit vector $u \in \mathcal{L}_\theta$ with $\psi_\Gamma^\theta(u) > 0$. Since $p_\theta^{-1}(u) \cap \mathcal{L}_\Theta$ is a compact subset and ψ_Γ^Θ is upper semi-continuous, we have

$$\sup_{v \in p_\theta^{-1}(u)} \psi_\Gamma^\Theta(v) = \max_{v \in p_\theta^{-1}(u) \cap \mathcal{L}_\Theta} \psi_\Gamma^\Theta(v).$$

For all sufficiently small $\varepsilon > 0$ and each $v \in p_\theta^{-1}(u)$, there exists $0 < \varepsilon_v < \varepsilon$ such that

$$\|v\| \tau^\Theta_{\mathcal{C}(v,\varepsilon_v)} < \psi_\Gamma^\Theta(v) + \varepsilon \tag{3.10}$$

where $\mathcal{C}(v, \varepsilon_v)$ is as defined in (3.8). Since $p_\theta^{-1}(u) \cap \mathcal{L}_\Theta$ is compact, there exist $v_1, \cdots, v_n \in p_\theta^{-1}(u)$ such that

$$p_\theta^{-1}(u) \cap \mathcal{L}_\Theta \subset \bigcup_{i=1}^n \mathcal{C}(v_i, \varepsilon_{v_i}).$$

Take an open cone $\mathcal{C} \subset \mathfrak{a}_\theta$ containing u such that

$$p_\theta^{-1}(u) \cap \mathcal{L}_\Theta \subset p_\theta^{-1}(\mathcal{C}) \cap \mathcal{L}_\Theta \subset \bigcup_{i=1}^n \mathcal{C}(v_i, \varepsilon_{v_i}).$$

This is indeed possible; if not, there is a sequence of unit vectors $u_j \in \mathfrak{a}_\theta$ converging to u as $j \to \infty$ such that for each j, there exists $w_j \in p_\theta^{-1}(u_j) \cap \mathcal{L}_\Theta$ that does not belong to $\bigcup_{i=1}^n \mathcal{C}(v_i, \varepsilon_{v_i})$. Since $p_\theta|_{\mathcal{L}_\Theta}$ is proper and the unit sphere in $\mathfrak{a}_\theta$ is compact, we may assume that the sequence w_j converges to some $w \in \mathcal{L}_\Theta$ after passing to a subsequence. Since $p_\theta(w_j) = u_j \to u$ as $j \to \infty$, we have $p_\theta(w) = u$, and hence $w \in p_\theta^{-1}(u) \cap \mathcal{L}_\Theta$. It implies that $w_j \in \bigcup_{i=1}^n \mathcal{C}(v_i, \varepsilon_{v_i})$ for all large j, contradiction.

Since $\mu_\Theta(\Gamma)$ is within a bounded distance from $\mathcal{L}_\Theta$ (Lemma 3.4), there are only finitely many elements of $\mu_\Theta(\Gamma) \cap p_\theta^{-1}(\mathcal{C})$ outside of $\bigcup_{i=1}^n \mathcal{C}(v_i, \varepsilon_{v_i})$. Hence for each $s \geq 0$, we have

$$\begin{aligned}\sum_{\gamma \in \Gamma, \mu_\theta(\gamma) \in \mathcal{C}} e^{-s\|\mu_\theta(\gamma)\|} &\ll \sum_{i=1}^n \sum_{\gamma \in \Gamma, \mu_\Theta(\gamma) \in \mathcal{C}(v_i, \varepsilon_{v_i})} e^{-s\|\mu_\theta(\gamma)\|} \\ &\leq \sum_{i=1}^n \sum_{\gamma \in \Gamma, \mu_\Theta(\gamma) \in \mathcal{C}(v_i, \varepsilon_{v_i})} e^{-s \frac{\|\mu_\Theta(\gamma)\|}{\|v_i\| + \varepsilon_{v_i}}}.\end{aligned}$$

Here and afterwards, the notation $f(s) \ll g(s)$ means that for some uniform constant $C \geq 1$, $f(s) \leq C g(s)$ for all s at hand. Since $\tau_\mathcal{C}^\theta \geq \psi_\Gamma^\theta(u) > 0$ is positive, it follows that

$$\tau_\mathcal{C}^\theta \leq \max_i (\|v_i\| + \varepsilon_{v_i}) \tau^\Theta_{\mathcal{C}(v_i, \varepsilon_{v_i})}.$$

Therefore, together with $0 < \varepsilon_{v_i} < \varepsilon$ and (3.10), we get

$$\psi_\Gamma^\theta(u) \leq \tau_\mathcal{C}^\theta \leq \frac{\|v_i\| + \varepsilon_{v_i}}{\|v_i\|} \left(\max_i \psi_\Gamma^\Theta(v_i) + \varepsilon \right) \leq \frac{\|v_i\| + \varepsilon}{\|v_i\|} \left(\max_{v \in p_\theta^{-1}(u)} \psi_\Gamma^\Theta(v) + \varepsilon \right).$$

Since $0 < \varepsilon < 1$ was arbitrary, this proves the claim by Lemma 3.13. ☐

Example 3.15. We discuss some explicit upper bounds for ψ_Γ^θ when $G = \mathrm{PSL}_d(\mathbb{R})$. Identify $\mathfrak{a}^+ = \{(t_1, \cdots, t_d) : t_1 \geq \cdots \geq t_d, t_1 + \cdots + t_d = 0\}$. Let $\alpha_i(t_1, ..., t_d) = t_i - t_{i+1}$ for $i = 1, 2, ..., d-1$. Let

$$w_i = \left(\tfrac{d-i}{d}, \cdots, \tfrac{d-i}{d}, -\tfrac{i}{d}, \cdots, -\tfrac{i}{d} \right),$$

where the first i coordinates are $\frac{d-i}{d}$'s and the last $d-i$ coordinates are $-\frac{i}{d}$'s, so that $\mathfrak{a}_{\alpha_i} = \mathbb{R} w_i$ and $\alpha_i(w_i) = 1$. We compute that

$$p_{\alpha_i}(t_1, \cdots, t_d) = \frac{d(t_1 + \cdots + t_i)}{i(d-i)} w_i$$

and hence

$$p_{\alpha_i}^{-1}(w_i) \cap \mathfrak{a}^+ = \{(t_1, \cdots, t_d) \in \mathfrak{a}^+ : d(t_1 + \cdots + t_i) = i(d-i)\}.$$

For any non-lattice discrete subgroup $\Gamma < \mathrm{PSL}_d(\mathbb{R})$, we have

$$\psi_\Gamma(t_1, \cdots, t_d) \leq \sum_{i<j} (t_i - t_j) - \frac{1}{2} \sum_{i=1}^{\lfloor d/2 \rfloor} (t_i - t_{d+1-i}) \tag{3.11}$$

by ([38], [31], [30, Theorem 7.1]). By Lemma 3.14, for any discrete non-lattice subgroups, we get

$$\psi_\Gamma^{\alpha_i}(w_i) \leq \max \sum_{i<j} (t_i - t_j) - \frac{1}{2} \sum_{i=1}^{\lfloor d/2 \rfloor} (t_i - t_{d+1-i}) \tag{3.12}$$

where the maximum is taken over all $(t_1, \cdots, t_d) \in \mathfrak{a}^+$ such that $d(t_1 + \cdots + t_i) = i(d-i)$.

For instance, for $d=3$, the right hand side is always 3 and hence for each $i=1,2$, $\psi_\Gamma^{\alpha_i}\le 3\alpha_i$ on $\mathbb{R}w_i$.

Hitchin subgroups. Let $\iota:\mathrm{PSL}_2(\mathbb{R})\to\mathrm{PSL}_d(\mathbb{R})$ be the irreducible representation, which is unique up to conjugations. A Hitchin subgroup is the image of a representation $\pi:\Sigma\to\mathrm{PSL}_d(\mathbb{R})$ of a non-elementary torsion-free geometrically finite subgroup $\Sigma<\mathrm{PSL}_2(\mathbb{R})$, which is a type-preserving deformation of $\iota|_\Sigma$. Hitchin subgroups are Π-transverse, as defined in the introduction, by [7] and hence α_i-discrete for each $i=1,\cdots,d-1$. For a Zariski dense Hitchin subgroup Γ, it follows from Lemma 4.5 that if δ_{α_i} denotes the abscissa of convergence of $s\mapsto\sum_{\gamma\in\Gamma}e^{-s\alpha_i(\mu(\gamma))}$, then

$$\psi_\Gamma^{\alpha_i}(w_i)\le\delta_{\alpha_i}\cdot\alpha_i(w_i)=\delta_{\alpha_i}.$$

For Hitchin subgroups, it was proved by Potrie and Sambarino [33] for Δ cocompact and Canary, Zhang and Zimmer [8] for Δ geometrically finite that

$$\delta_{\alpha_i}\le 1$$

for all i (see also [34]). Hence $\max_{1\le i\le d-1}\psi_\Gamma^{\alpha_i}(w_i)\le 1$. We get a sharper bound in the following:

Corollary 3.16. *Let $\Gamma<\mathrm{PSL}_d(\mathbb{R})$ be a Zariski dense Hitchin subgroup. For each $i=1,\cdots,d-1$,*

$$\psi_\Gamma^{\alpha_i}<\frac{\max(i,d-i)}{d-1}\alpha_i\quad\text{on }\mathfrak{a}_{\alpha_i}-\{0\}.$$

Proof. For a Zariski dense Hitchin subgroup $\Gamma<G$, it is shown in [25, Corollary 1.10] that

$$\psi_\Gamma(t_1,\cdots,t_d)<\frac{1}{d-1}(t_1-t_d)\quad\text{for }(t_1,\cdots,t_d)\in\mathfrak{a}^+-\{0\}.\tag{3.13}$$

Indeed, [25, Corollary 1.10] is stated only for Σ cocompact. However in view of [8] mentioned above, this bound holds for a general Hitchin subgroup. Hence by Lemma 3.14, we get

$$\psi_\Gamma^{\alpha_i}(w_i)<\frac{1}{d-1}\max\,(t_1-t_d)\tag{3.14}$$

where the maximum is taken over all $t_1\ge\cdots\ge t_d$ such that $d\sum_{j=1}^i t_j=i(d-i)$ and $\sum_{j=1}^d t_j=0$. Suppose that this maximum is realized at $(t_1,\cdots,t_d)$. Since t_1-t_d does not involve any t_j, $2\le j\le d-1$, we may assume that $t_2=\cdots=t_i$ and $t_{i+1}=\cdots=t_{d-1}$, which we denote by x and y respectively. Since $\sum_{j=1}^i t_j=\frac{i(d-i)}{d}$ and $\sum_{j=i+1}^d t_j=-\frac{i(d-i)}{d}$, we then have

$$t_1=\tfrac{i(d-i)}{d}-(i-1)x\quad\text{and}\quad t_d=-\tfrac{i(d-i)}{d}-(d-1-i)y.$$

Therefore

$$t_1-t_d=\tfrac{2i(d-i)}{d}-((i-1)x-(d-1-i)y).\tag{3.15}$$

It follows from $t_j \geq t_{j+1}$ for all j that $\frac{d-i}{d} \geq x \geq y \geq -\frac{i}{d}$. Therefore, for each fixed x, the maximum in (3.15) is obtained when $y = x$. Hence we have

$$\begin{aligned}\psi_\Gamma^{\alpha_i}(w_i) &< \frac{1}{d-1} \max_{x \in [-i/d, (d-i)/d]} \frac{2i(d-i)}{d} - (2i-d)x \\ &= \frac{1}{d-1} \max(i, d-i).\end{aligned}$$

□

4. On the proper and critical linear forms

Let Γ be a θ-discrete infinite subgroup of G.

Definition 4.1. A linear form $\psi \in \mathfrak{a}_\theta^*$ is called (Γ, θ)-proper if $\operatorname{Im}(\psi \circ \mu_\theta) \subset [-\varepsilon, \infty)$ and $\psi \circ \mu_\theta : \Gamma \to [-\varepsilon, \infty)$ is proper for some $\varepsilon > 0$.

Consider the series $\mathcal{P}_\psi = \mathcal{P}_{\Gamma, \psi}$ given by

$$\mathcal{P}_\psi(s) = \sum_{\gamma \in \Gamma} e^{-s\psi(\mu_\theta(\gamma))}. \tag{4.1}$$

The abscissa of convergence of $\mathcal{P}_\psi$ is well-defined for a (Γ, θ)-proper linear form:

Lemma 4.2. *If ψ is (Γ, θ)-proper, the following $\delta_\psi = \delta_\psi(\Gamma)$ is well-defined (possibly $+\infty$):*

$$\delta_\psi := \sup\{s \in \mathbb{R} : \mathcal{P}_\psi(s) = \infty\} = \inf\{s \in \mathbb{R} : \mathcal{P}_\psi(s) < \infty\} \in [0, \infty]. \tag{4.2}$$

Moreover, if Γ is Zariski dense, then

$$0 < \delta_\psi = \limsup_{t \to \infty} \frac{\log \#\{\gamma \in \Gamma : \psi(\mu_\theta(\gamma)) \leq t\}}{t}.$$

Proof. Since ψ is (Γ, θ)-proper, $\psi(\mu_\theta(\gamma)) > 0$ for all but finitely many $\gamma \in \Gamma$. Hence we may replace $\mathcal{P}_\psi(s)$ by the series $\mathcal{P}_\psi^+(s) = \sum_{\gamma \in \Gamma, \psi(\mu_\theta(\gamma)) > 0} e^{-s\psi(\mu_\theta(\gamma))}$ in proving this claim. Since $\mathcal{P}_\psi^+(s)$ is a decreasing function of $s \in \mathbb{R}$, $I_1 := \{\mathcal{P}_\psi(s) = \infty\}$ and $I_2 := \{\mathcal{P}_\psi(s) < \infty\}$ are disjoint intervals. Since Γ is infinite, $0 \in I_1$, and hence $\delta_\psi = \overline{I}_1 \cap \overline{I}_2 \in [0, \infty]$ is well-defined.

Now suppose that Γ is Zariski dense. By Lemma 3.4, $\operatorname{int} \mathcal{L}_\theta \neq \emptyset$. To show $\delta_\psi > 0$, fix $u \in \operatorname{int} \mathcal{L}_\theta$. Then $\psi(u) > 0$ by Lemma 4.3. Since $\psi_\Gamma^\theta(u) > 0$ as well by Theorem 3.3(3), we have $s_0 \psi(u) < \psi_\Gamma^\theta(u)$ for some $0 < s_0 < \infty$. By [35, Lemma 3.1.3], we have $\mathcal{P}_\psi(s_0) = \infty$, and therefore $\delta_\psi \geq s_0 > 0$. The last claim follows by [35, Lemma 3.1.1] since the counting measure on $\psi(\mu_\theta(\Gamma))$ is locally finite and $\delta_\psi > 0$. □

Hence for a (Γ, θ)-proper form $\psi \in \mathfrak{a}_\theta^*$, $0 < \delta_\psi \leq \infty$ is the abscissa of convergence of $\mathcal{P}_\psi(s)$.

Lemma 4.3. *We have:*

(1) *If $\psi > 0$ on $\mathcal{L}_\theta - \{0\}$, then ψ is (Γ, θ)-proper and $\delta_\psi < \infty$.*

(2) *If ψ is (Γ,θ)-proper, then $\psi \geq 0$ on $\mathcal{L}_\theta$ and $\psi > 0$ on* $\operatorname{int}\mathcal{L}_\theta$.

Proof. If ψ is positive on $\mathcal{L}_\theta - \{0\}$, then $\psi > 0$ on some open cone $\mathcal{C}$ containing $\mathcal{L}_\theta - \{0\}$. Then for some $c > 1$, $c^{-1}\|u\| \leq \psi(u) \leq c\|u\|$ for all $u \in \mathcal{C}$. Since there can be only finitely many points of $\mu_\theta(\Gamma)$ outside $\mathcal{C}$ by Lemma 3.4, this implies that ψ is (Γ,θ)-proper. Since $\delta_\Gamma^\theta < \infty$ by Proposition 3.7, we also have $\delta_\psi < \infty$.

To prove (2), suppose to the contrary that $\psi(u) < 0$ for some $u \in \mathcal{L}_\theta$. Then there exists an open cone $\mathcal{C} \subset \mathfrak{a}_\theta$ containing u so that $\psi < 0$ on $\mathcal{C} - \{0\}$. In particular, there are infinitely many $\gamma_i \in \Gamma$ such that $\psi(\mu_\theta(\gamma_i)) < 0$, which contradicts (Γ,θ)-properness of ψ. Therefore, $\psi \geq 0$ on $\mathcal{L}_\theta$. Since $\ker\psi$ is a hyperplane in $\mathfrak{a}_\theta$, it follows $\psi > 0$ on $\operatorname{int}\mathcal{L}_\theta$. $\square$

Critical forms. Analogous to the critical exponent of a discrete subgroup of a rank one Lie group, we define:

Definition 4.4. A linear form $\psi \in \mathfrak{a}_\theta^*$ is (Γ,θ)-critical if it is tangent to ψ_Γ^θ.

The following lemma can be proved by adapting the proof of [25, Theorem 2.5] replacing ψ_Γ by ψ_Γ^θ.

Lemma 4.5. *Suppose that Γ is Zariski dense. If a (Γ,θ)-proper $\psi \in \mathfrak{a}_\theta^*$ satisfies $\delta_\psi < \infty$, then $\delta_\psi\psi$ is (Γ,θ)-critical; in particular,*

$$\psi_\Gamma^\theta \leq \delta_\psi \psi.$$

Proof. Suppose that $\delta_\psi < \infty$. By Lemma 4.2, $\delta_\psi > 0$. We first claim

$$\psi_\Gamma^\theta(v) \leq \delta_\psi\psi(v) \quad \text{for all } v \in \operatorname{int}\mathcal{L}_\theta. \tag{4.3}$$

Fix $v \in \operatorname{int}\mathcal{L}_\theta$ and $\varepsilon > 0$. Since ψ is (Γ,θ)-proper, $\psi(v) > 0$ by Lemma 4.3.

We then consider

$$\mathcal{C}_\varepsilon(v) = \left\{ w \in \mathfrak{a}_\theta : \psi(w) > 0 \text{ and } \left\| \frac{w}{\psi(w)} - \frac{v}{\psi(v)} \right\| < \varepsilon \right\};$$

since $\psi(v) > 0$, this is a well-defined open cone containing v. Therefore by the definition of ψ_Γ^θ, we have

$$\psi_\Gamma^\theta(v) \leq \|v\| \tau^\theta_{\mathcal{C}_\varepsilon(v)}. \tag{4.4}$$

Observe that for any $s \geq 0$,

$$\begin{aligned}\sum_{\gamma\in\Gamma, \mu_\theta(\gamma)\in\mathcal{C}_\varepsilon(v)} e^{-s\|\mu_\theta(\gamma)\|} &\leq \sum_{\gamma\in\Gamma, \mu_\theta(\gamma)\in\mathcal{C}_\varepsilon(v)} e^{-s\psi(\mu_\theta(\gamma))\left(\frac{\|v\|}{\psi(v)} - \varepsilon\right)} \\ &\leq \sum_{\gamma\in\Gamma} e^{-s\psi(\mu_\theta(\gamma))\left(\frac{\|v\|}{\psi(v)} - \varepsilon\right)}.\end{aligned}$$

It follows from the definitions of $\tau^\theta_{\mathcal{C}_\varepsilon(v)}$ and δ_ψ that

$$\tau^\theta_{\mathcal{C}_\varepsilon(v)} \leq \frac{\delta_\psi}{\|v\|\psi(v)^{-1} - \varepsilon} = \frac{\delta_\psi\psi(v)}{\|v\| - \varepsilon\psi(v)},$$

and hence

$$\psi_\Gamma^\theta(v) \le \|v\| \frac{\delta_\psi \psi(v)}{\|v\| - \varepsilon\psi(v)}.$$

Since $\varepsilon > 0$ is arbitrary, we get $\psi_\Gamma^\theta(v) \le \delta_\psi \psi(v)$, proving the claim (4.3).

We now claim that the inequality (4.3) also holds for any v in the boundary $\partial\mathcal{L}_\theta$. Choose any $v_0 \in \operatorname{int}\mathcal{L}_\theta$. From the concavity of ψ_Γ^θ (Theorem 3.3), we have

$$t\psi_\Gamma^\theta(v_0) + (1-t)\psi_\Gamma^\theta(v) \le \psi_\Gamma^\theta(tv_0 + (1-t)v) \quad \text{for all } 0 < t < 1.$$

Since $\mathcal{L}_\theta$ is convex, $tv_0 + (1-t)v \in \operatorname{int}\mathcal{L}_\theta$ for all $0 < t < 1$. As we have already shown $\psi_\Gamma^\theta \le \delta_\psi\psi$ on $\operatorname{int}\mathcal{L}_\theta$, we get

$$t\psi_\Gamma^\theta(v_0) + (1-t)\psi_\Gamma^\theta(v) \le \delta_\psi\psi(tv_0 + (1-t)v) \quad \text{for all } 0 < t < 1.$$

By sending $t \to 0^+$, we get

$$\psi_\Gamma^\theta(v) \le \delta_\psi \cdot \psi(v).$$

Since $\psi_\Gamma^\theta = -\infty$ outside $\mathcal{L}_\theta$, we have established $\psi_\Gamma^\theta \le \delta_\psi\psi$. Suppose that $\psi_\Gamma^\theta < \delta_\psi\psi$ on $\mathfrak{a} - \{0\}$. Then the abscissa of convergence of the series $s \mapsto \sum_{\gamma\in\Gamma} e^{-s\delta_\psi\psi(\mu_\theta(\gamma))}$ is strictly less than 1 by [35, Lemma 3.1.3]. However the abscissa of convergence of this series is equal to 1 by the definition of δ_ψ. Therefore $\delta_\psi\psi$ is tangent to ψ_Γ^θ, finishing the proof. □

Corollary 4.6. *Suppose that Γ is Zariski dense. A (Γ,θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$ with $\delta_\psi = 1$ is (Γ,θ)-critical. Moreover, if $\psi > 0$ on $\mathcal{L}_\theta$, then ψ is (Γ,θ)-critical if and only if $\delta_\psi = 1$.*

Via the identification $\mathfrak{a}_\theta^* = \{\psi \in \mathfrak{a}^* : \psi = \psi \circ p_\theta\}$, we can consider $\mathfrak{a}_\theta^*$ as a subspace of $\mathfrak{a}^*$. Lemma 3.14 implies the following identity:

Corollary 4.7. *Suppose that Γ is Zariski dense. If $p_\theta|_{\mathcal{L}}$ is a proper map, then*

$$\{\psi \in \mathfrak{a}_\theta^* : \psi \textit{ is } (\Gamma,\theta)\textit{-critical}\} = \{\psi \in \mathfrak{a}^* : \psi = \psi\circ p_\theta, \psi \textit{ is } (\Gamma,\Pi)\textit{-critical}\}.$$

Proof. To show the inclusion $\supset$, suppose $\psi = \psi \circ p_\theta$ and ψ is (Γ,Π)-critical. Then for any $u \in \mathfrak{a}_\theta$ and any $v' \in p_\theta^{-1}(u)$, $\psi(u) = \psi(v') \ge \psi_\Gamma(v')$ and hence $\psi(u) \ge \psi_\Gamma^\theta(u)$ by Lemma 3.14. Moreover, if $\psi(v) = \psi_\Gamma(v)$, then for $u = p_\theta(v)$, $\psi(u) \ge \psi_\Gamma^\theta(u) \ge \psi_\Gamma(v) = \psi(v) = \psi(u)$ and hence $\psi(u) = \psi_\Gamma^\theta(u)$, proving ψ is (Γ,θ)-critical. For the other inclusion $\subset$, suppose that $\psi \ge \psi_\Gamma^\theta$ on $\mathfrak{a}_\theta^+$ and $\psi(u) = \psi_\Gamma^\theta(u)$ for some $u \in \mathfrak{a}_\theta^+$. Then for any $v \in \mathfrak{a}^+$, $\psi(v) = \psi(p_\theta(v)) \ge \psi_\Gamma^\theta(p_\theta(v)) \ge \psi_\Gamma(v)$ by Lemma 3.13. Let $v \in p_\theta^{-1}(u)$ be such that $\psi_\Gamma^\theta(u) = \psi_\Gamma(v)$ given by Lemma 3.14. Then $\psi(v) = \psi(u) = \psi_\Gamma^\theta(u) = \psi_\Gamma(v)$; so ψ is (Γ,Π)-critical. □

5. Limit set, θ-conical set, and conformal measures

Let $\Gamma < G$ be a closed subgroup.

Definition 5.1 (θ-limit set)**.** We define the θ-limit set of Γ as follows:

$$\Lambda_\theta = \Lambda_\theta(\Gamma) := \{\lim \gamma_i \in \mathcal{F}_\theta : \gamma_i \in \Gamma\}$$

where $\lim \gamma_i$ is defined as in Definition 2.2.

This is a Γ-invariant closed subset of $\mathcal{F}_\theta$, which may be empty in general. Set $\Lambda = \Lambda_\Pi$. Denote by Leb_θ the K-invariant probability measure on $\mathcal{F}_\theta$. This definition of Λ_θ coincides with that of Benoist:

Lemma 5.2 ([3], [36, Corollary 5.2, Lemma 6.3, Theorem 7.2], [29, Lemma 2.13])**.** *If Γ is Zariski dense in G, we have the following:*

1. $\Lambda_\theta = \{\xi \in \mathcal{F}_\theta : (\gamma_i)_* \mathrm{Leb}_\theta \to D_\xi$ *for some infinite sequence* $\gamma_i \in \Gamma\}$ *where D_ξ is the Dirac measure at ξ;*
2. $\Lambda_\theta = \pi_\theta(\Lambda)$*;*
3. Λ_θ *is the unique Γ-minimal subset of $\mathcal{F}_\theta$.*

In the rest of this section, suppose that Γ is discrete.

Definition 5.3 (θ-conical set)**.** We define the θ-conical set of Γ as

$$\Lambda_\theta^{\mathsf{con}} = \left\{ gP_\theta \in \mathcal{F}_\theta : \limsup_{\gamma \in \Gamma} \gamma g M_\theta A^+ \neq \emptyset \right\}. \tag{5.1}$$

For $\theta = \Pi$, $\Lambda_\Pi^{\mathsf{con}} = \{gP \in \mathcal{F} : \limsup_{\gamma\in\Gamma} \gamma g A^+ \neq \emptyset\}$ because $M_\Pi = M$ commutes with A. Note that the conical set is not contained in the limit set Λ in general even for $\theta = \Pi$. For example, if $G = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ and $\Gamma = \Gamma_1 \times \Gamma_2$ is a product of two convex cocompact subgroups, then $\Lambda = \Lambda(\Gamma_1) \times \Lambda(\Gamma_2)$ while $\Lambda^{\mathsf{con}} = (\Lambda(\Gamma_1) \times \mathbb{S}^1) \cup (\mathbb{S}^1 \times \Lambda(\Gamma_2))$.

θ-shadows. For $q \in X$ and $R > 0$, let $B(q,R) = \{x \in X : d(x,q) \le R\}$. For $p \in X$, the θ-shadow $O_R^\theta(p,q) \subset \mathcal{F}_\theta$ of $B(q,R)$ viewed from p is defined as

$$\begin{aligned} O_R^\theta(p,q) &= \{gP_\theta \in \mathcal{F}_\theta : g \in G,\ go = p,\ gA^+o \cap B(q,R) \neq \emptyset\} \\ &= \{gP_\theta \in \mathcal{F}_\theta : g \in G,\ go = p,\ gM_\theta A^+o \cap B(q,R) \neq \emptyset\}. \end{aligned} \tag{5.2}$$

Clearly, for $O_R(p,q) = O_R^\Pi(p,q)$, we have

$$O_R^\theta(p,q) := \pi_\theta(O_R(p,q)).$$

Lemma 5.4. *We have $\xi \in \Lambda_\theta^{\mathsf{con}}$ if and only if there exist an infinite sequence $\gamma_i \in \Gamma$ and $N > 0$ such that $\xi \in \bigcap_i O_N^\theta(o, \gamma_i o)$.*

Proof. The direction $\Rightarrow$ is clear. To see the other direction, suppose that $\xi \in \bigcap_i O_N^\theta(o,\gamma_i o)$ for some $N > 0$ and an infinite sequence $\gamma_i \in \Gamma$, that is, there exist sequences $k_i \in K$ and $a_i \to \infty$ in A^+ such that $\xi = k_i P_\theta$ and the sequence $\gamma_i^{-1} k_i a_i$ is bounded. By passing to a subsequence, we may assume k_i converges to some $k \in K$. Since $\xi = k_i P_\theta$ for all i, we have $\xi = kP_\theta$.

Since $k_i P_\theta = kP_\theta$ and $M_\theta = P_\theta \cap K$, we have $k_i = km_i$ for some $m_i \in M_\theta$. Since $\gamma_i^{-1} km_i a_i = \gamma_i^{-1} k_i a_i$ is bounded, we have $\xi = kP_\theta \in \Lambda_\theta^{\mathsf{con}}$. $\square$

We remark that we may replace o by any $p \in X$ in the above lemma.

For each $N > 0$, we set

$$\Lambda_\theta^N := \left\{ \xi \in \mathcal{F}_\theta : \text{ there exists } \gamma_i \to \infty \text{ in } \Gamma \text{ such that } \xi \in \bigcap_i O_N^\theta(o, \gamma_i o) \right\}.$$

By Lemma 5.4, we have

$$\Lambda_\theta^{\mathsf{con}} = \bigcup_{N=1}^{\infty} \Lambda_\theta^N. \tag{5.3}$$

Definition 5.5. For a θ-discrete subgroup Γ, we say that Γ is *θ-regular* if for any sequence $\gamma_i \to \infty$ in Γ, we have

$$\min_{\alpha \in \theta} \alpha(\mu(\gamma_i)) \to \infty.$$

Observe that θ-regularity is same as $\theta \cup \mathrm{i}(\theta)$-regularity by (2.1) and that not every θ-discrete subgroup is θ-regular.

Proposition 5.6. *If Γ is θ-regular, then*

1. $\Lambda_\theta^{\mathsf{con}} \subset \Lambda_\theta$*;*
2. *for any compact subset $Q \subset G$, the union $\Gamma Q \cup \Lambda_\theta$ is compact where the topology is given by the convergence in Definition 2.2; that is, any infinite sequence has a limit.*

Proof. To show (1), let $\xi \in \Lambda_\theta^{\mathsf{con}}$. Then there exist $g \in G$, a sequence $\gamma_i \in \Gamma$, $m_i \in M_\theta$ and $a_i \in A^+$ such that $\xi = g\xi_\theta$ and $d(gm_i a_i o, \gamma_i o)$ is uniformly bounded. Since $\mu(\gamma_i) - \log a_i$ is uniformly bounded by Lemma 2.1, and $\min_{\alpha \in \theta} \alpha(\mu(\gamma_i)) \to \infty$ by the θ-regularity, we have $\min_{\alpha \in \theta} \alpha(\log a_i) \to \infty$ as $i \to \infty$. We may assume that m_i converges to some $m \in M_\theta$ by passing to a subsequence. Therefore as $i \to \infty$, $gm_i a_i o \to gm\xi_\theta = g\xi_\theta$ by Lemma 2.8. This implies that $\gamma_i o \to g\xi_\theta$ by Lemma 2.7. Hence $\xi \in \Lambda_\theta$. For (2), if $\gamma_i \in \Gamma$ is an infinite sequence and $q_i \in Q$, then $\min_{\alpha \in \theta} \alpha(\mu(\gamma_i q_i)) \to \infty$ by the θ-regularity of Γ and Lemma 2.1. Hence the claim is now immediate from Definition 2.2 and Lemma 2.7. $\square$

Conical convergence. From the viewpoint of Lemma 5.4, we define the conical convergence as follows.

Definition 5.7. We say that a sequence $g_i \in G$ converges to $\xi \in \mathcal{F}_\theta$ conically if $g_i \to \xi$ in the sense of Definition 2.2 and there exists $R > 0$ such that $\xi \in O_R^\theta(o, g_i o)$ for all $i \geq 1$. Note that if $\gamma_i \in \Gamma$ converges to $\xi \in \mathcal{F}_\theta$ conically, then $\xi \in \Lambda_\theta^{\mathsf{con}}$.

The following lemma is stated in [22, Lemma 5.29] in a different language. We give a more direct proof.

Lemma 5.8. *Let $g_i \in G$ be a sequence which converges to $\xi \in \mathcal{F}_\theta$. Then the following are equivalent:*

(1) *The convergence $g_i \to \xi$ is conical.*
(2) *For any $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ such that $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$, the sequence $g_i^{-1}(\xi, \eta)$ is precompact in $\mathcal{F}_\theta^{(2)}$.*
(3) *For some $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ such that $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$, the sequence $g_i^{-1}(\xi, \eta)$ is precompact in $\mathcal{F}_\theta^{(2)}$.*

Proof. The map $gL_\theta \to (gP_\theta, gw_0P_{\mathrm{i}(\theta)})$ is a G-equivariant homeomorphism from G/L_θ to $\mathcal{F}_\theta^{(2)}$. We first prove (1) $\Rightarrow$ (2). Suppose (1). Then there exist sequences $k_i \in K$ and $a_i \to \infty$ in A^+ such that $\xi = k_iP_\theta$ for all i and the sequence $g_i^{-1}k_ia_i$ is bounded. If $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$, then $\xi = hP_\theta$ and $\eta = hw_0P_{\mathrm{i}(\theta)}$ for some $h \in G$. Since $hP_\theta = k_iP_\theta$, $h = k_ip_im_i$ for some $p_i \in P$ and $m_i \in M_\theta$, by using $P_\theta = PM_\theta$. In other words, we have $k_i^{-1}hm_i^{-1} = p_i$ and hence p_i is a bounded sequence in P since k_i and m_i are bounded sequences. In particular, it follows from $a_i \in A^+$ that the sequence $a_i^{-1}p_ia_i$ is bounded by Lemma 2.3. Therefore the sequence $g_i^{-1}hL_\theta = g_i^{-1}k_ip_iL_\theta = (g_i^{-1}k_ia_i)(a_i^{-1}p_ia_i)L_\theta$ is precompact in G/L_θ, which is equivalent to saying that $g_i^{-1}(\xi, \eta)$ is precompact, proving (2). The implication (2) $\Rightarrow$ (3) is clear.

Now (3) $\Rightarrow$ (1) follows from Lemma 5.9 below applied to the constant sequence $(\xi_i, \eta_i) = (\xi, \eta)$. $\square$

Lemma 5.9. *Let $g_i \in G$ and $\xi_i \in \mathcal{F}_\theta$ be sequences both converging to some $\xi \in \mathcal{F}_\theta$. Suppose that there exists a sequence $\eta_i \in \mathcal{F}_{\mathrm{i}(\theta)}$ converging to some $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ such that $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$ and the sequence $g_i^{-1}(\xi_i, \eta_i)$ is precompact in $\mathcal{F}_\theta^{(2)}$. Then there exists $R > 0$ such that*

$$\xi_i \in O_R^\theta(o, g_io) \quad \textit{for all } i \geq 1.$$

Proof. Under the identification $G/L_\theta = \mathcal{F}_\theta^{(2)}$ given by $gL_\theta = (gP_\theta, gw_0P_{\mathrm{i}(\theta)})$, the hypothesis implies that there exists a sequence $h_i \in G$ with the limit $h \in G$ so that $(\xi_i, \eta_i) = h_iL_\theta$ for all $i \geq 1$ and $(\xi, \eta) = hL_\theta$. It follows from the precompactness of $g_i^{-1}(\xi_i, \eta_i)$ that there exists a sequence $\ell_i \in L_\theta$ such that $g_i^{-1}h_i\ell_i$ is a bounded sequence.

Since $L_\theta = M_\theta AM_\theta$, we can write $\ell_i = m_ia_i'm_i' \in M_\theta AM_\theta$, and hence we have $g_i^{-1}h_im_ia_i'$ is bounded. For each i, let $w_i \in K$ be a representative of a Weyl element such that $w_i^{-1}a_i'w_i \in A^+$. After passing to a subsequence, we may assume that the sequence m_i converges to some $m \in M_\theta$ and w_i is a constant sequence, say w. We claim that $w \in M_\theta$. Denoting by $a_i = w^{-1}a_i'w \in A^+$,

$$\text{the sequence } g_i^{-1}h_im_iwa_i \text{ is bounded.} \tag{5.4}$$

Moreover, since $\min_{\alpha\in\theta} \alpha(\mu(g_i)) \to \infty$, we have $\min_{\alpha\in\theta} \alpha(\log a_i) \to \infty$ as $i \to \infty$ by Lemma 2.1. Since $h_im_iwa_i = g_i(g_i^{-1}h_im_iwa_i)$, $g_i \to \xi$, and

$g_i^{-1}h_im_iwa_i$ is a bounded sequence by (5.4), we have as $i \to \infty$,

$$h_im_iwa_i \to \xi$$

by Lemma 2.7. On the other hand, by Lemma 2.8, we have that as $i \to \infty$,

$$h_im_iwa_i \to hmwP_\theta.$$

Hence we have $hmwP_\theta = \xi = hP_\theta$. Since $m \in M_\theta$, it follows that

$$w \in K \cap P_\theta = M_\theta.$$

In particular, $\xi_i = h_im_iwP_\theta$ for all i.

For each i, write $h_im_iw = k_ib_in_i \in KAN$ in the Iwasawa decomposition. We then have $\xi_i = h_im_iwP_\theta = k_iP_\theta$. Since the sequence h_im_iw is convergent and the product map $K \times A \times N \to G$ is a diffeomorphism, the sequences b_i and n_i are bounded. Since $a_i \in A^+$, the sequence $a_i^{-1}n_ia_i$ is bounded by Lemma 2.3, and so is the sequence $b_ia_i^{-1}n_ia_i$. On the other hand, (5.4) implies that

$$\text{the sequence } g_i^{-1}k_ib_in_ia_i = (g_i^{-1}k_ia_i)(b_ia_i^{-1}n_ia_i) \text{ is bounded.} \tag{5.5}$$

Therefore it follows that $g_i^{-1}k_ia_i$ is bounded. This mean that for some $R > 0$, $\xi_i = k_iP_\theta \in O_R^\theta(o, g_io)$ for all i, as desired. $\square$

Conformal measures. The $\mathfrak{a}$-valued Busemann map $\beta : \mathcal{F} \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi)$$

where $\sigma(g^{-1}, \xi) \in \mathfrak{a}$ is the unique element such that we have the Iwasawa decomposition $g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N$ for any $k \in K$ with $\xi = kP$. We define the $\mathfrak{a}_\theta$-valued Busemann map $\beta^\theta : \mathcal{F}_\theta \times G \times G \to \mathfrak{a}_\theta$ as follows: for $(\xi, g, h) \in \mathcal{F}_\theta \times G \times G$, we set

$$\beta_\xi^\theta(g, h) := p_\theta(\beta_{\xi_0}(g, h)) \quad \text{for } \xi_0 \in \pi_\theta^{-1}(\xi); \tag{5.6}$$

this is well-defined independent of the choice of ξ_0 [36, Lemma 6.1].

The following was shown for $\theta = \Pi$ in [29, Lemma 5.7] which directly implies the statement for general θ since p_θ is norm-decreasing.

Lemma 5.10. *There exists $\kappa > 0$ such that for any $g, h \in G$ and $R > 0$, we have*

$$\sup_{\xi \in O_R^\theta(go, ho)} \|\beta_\xi^\theta(g, h) - \mu_\theta(g^{-1}h)\| \le \kappa R.$$

Following the work of Patterson-Sullivan ([32], [42]) in rank one, Quint [36] has introduced the notion of conformal measures in general.

Definition 5.11 (Conformal measures)**.** For a linear form $\psi \in \mathfrak{a}_\theta^*$ and a closed subgroup $\Gamma < G$, a Borel probability measure ν on $\mathcal{F}_\theta$ is called a (Γ, ψ)-*conformal measure* if

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi^\theta(e, \gamma))} \quad \text{for all } \gamma \in \Gamma \text{ and } \xi \in \mathcal{F}_\theta.$$

Proposition 5.12. *Suppose that Γ is Zariski dense and θ-discrete. For any linear form $\psi \in \mathfrak{a}_\theta^*$ which is tangent to ψ_Γ^θ at an interior direction of $\mathfrak{a}_\theta^+$, there exists a (Γ, ψ)-conformal measure supported on Λ_θ.*

Proof. For $\theta = \Pi$, this was shown by Quint using the concavity of ψ_Γ [36, Theorem 8.4]. Now that we established the concavity of the θ-growth indicator ψ_Γ^θ (Proposition 3.10), the same proof works for general θ. $\square$

As in the Patterson-Sullivan construction, the conformal measure in the above proposition can be obtained as a limit of a sequence of certain weighted counting measures on Γo. The assumption that ψ is tangent to ψ_Γ^θ at an *interior* direction of $\mathfrak{a}_\theta^+$ is needed to guarantee that the limiting measure is supported on the limit set Λ_θ. For a θ-regular subgroup Γ, the union $\Gamma o \cup \Lambda_\theta$ is a compact space, and hence the assumption that the tangent direction belongs to $\operatorname{int} \mathfrak{a}_\theta^+$ is unnecessary. The proof below is an easy adaptation of the standard construction of Patterson-Sullivan (see also [24, Section 2], [41, Section 5], [9]).

Proposition 5.13. *Suppose that Γ is θ-regular. For any (Γ, θ)-proper $\psi \in \mathfrak{a}_\theta^*$ such that $\delta_\psi = 1$ and $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$, there exists a (Γ, ψ)-conformal measure supported on Λ_θ.*

Proof. By Proposition 5.6, $\Gamma o \cup \Lambda_\theta$ is a compact space. Recall that $\mathcal{P}_\psi(s) = \sum_{\gamma \in \Gamma} e^{-s\psi(\mu_\theta(\gamma))}$. As $\delta_\psi = 1$, $\mathcal{P}_\psi(s) < \infty$ for $s > 1$. and hence we may consider the probability measure on $\Gamma o \cup \Lambda_\theta$ given by

$$\nu_{\psi,s} := \frac{1}{\mathcal{P}_\psi(s)} \sum_{\gamma \in \Gamma} e^{-s\psi(\mu_\theta(\gamma))} D_{\gamma o} \tag{5.7}$$

where $D_{\gamma o}$ is the point mass at γo.

Since the space of probability measures on a compact metric space a weak* compact space, by passing to a subsequence, as $s \to 1$, $\nu_{\psi,s}$ weakly converges to a probability measure, say $\tilde{\nu}_\psi$, on $\Gamma o \cup \Lambda_\theta$. Since $\mathcal{P}_\psi(1) = \infty$, ν_ψ is supported on Λ_θ. It is standard to check that ν_ψ is a (Γ, ψ)-conformal measure. $\square$

Although we will not be using this generality, Proposition 5.13 holds without the hypothesis $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$ (see [9, Proposition 3.2]).

6. Transverse subgroups and multiplicity of θ-shadows

We say that a discrete subgroup $\Gamma < G$ is θ-*antipodal* if any two distinct points $\xi \neq \eta$ in $\Lambda_{\theta \cup \mathrm{i}(\theta)}$ are in general position, i.e.,

$$\xi = g P_{\theta \cup \mathrm{i}(\theta)} \quad \text{and} \quad \eta = g w_0 P_{\theta \cup \mathrm{i}(\theta)}$$

for some $g \in G$. Recall that a discrete subgroup $\Gamma < G$ is called θ-transverse if Γ is both θ-regular and θ-antipodal. A θ-transverse subgroup $\Gamma < G$ is called *non-elementary* if $\# \Lambda_\theta \geq 3$. Note that for $\theta_1 \subset \theta_2$, θ_2-transverse implies θ_1-transverse.

Remark 6.1. We may try to define Γ to be θ-*Antipodal* if for any $(\xi, \eta) \in \Lambda_\theta \times \Lambda_{\mathrm{i}(\theta)}$ such that $\pi_\theta^{-1}(\xi) \cap \pi_{\mathrm{i}(\theta)}^{-1}(\eta) = \emptyset$, (ξ, η) is in general position, i.e., $\xi = gP_\theta$ and $\eta = gw_0 P_{\mathrm{i}(\theta)}$ for some $g \in G$. While the θ-antipodality implies the θ-Antipodality, the converse direction is not true in general; for instance, any lattice of $\mathrm{PSL}_3(\mathbb{R})$ is $\{\alpha_1\}$-Antipodal but not $\{\alpha_1, \alpha_2\}$-Antipodal, i.e., not $\{\alpha_1\}$-antipodal, where $\alpha_i(\mathrm{diag}(u_1, u_2, u_3)) = u_i - u_{i+1}$ for $i = 1, 2$.

The main aim of this section is to prove the following proposition, which is the essential reason why the main results of this paper are proved for θ-transverse subgroups.

Proposition 6.2 (Bounded multiplicity of θ-shadows)**.** *Assume that Γ is θ-transverse. Let $\phi \in \mathfrak{a}_\theta^*$ be a (Γ, θ)-proper linear form. Then for any $R, D > 0$, there exists $q = q(\phi, R, D) > 0$ such that for any $T > 0$, the collection of shadows*

$$\left\{ O_R^\theta(o, \gamma o) \subset \mathcal{F}_\theta : T \le \phi(\mu_\theta(\gamma)) \le T + D \right\}$$

have multiplicity at most q, i.e., for and $\xi \in \mathcal{F}_\theta$ and $T > 0$, there are at most q number of shadows from the above collection that contain ξ.

The following lemma is a key ingredient of the proof of Proposition 6.2.

Lemma 6.3. *Assume that Γ is θ-transverse. For any compact subset Q of G, there exists $C_0 = C_0(Q) > 0$ such that if $\gamma_1, \gamma_2 \in \Gamma$ are such that $Q \cap \gamma_1 Q a^{-1} \cap \gamma_2 Q b^{-1} m^{-1} \neq \emptyset$ for some $a, b \in A^+$ and $m \in M_\theta$, then*

$$\min\{\|\mu_\theta(\gamma_2) - \mu_\theta(\gamma_1) - \mu_\theta(\gamma_1^{-1}\gamma_2)\|, \|\mu_\theta(\gamma_1) - \mu_\theta(\gamma_2) - \mu_\theta(\gamma_2^{-1}\gamma_1)\|\} \le C_0. \tag{6.1}$$

Proof. Since $\|p_\theta(u)\| \le \|p_{\theta \cup \mathrm{i}(\theta)}(u)\|$ for all $u \in \mathfrak{a}$, it suffices to prove the lemma for $\theta \cup \mathrm{i}(\theta)$ in place of θ. Therefore we may assume without loss of generality that $\mathrm{i}(\theta) = \theta$ by replacing θ with $\theta \cup \mathrm{i}(\theta)$.

We prove by contradiction. Suppose to the contrary that there exist sequences $q_{0,i}, q_{1,i}, q_{2,i} \in Q$, $a_i, b_i \in A^+$, $m_i \in M_\theta$ and $\gamma_{1,i}, \gamma_{2,i} \in \Gamma$ such that

$$q_{0,i} = \gamma_{1,i}\, q_{1,i}\, a_i^{-1} = \gamma_{2,i}\, q_{2,i} b_i^{-1} m_i^{-1}; \tag{6.2}$$

$$\|\mu_\theta(\gamma_{2,i}) - \mu_\theta(\gamma_{1,i}) - \mu_\theta(\gamma_{1,i}^{-1}\gamma_{2,i})\| \to \infty; \tag{6.3}$$

$$\|\mu_\theta(\gamma_{1,i}) - \mu_\theta(\gamma_{2,i}) - \mu_\theta(\gamma_{2,i}^{-1}\gamma_{1,i})\| \to \infty. \tag{6.4}$$

By Lemma 2.1, it follows that all sequences $\gamma_{1,i}$, $\gamma_{2,i}$, $\gamma_{1,i}^{-1}\gamma_{2,i}$ and $\gamma_{2,i}^{-1}\gamma_{1,i}$ are unbounded. Without loss of generality, we assume that each of these sequences tends to infinity. By (6.2) and Lemma 2.1, there exists $C' = C'(Q) > 1$ such that

$$\sup_i \{\|\mu_\theta(\gamma_{1,i}) - \mu_\theta(a_i)\|, \|\mu_\theta(\gamma_{2,i}) - \mu_\theta(b_i)\|\} \le C' \tag{6.5}$$

As Γ is θ-regular, as $i \to \infty$,

$$\min_{\alpha \in \theta} \alpha(\log a_i),\ \min_{\alpha \in \theta} \alpha(\log b_i) \to \infty.$$

Note that $\alpha(\log w_0^{-1}a^{-1}w_0) = \alpha(\mathrm{i}(\log a)) = \mathrm{i}(\alpha)(\log a)$ for all $a \in A$ and all $\alpha \in \Phi$. Since θ is symmetric, it follows that

$$\min_{\alpha\in\theta} \alpha(\log(w_0^{-1}a_i^{-1}w_0)),\ \min_{\alpha\in\theta} \alpha(\log(w_0^{-1}b_i^{-1}w_0)) \to \infty. \tag{6.6}$$

Passing to a subsequence, we may assume that $q_{1,i}$ converges to some $q_1 \in Q$. We claim that

$$q_1 w_0 \xi_\theta \in \Lambda_\theta \quad \text{and} \quad q_1 m_1 w \xi_\theta \in \Lambda_\theta \tag{6.7}$$

for some $m_1 \in M_\theta$ and $w \in N_K(A)$. By Lemma 5.6, we may also assume that $\gamma_{1,i}^{-1} q_{0,i} o$ converges to some $\xi \in \Lambda_\theta$ as $i \to \infty$. Since $\gamma_{1,i}^{-1} q_{0,i} o = q_{1,i} a_i^{-1} o = q_{1,i} w_0 (w_0^{-1} a_i^{-1} w_0) o$, it follows from Lemma 2.8 and (6.6) that $\xi = q_1 w_0 \xi_\theta$. Therefore

$$q_1 w_0 \xi_\theta \in \Lambda_\theta.$$

Since $A = A_\theta B_\theta$, we may write $a_i = a_{1,i} a_{2,i} \in A_\theta^+ B_\theta^+$ and $b_i = b_{1,i} b_{2,i} \in A_\theta^+ B_\theta^+$. Using $S_\theta = M_\theta B_\theta^+ M_\theta$, write

$$a_{2,i}^{-1} m_i b_{2,i} = m_{1,i} c_i m_{2,i} \in M_\theta B_\theta^+ M_\theta.$$

Then

$$\begin{aligned}\gamma_{1,i}^{-1}\gamma_{2,i}\, q_{2,i} &= q_{1,i}\, a_i^{-1} m_i b_i \\ &= q_{1,i}(a_{1,i}^{-1} b_{1,i})(a_{2,i}^{-1} m_i b_{2,i}) = q_{1,i} m_{1,i} (a_{1,i}^{-1} b_{1,i} c_i) m_{2,i}.\end{aligned}$$

By passing to a subsequence, we have $w \in N_K(A)$ such that for all $i \geq 1$,

$$d_i := w^{-1} a_{1,i}^{-1} b_{1,i} c_i w \in A^+. \tag{6.8}$$

Then we have the following:

$$\gamma_{1,i}^{-1}\gamma_{2,i} q_{2,i} = q_{1,i}(m_{1,i} w) d_i (w^{-1} m_{2,i}) \in q_{1,i} K A^+ K. \tag{6.9}$$

Since $\gamma_{1,i}^{-1}\gamma_{2,i} \to \infty$, by the θ-regularity of Γ, we have $\min_{\alpha\in\theta} \alpha(\log d_i) \to \infty$. We may assume that $m_{1,i} \to m_1 \in M_\theta$. By Lemma 5.6 and Lemma 2.8, we get

$$\lim_{i\to\infty} \gamma_{1,i}^{-1}\gamma_{2,i} q_{2,i} = q_1 m_1 w \xi_\theta \in \Lambda_\theta$$

by passing to a subsequence. Hence the claim (6.7) is proved.

By the θ-antipodal property of Γ, two distinct points of Λ_θ must be in general position; hence (6.7) implies that we must have either

$$w_0 \xi_\theta = m_1 w \xi_\theta \quad \text{or} \quad m_1 w \xi_\theta \in N_\theta^+ \xi_\theta.$$

First suppose that $(m_1 w)\xi_\theta \in N_\theta^+ \xi_\theta$. By Corollary 2.6, this implies that $w \in M_\theta$. As $a_{1,i}^{-1} b_{1,i} \in A_\theta$, using the commutativity of M_θ and A_θ, we get from (6.8) that $d_i = (a_{1,i}^{-1} b_{1,i})(w^{-1} c_i w)$. Since $d_i \in A^+$, $a_{1,i}^{-1} b_{1,i} \in A_\theta$, and $w^{-1} c_i w \in B_\theta$, it follows that $a_{1,i}^{-1} b_{1,i} \in A_\theta^+$. Hence

$$\mu_\theta(d_i) = \log a_{1,i}^{-1} b_{1,i} = -\log a_{1,i} + \log b_{1,i} = -\mu_\theta(a_i) + \mu_\theta(b_i). \tag{6.10}$$

Since $\|\mu_\theta(\gamma_{1,i}^{-1}\gamma_{2,i}) - \mu_\theta(d_i)\|$ is uniformly bounded by (6.9) and Lemma 2.1, (6.10) and (6.5) imply that the sequence $\|\mu_\theta(\gamma_{1,i}^{-1}\gamma_{2,i}) + \mu_\theta(\gamma_{1,i}) - \mu_\theta(\gamma_{2,i})\|$ is uniformly bounded. This contradicts (6.3).

Now suppose the other case that $w_0\xi_\theta = m_1 w\xi_\theta$. In this case, we have

$$w\xi_\theta = m_1^{-1}w_0\xi_\theta = w_0(w_0^{-1}m_1^{-1}w_0)\xi_\theta = w_0\xi_\theta$$

since $m_1 \in M_\theta$ and $w_0^{-1}M_\theta w_0 = M_\theta$ by the symmetricity of θ. Hence we have $w \in w_0(P_\theta \cap K) = w_0 M_\theta = M_\theta w_0$, and thus $ww_0^{-1} \in M_\theta$. Since $ww_0^{-1} \in M_\theta$, we may write using (6.8) that

$$\begin{aligned} w_0 d_i^{-1} w_0^{-1} &= (ww_0^{-1})^{-1} a_{1,i} b_{1,i}^{-1} c_i^{-1} (ww_0^{-1}) \\ &= (a_{1,i}b_{1,i}^{-1})((ww_0^{-1})^{-1} c_i^{-1} (ww_0^{-1})) \in A_\theta B_\theta \end{aligned}$$

Since $d_i \in A^+$, we have $w_0 d_i^{-1} w_0^{-1} \in A^+$. It follows that $a_{1,i}b_{1,i}^{-1} \in A_\theta^+$. Hence we have

$$\mu_\theta(d_i^{-1}) = p_\theta(\log(w_0 d_i^{-1} w_0^{-1})) = \log a_{1,i} - \log b_{1,i} = \mu_\theta(a_i) - \mu_\theta(b_i). \quad (6.11)$$

Similarly, (6.9) implies that the sequence $\|\mu_\theta(\gamma_{2,i}^{-1}\gamma_{1,i}) - \mu_\theta(d_i^{-1})\|$ is uniformly bounded. Hence it follows from (6.11) and (6.5) that the sequence $\|\mu_\theta(\gamma_{2,i}^{-1}\gamma_{1,i}) - \mu_\theta(\gamma_{1,i}) + \mu_\theta(\gamma_{2,i})\|$ is uniformly bounded, contradicting (6.4). This completes the proof. $\square$

Proof of Proposition 6.2. Suppose that there exists $\xi \in \bigcap_{i=1}^n O_R^\theta(o, \gamma_i o)$ and $T \leq \phi(\mu_\theta(\gamma_i)) \leq T + D$ for some distinct $\gamma_i \in \Gamma$, $i = 1, \cdots, n$. Setting $Q = KA_RK$ where $A_R := \{a \in A : d(o, ao) < R\}$, let $C_0 = C_0(Q)$ be as in Lemma 6.3. Note also that $Q = \{g \in G : d(o, go) < R\}$. Set

$$D' = D'(\phi, Q, D) := \|\phi\| C_0 + D$$

where $\|\phi\|$ is the operator norm of $\phi : \mathfrak{a}_\theta \to \mathbb{R}$. Then the following number

$$q := \#\{\gamma \in \Gamma : \phi(\mu_\theta(\gamma)) \leq D'\}$$

is finite by the (Γ, θ)-properness of ϕ. We claim that

$$n \leq 2q;$$

this proves the proposition. It suffices to show that

$$\max_i \min\{\phi(\mu_\theta(\gamma_1^{-1}\gamma_i)), \phi(\mu_\theta(\gamma_i^{-1}\gamma_1))\} \leq D', \quad (6.12)$$

as this implies that

$$n = \#\{\gamma_1, \cdots, \gamma_n\} \leq \#\{\gamma_1\gamma, \gamma_1\gamma^{-1} : \gamma \in \Gamma, \phi(\mu_\theta(\gamma)) \leq D'\} \leq 2q.$$

To prove (6.12), for each $i = 1, \cdots, n$, there exist $k_i \in K$ and $a_i \in A^+$ such that $\xi = k_i\xi_\theta$ and $d(k_i a_i o, \gamma_i o) < R$. Then $k_i = k_1 m_i$ for some $m_i \in K \cap P_\theta = M_\theta$. Hence we have $d(\gamma_1^{-1}k_1 a_1 o, o) < R$ and $d(\gamma_i^{-1}k_1 m_i a_i o, o) < R$, which implies

$$k_1 \in Q \cap \gamma_1 Q a_1^{-1} \cap \gamma_i Q a_i^{-1} m_i^{-1}.$$

By Lemma 6.3, we have

$$\|\mu_\theta(\gamma_i)-\mu_\theta(\gamma_1)-\mu_\theta(\gamma_1^{-1}\gamma_i)\| \le C_0 \quad \text{or} \quad \|\mu_\theta(\gamma_1)-\mu_\theta(\gamma_i)-\mu_\theta(\gamma_i^{-1}\gamma_1)\| \le C_0.$$

Suppose first that $\|\mu_\theta(\gamma_i) - \mu_\theta(\gamma_1) - \mu_\theta(\gamma_1^{-1}\gamma_i)\| \le C_0$. Now we have

$$\begin{aligned}\phi(\mu_\theta(\gamma_1^{-1}\gamma_i)) &= \phi(\mu_\theta(\gamma_1^{-1}\gamma_i) - (\mu_\theta(\gamma_i) - \mu_\theta(\gamma_1))) + \phi(\mu_\theta(\gamma_i) - \mu_\theta(\gamma_1)) \\ &\le \|\phi\|C_0 + |\phi(\mu_\theta(\gamma_i)) - \phi(\mu_\theta(\gamma_1))| \\ &\le \|\phi\|C_0 + D = D'\end{aligned}$$

where the last inequality follows from $\phi(\mu_\theta(\gamma_1)), \phi(\mu_\theta(\gamma_i)) \in [T, T+D]$. When $\|\mu_\theta(\gamma_1) - \mu_\theta(\gamma_i) - \mu_\theta(\gamma_i^{-1}\gamma_1)\| \le C_0$, similarly, we have

$$\phi(\mu_\theta(\gamma_i^{-1}\gamma_1)) \le \|\phi\|C_0 + D = D'.$$

Therefore (6.12) follows. □

7. Dimensions of conformal measures and growth indicators

For a general Zariski dense discrete subgroup $\Gamma < G$, Quint [36, Theorem 8.1] showed that if there exists a (Γ, ψ)-conformal measure on $\mathcal{F}_\Pi$ for $\psi \in \mathfrak{a}^*$, then

$$\psi \ge \psi_\Gamma.$$

The main aim of this section is to prove the following analogous inequality for θ-transverse subgroups, using Theorem 7.3 whose key ingredient is the control on multiplicity of shadows obtained in Proposition 6.2.

Theorem 7.1. *Let Γ be a Zariski dense θ-transverse subgroup of G. If there exists a (Γ, ψ)-conformal measure ν on $\mathcal{F}_\theta$ for a (Γ, θ)-proper $\psi \in \mathfrak{a}_\theta^*$, then*

$$\psi \ge \psi_\Gamma^\theta. \tag{7.1}$$

Moreover if $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$ in addition, then $\delta_\psi = 1$ and ψ is (Γ, θ)-critical.

Lemma 7.2 (θ-shadow lemma)**.** *Let $\Gamma < G$ be a discrete subgroup. Let ν be a (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$ for $\psi \in \mathfrak{a}_\theta^*$. Suppose that* $\operatorname{supp}\nu$ *is not contained in $\mathcal{F}_\theta - \ell N_\theta^+ P_\theta$ for any $\ell \in K$. Then we have the following:*

1. *for some $R = R(\nu) > 0$, we have $c := \inf_{\gamma\in\Gamma} \nu(O_R^\theta(\gamma o, o)) > 0$; and*
2. *for all $r \ge R$ and for all $\gamma \in \Gamma$,*
$$ce^{-\|\psi\|\kappa r} e^{-\psi(\mu_\theta(\gamma))} \le \nu(O_r^\theta(o, \gamma o)) \le e^{\|\psi\|\kappa r} e^{-\psi(\mu_\theta(\gamma))} \tag{7.2}$$
where $\kappa > 0$ is a constant given in Lemma 5.10.

In particular, if Γ is Zariski dense, (7.2) *holds for any (Γ, ψ)-conformal measure ν.*

Moreover, if Γ is a θ-transverse subgroup, then (7.2) *holds for any (Γ, ψ)-conformal measure ν on $\mathcal{F}_\theta$ such that*

$$(\operatorname{supp}\nu, \eta) \cap \mathcal{F}_\theta^{(2)} \ne \emptyset \quad \textit{for any } \eta \in \Lambda_{\mathrm{i}(\theta)}.$$

Proof. This lemma was proved in [29, Lemma 7.8] for $\theta = \Pi$, and a general case can be proved verbatim, by replacing P and N by P_θ and N_θ respectively and noting that the projection $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ is a Lipschitz map. We provide a proof for completeness. To prove (1), suppose not. Then there exist $R_i \to \infty$ and $\gamma_i \in \Gamma$ such that $\nu(O^\theta_{R_i}(\gamma_i^{-1}o, o)) < 1/i$ for all $i \geq 1$. We write the Cartan decomposition $\gamma_i = k_i' a_i \ell_i^{-1} \in KA^+K$ and after passing to a subsequence, we may assume that $k_i' \to k'$ and $\ell_i \to \ell$ as $i \to \infty$. We claim that $N_\theta^+ P_\theta \subset \limsup O^\theta_{R_i}(a_i^{-1}o, o)$. Let $h \in N_\theta^+$ and write $a_i h = k_i b_i n_i \in KAN$. Since $a_i h a_i^{-1}$ is bounded by Lemma 2.3 and $a_i h a_i^{-1} = k_i (b_i a_i^{-1})(a_i n_i a_i^{-1}) \in KAN$, it follows that both sequences $b_i a_i^{-1}$ and n_i are bounded. Hence for all large $i \geq 1$, $h n_i^{-1} b_i^{-1} a_i o \in B(o, R_i)$ and hence $hP_\theta = h n_i^{-1} b_i^{-1} P_\theta \in O^\theta_{R_i}(h n_i^{-1} b_i^{-1} o, o)$. Since $h n_i^{-1} b_i^{-1} = a_i^{-1} k_i$, we have $hP_\theta \in O^\theta_{R_i}(a_i^{-1}o, o)$, proving the claim.

Since $O^\theta_{R_i}(\gamma_i^{-1}o, o) = \ell_i O^\theta_{R_i}(a_i^{-1}o, o)$ and $\ell_i \to \ell$, it follows that $\nu(\ell N_\theta^+ P_\theta) = 0$. Since $\ell N_\theta^+ P_\theta$ is Zariski open in $\mathcal{F}_\theta$, it follows that $\operatorname{supp} \nu \cap \ell N_\theta^+ P_\theta = \emptyset$. This is a contradiction to the hypothesis. Hence this proves (1). To see (2), let $\gamma \in \Gamma$ and $r \geq R$. By Lemma 5.10, for all $\xi \in O^\theta_r(\gamma^{-1}o, o)$, we have $\|\beta^\theta_\xi(\gamma^{-1}o, o) - \mu_\theta(\gamma)\| \leq \kappa r$. Since $\nu(O^\theta_r(o, \gamma o)) = \int_{O^\theta_r(\gamma^{-1}o, o)} e^{-\psi(\beta^\theta_\xi(\gamma^{-1}o, o))} d\nu(\xi)$, (2) follows from (1).

If Γ is Zariski dense, then Λ_θ is Zariski dense in $\mathcal{F}_\theta$ and is contained in $\operatorname{supp} \nu$. Hence any Γ-conformal measure ν satisfies the hypothesis.

For the last claim in the statement, letting Γ be a θ-transverse subgroup and ν a (Γ, ψ)-conformal measure such that for any $\eta \in \Lambda_{\mathrm{i}(\theta)}$, $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$ for some $\xi \in \operatorname{supp} \nu$, it suffices to show that $\inf_{\gamma \in \Gamma} \nu(O^\theta_R(\gamma o, o)) > 0$. If not, there exist $R_i \to \infty$ and $\gamma_i \in \Gamma$ such that $\nu(O^\theta_{R_i}(\gamma_i^{-1}o, o)) < 1/i$ for all $i \geq 1$. Write the Cartan decomposition $\gamma_i = k_i' a_i \ell_i^{-1} \in KA^+K$ and assume $\ell_i \to \ell \in K$ as $i \to \infty$. By the same argument as above, we have $\operatorname{supp} \nu \cap \ell N_\theta^+ P_\theta = \emptyset$. By (2.4), this implies that every element of $\operatorname{supp} \nu$ is not in general position with $\ell w_0 P_{\mathrm{i}(\theta)}$. On the other hand, it follows from $\gamma_i^{-1} = \ell_i w_0 (w_0^{-1} a_i^{-1} w_0) w_0^{-1} k_i'^{-1}$ for all $i \geq 1$ that $\ell w_0 P_{\mathrm{i}(\theta)} = \lim_i \gamma_i^{-1} \in \Lambda_{\mathrm{i}(\theta)}$. By the hypothesis on $\operatorname{supp} \nu$, there exists an element of $\operatorname{supp} \nu$ in general position with $\ell w_0 P_{\mathrm{i}(\theta)}$. This contradicts $\operatorname{supp} \nu \cap \ell N_\theta^+ P_\theta = \emptyset$. This finishes the proof. $\square$

Theorem 7.3. *Let Γ be a Zariski dense θ-transverse subgroup of G. If there exists a (Γ, ψ)-conformal measure ν on $\mathcal{F}_\theta$ for a (Γ, θ)-proper $\psi \in \mathfrak{a}_\theta^*$, then*

$$\delta_\psi \leq 1.$$

Proof. For each $n \in \mathbb{Z}$, we set

$$\Gamma_{\psi, n} := \{\gamma \in \Gamma : n \leq \psi(\mu_\theta(\gamma)) < n + 1\}.$$

Since ψ is (Γ, θ)-proper, $\bigcup_{n<0} \Gamma_{\psi, n}$ is a finite subset, and hence can be ignored in the arguments below. Let ν be a (Γ, ψ)-conformal measure. We fix a sufficiently large $R > 0$ satisfying the conclusion of Lemma 7.2 for ν.

Since ψ is a (Γ, θ)-proper linear form, by Proposition 6.2, we have that for all $n \in \mathbb{N}$,

$$1 \gg \sum_{\gamma \in \Gamma_{\psi,n}} \nu(O_R^\theta(o, \gamma o)) \gg \sum_{\gamma \in \Gamma_{\psi,n}} e^{-\psi(\mu_\theta(\gamma))} \geq e^{-(n+1)} \# \Gamma_{\psi,n}$$

where the implied constants do not depend on n. It implies

$$\# \Gamma_{\psi,n} \ll e^{n+1} \quad \text{for each } n \geq 0.$$

Therefore, we have (cf. [35, Lemma 3.1.1])

$$\begin{aligned} \delta_\psi &\leq \limsup_{N \to \infty} \frac{\log \# \{\gamma \in \Gamma : \psi(\mu_\theta(\gamma)) < N\}}{N} \\ &\leq \limsup_{N \to \infty} \frac{1}{N} \log \sum_{0 \leq n < N} e^{n+1} = 1. \end{aligned} \tag{7.3}$$

Hence the claim follows. $\square$

Proof of Theorem 7.1. By Lemma 4.5 and Theorem 7.3, we have that $\delta_\psi \leq 1$ and $\delta_\psi \psi$ is tangent to ψ_Γ^θ, and therefore we have

$$\delta_\psi \psi \geq \psi_\Gamma^\theta.$$

Since ψ is (Γ, θ)-proper, $\psi \geq 0$ on $\mathcal{L}_\theta$ by Lemma 4.3 and hence $\psi \geq \delta_\psi \psi$ on $\mathcal{L}_\theta$. Therefore $\psi \geq \psi_\Gamma^\theta$ on $\mathcal{L}_\theta$. Since $\psi_\Gamma^\theta = -\infty$ outside of $\mathcal{L}_\theta$, $\psi \geq \psi_\Gamma^\theta$ on $\mathfrak{a}_\theta$. If $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$ in addition, then $\delta_\psi \geq 1$ and hence $\delta_\psi = 1$. In particular, $\psi = \delta_\psi \psi$ is tangent to ψ_Γ^θ. Therefore this finishes the proof. $\square$

8. Divergence of Poincaré series and conical sets

Let $\psi \in \mathfrak{a}_\theta^*$ and $\Gamma < G$ be discrete subgroup. Denote by $\mathsf{M}_\psi^\theta = \mathsf{M}_{\Gamma,\psi}^\theta$ the collection of all (Γ, ψ)-conformal (probability) measures on $\mathcal{F}_\theta$. Define the following subset of M_ψ^θ:

$$\mathsf{N}_{\Gamma,\psi}^\theta = \mathsf{N}_\psi^\theta := \left\{ \nu \in \mathsf{M}_\psi^\theta : \begin{array}{c} \text{either } \nu(\Lambda_\theta) = 1 \text{ and } \# \operatorname{supp} \nu \geq 2, \quad \text{or} \\ (\operatorname{supp} \nu|_{\mathcal{F}_\theta - \Lambda_\theta}, \eta) \cap \mathcal{F}_\theta^{(2)} \neq \emptyset \text{ for all } \eta \in \Lambda_{\mathrm{i}(\theta)} \end{array} \right\}.$$

The reason for this definition is to guarantee that for Γ θ-transverse, the shadow lemma (Lemma 7.2) holds for any $\nu \in \mathsf{N}_\psi^\theta$ as well as its restriction $\nu|_{\mathcal{F}_\theta - \Lambda_\theta}$ (if non-zero).

Lemma 8.1. (1) *If Γ is Zariski dense, then*

$$\mathsf{N}_\psi^\theta = \mathsf{M}_\psi^\theta.$$

(2) *If Γ is non-elementary θ-transverse, then*

$$\mathsf{N}_\psi^\theta \supset \{\nu \in \mathsf{M}_\psi^\theta : \nu(\Lambda_\theta) = 1\}.$$

Proof. Since $\#\Lambda_\theta \geq 3$ for a non-elementary θ-transverse subgroup Γ, (2) is straightforward. For (1), suppose that Γ is a Zariski dense discrete subgroup. Then for any $\nu \in \mathsf{M}_\psi^\theta$, $\operatorname{supp}\nu$ is a closed Γ-invariant set, and hence is Zariski dense in $\mathcal{F}_\theta$ (Lemma 5.2). Therefore if $\nu(\Lambda_\theta) = 1$, then $\nu \in \mathsf{N}_\psi^\theta$. Otherwise, we have $\nu(\mathcal{F}_\theta - \Lambda_\theta) > 0$, and $\operatorname{supp}\nu|_{\mathcal{F}_\theta - \Lambda_\theta}$ is a non-empty closed Γ-invariant set, and thus Zariski dense in $\mathcal{F}_\theta$. Given $\eta \in \Lambda_{\mathrm{i}(\theta)}$, the set $\{\xi \in \mathcal{F}_\theta : (\xi, \eta) \in \mathcal{F}_\theta^{(2)}\}$ is a Zariski open subset of $\mathcal{F}_\theta$ and hence $(\operatorname{supp}\nu|_{\mathcal{F}_\theta - \Lambda_\theta}, \eta) \cap \mathcal{F}_\theta^{(2)} \neq \emptyset$, finishing the proof. □

The main goal of this section is to prove the following theorem and discuss its applications. Note that we do not assume that ψ is (Γ, θ)-proper in the following theorem.

Theorem 8.2. *Let Γ be any θ-transverse subgroup (which may be elementary). Then the following are equivalent:*

1. $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$ *(resp.* $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} < \infty$*)*
2. $\nu(\Lambda_\theta^{\mathsf{con}}) = 1$ *for all* $\nu \in \mathsf{N}_\psi^\theta$ *(resp.* $\nu(\Lambda_\theta^{\mathsf{con}}) = 0$ *for all* $\nu \in \mathsf{N}_\psi^\theta$*).*

In the rest of this section, suppose that Γ is θ-transverse. We make the following simple observation:

Lemma 8.3. *Suppose that $\nu(\Lambda_\theta^{\mathsf{con}}) > 0$ for all $\nu \in \mathsf{N}_\psi^\theta$. Then*

$$\nu(\Lambda_\theta^{\mathsf{con}}) = 1 \quad \text{for all } \nu \in \mathsf{N}_\psi^\theta.$$

Proof. Suppose that for some $\nu \in \mathsf{N}_\psi^\theta$, we have $0 < \nu(\Lambda_\theta^{\mathsf{con}}) < 1$. Then $\nu' := \frac{1}{\nu(\mathcal{F}_\theta - \Lambda_\theta^{\mathsf{con}})} \nu|_{\mathcal{F}_\theta - \Lambda_\theta^{\mathsf{con}}}$ belongs to M_ψ^θ. We now show that $\nu' \in \mathsf{N}_\psi^\theta$. There are two cases:

1. If ν satisfies that $(\operatorname{supp}\nu|_{\mathcal{F}_\theta - \Lambda_\theta}, \eta) \cap \mathcal{F}_\theta^{(2)} \neq \emptyset$ for all $\eta \in \Lambda_{\mathrm{i}(\theta)}$, then the same holds for ν', so $\nu' \in \mathsf{N}_\psi^\theta$.
2. Otherwise, $\nu(\Lambda_\theta) = 1$ and $\# \operatorname{supp}\nu \geq 2$. We consider the following two subcases:
   - If Γ is non-elementary, then $\operatorname{supp}\nu' = \Lambda_\theta$. Since $\#\Lambda_\theta = \infty$ in this case, we have $\nu \in \mathsf{N}_\psi^\theta$.
   - If Γ is elementary, then $\#\Lambda_\theta \leq 2$. Since $\nu(\Lambda_\theta) = 1$ and $\#\operatorname{supp}\nu \geq 2$, we have $\operatorname{supp}\nu = \Lambda_\theta$ and $\#\Lambda_\theta = 2$. Since $0 < \nu(\Lambda_\theta^{\mathsf{con}}) < 1$, we must have that $\#\Lambda_\theta^{\mathsf{con}} = 1$. Since both Λ_θ and $\Lambda_\theta^{\mathsf{con}}$ are Γ-invariant, this implies that each point of Λ_θ is Γ-invariant. Now for $\xi \in \Lambda_\theta - \Lambda_\theta^{\mathsf{con}}$, let $\gamma_i \in \Gamma$ be a sequence that converges to ξ. For $\eta \in \Lambda_{\mathrm{i}(\theta)}$ such that $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$, we have $\gamma_i^{-1}(\xi, \eta) = (\xi, \eta) \in \mathcal{F}_\theta^{(2)}$, and therefore the convergence $\gamma_i \to \xi$ is conical by Lemma 5.8. This contradicts $\xi \notin \Lambda_\theta^{\mathsf{con}}$.

Therefore, in any case, we have $\nu' \in \mathsf{N}_\psi^\theta$. On the other hand, $\nu'(\Lambda_\theta^{\mathsf{con}}) = 0$, contradicting the hypothesis. This finishes the proof. □

We will use the following:

Lemma 8.4 (Kochen-Stone Lemma [27]). *Let (Z,ν) be a finite measure space. If $\{\mathsf{A}_n\}$ is a sequence of measurable subsets of Z such that*

$$\sum_{n=1}^{\infty} \nu(\mathsf{A}_n) = \infty \quad \textit{and} \quad \liminf_{N\to\infty} \frac{\sum_{m=1}^{N}\sum_{n=1}^{N} \nu(\mathsf{A}_n \cap \mathsf{A}_m)}{\left(\sum_{n=1}^{N} \nu(\mathsf{A}_n)\right)^2} < \infty, \tag{8.1}$$

then $\nu(\limsup_n \mathsf{A}_n) > 0$.

Proof of Theorem 8.2. Suppose that $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$. By Lemma 8.3, it suffices to show that $\nu(\Lambda_\theta^{\mathsf{con}}) > 0$ for all $\nu \in \mathsf{N}_\psi^\theta$. Let $\nu \in \mathsf{N}_\psi^\theta$. Since Γ is θ-transverse, it follows from the definition of N_ψ^θ that ν satisfies Lemma 7.2.

We fix $\alpha \in \theta$. Since Γ is θ-regular, $\alpha \in \theta$ is (Γ, θ)-proper; in particular, $\alpha(\mu_\theta(\Gamma))$ is a discrete closed subset of $[0, \infty)$. Therefore we may enumerate $\Gamma = \{\gamma_1, \gamma_2, \cdots\}$ so that $\alpha(\mu_\theta(\gamma_n)) \le \alpha(\mu_\theta(\gamma_{n+1}))$ for all $n \in \mathbb{N}$. Fix a sufficiently large R which satisfies the conclusion of Lemma 7.2. Setting $\mathsf{A}_n := O_R^\theta(o, \gamma_n o)$, we then have

$$\sum_{n=1}^{\infty} \nu(\mathsf{A}_n) \gg \sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$$

where the implied constant depends only on R. Since $\limsup_n \mathsf{A}_n \subset \Lambda_\theta^{\mathsf{con}}$, by Lemma 8.4, it suffices to show that

$$\liminf_{N\to\infty} \frac{\sum_{m=1}^{N}\sum_{n=1}^{N} \nu(\mathsf{A}_n \cap \mathsf{A}_m)}{\left(\sum_{n=1}^{N} \nu(\mathsf{A}_n)\right)^2} < \infty. \tag{8.2}$$

Set $Q := KA_R^+K$ where $A_R^+ = \{a \in A^+ : \|\log a\| \le R\}$ and $C_0 = C_0(Q)$ be as in Lemma 6.3. Note that $Q = \{g \in G : d(o, go) \le R\}$. Define

$$T_N := \max\{n \in \mathbb{N} : \alpha(\mu_\theta(\gamma_n)) \le \alpha(\mu_\theta(\gamma_N)) + \|\alpha\| C_0\}$$

for each $N \ge 1$. Clearly, $N \le T_N$. Unless mentioned otherwise, all implied constants in this proof are independent of N. Since Γ is θ-regular, $\alpha|_{\mathfrak{a}_\theta}$ is (Γ, θ)-proper. Proposition 6.2 implies that the collection A_{n}, $N \le n \le T_N$, has multiplicity at most $q = q(\alpha, R, \|\alpha\| C_0)$, and hence

$$\sum_{N \le n \le T_N} \nu(\mathsf{A}_{\mathsf{n}}) \le q \cdot \nu(\mathcal{F}_\theta).$$

Therefore by Lemma 7.2, we have that for all $N \ge 1$,

$$\left| \sum_{n=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_n))} - \sum_{n=1}^{N} e^{-\psi(\mu_\theta(\gamma_n))} \right| \ll \sum_{n=N+1}^{T_N} \nu(\mathsf{A}_{\mathsf{n}})$$

$$\ll \nu(\mathcal{F}_\theta) = e^{\psi(\mu_\theta(\gamma_1))} e^{-\psi(\mu_\theta(\gamma_1))} \le e^{\psi(\mu_\theta(\gamma_1))} \sum_{n=1}^{N} e^{-\psi(\mu_\theta(\gamma_n))}$$

with all implied constants independent of N. Therefore we have:

$$\sum_{n=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_n))} \ll \sum_{n=1}^{N} e^{-\psi(\mu_\theta(\gamma_n))}. \tag{8.3}$$

Fix $N \in \mathbb{N}$. If $\mathsf{A}_n \cap \mathsf{A}_m \neq \emptyset$ for some $n, m \leq N$, then there exist $k \in K$ and $m_\theta \in M_\theta$ such that $d(kA^+o, \gamma_n o) < R$ and $d(km_\theta A^+ o, \gamma_m o) < R$. Since $K \subset Q$, it follows that

$$Q \cap \gamma_n Q a_n^{-1} \cap \gamma_m Q a_m^{-1} m_\theta^{-1} \neq \emptyset$$

for some $a_n, a_m \in A^+$. Hence, setting

$$\begin{aligned} E_1 &= \{(n,m) : n, m \leq N \text{ and } \|\mu_\theta(\gamma_n) - (\mu_\theta(\gamma_m) + \mu_\theta(\gamma_m^{-1}\gamma_n))\| \leq C_0\}, \\ E_2 &= \{(n,m) : n, m \leq N \text{ and } \|\mu_\theta(\gamma_m) - (\mu_\theta(\gamma_n) + \mu_\theta(\gamma_n^{-1}\gamma_m))\| \leq C_0\}, \end{aligned}$$

we get from Lemma 6.3 that

$$\sum_{n,m \leq N} \nu(\mathsf{A}_n \cap \mathsf{A}_m) \leq \sum_{(n,m) \in E_1} \nu(\mathsf{A}_n) + \sum_{(n,m) \in E_2} \nu(\mathsf{A}_m). \tag{8.4}$$

For all $(n, m) \in E_1$, we have

$$\begin{aligned} \alpha(\mu_\theta(\gamma_m^{-1}\gamma_n)) &\leq \alpha(\mu_\theta(\gamma_m) + \mu_\theta(\gamma_m^{-1}\gamma_n)) \\ &= \alpha(\mu_\theta(\gamma_m) + \mu_\theta(\gamma_m^{-1}\gamma_n) - \mu_\theta(\gamma_n)) + \alpha(\mu_\theta(\gamma_n)) \\ &\leq \|\alpha\| C_0 + \alpha(\mu_\theta(\gamma_n)). \end{aligned} \tag{8.5}$$

Therefore, by Lemma 7.2,

$$\begin{aligned} \sum_{(n,m) \in E_1} \nu(\mathsf{A}_n) &\ll \sum_{(n,m) \in E_1} e^{-\psi(\mu_\theta(\gamma_n))} \\ &\ll \sum_{(n,m) \in E_1} e^{-\psi(\mu_\theta(\gamma_m))} e^{-\psi(\mu_\theta(\gamma_m^{-1}\gamma_n))} \\ &\leq \sum_{m=1}^{N} \sum_{j=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_m))} e^{-\psi(\mu_\theta(\gamma_j))}; \end{aligned} \tag{8.6}$$

the last inequality follows because, for each fixed $1 \leq m \leq N$, the correspondence $n \leftrightarrow \gamma_m^{-1}\gamma_n$ is one-to-one and when $(n,m) \in E_1$, $\gamma_j = \gamma_m^{-1}\gamma_n$ for some $j \leq T_n \leq T_N$ by (8.5). Similarly, we have

$$\sum_{(n,m) \in E_2} \nu(\mathsf{A}_m) \ll \sum_{n=1}^{N} \sum_{j=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_n))} e^{-\psi(\mu_\theta(\gamma_j))}.$$

By (8.4), we have

$$\begin{aligned}\sum_{n,m\le N}\nu(\mathsf{A}_n\cap\mathsf{A}_m) &\ll \sum_{n=1}^{N}\sum_{j=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_n))}e^{-\psi(\mu_\theta(\gamma_j))}\\ &= \left(\sum_{n=1}^{N} e^{-\psi(\mu_\theta(\gamma_n))}\right)\left(\sum_{n=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_n))}\right)\\ &\ll \left(\sum_{n=1}^{N} e^{-\psi(\mu_\theta(\gamma_n))}\right)^2 \ll \left(\sum_{n=1}^{N}\nu(\mathsf{A}_n)\right)^2\end{aligned}$$

where we have applied (8.3) for the second last inequality and Lemma 7.2 for the last inequality. Hence (8.2) is verified, completing the proof of the first statement.

We now suppose that $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))}<\infty$. Consider the following increasing sequence

$$\Lambda_\theta^N=\left\{\xi\in\mathcal{F}_\theta : \begin{array}{c}\exists \text{ infinite sequence } \gamma_n\in\Gamma \text{ s.t.}\\ \xi\in\bigcap_{n\ge1}O_N^\theta(o,\gamma_n o)\end{array}\right\},\quad N\ge 1.$$

Since $\Lambda_\theta^{\mathsf{con}}=\bigcup_N\Lambda_\theta^N$, it suffices to show $\nu(\Lambda_\theta^N)=0$ for all sufficiently large $N\ge 1$. Since

$$\Lambda_\theta^N\subset\bigcup_{\gamma\in\Gamma,\|\mu_\theta(\gamma)\|>t} O_N^\theta(o,\gamma o)$$

for any $t>0$, we get from Lemma 7.2 that for all $t>0$,

$$\nu(\Lambda_\theta^N)\ll\sum_{\gamma\in\Gamma,\|\mu_\theta(\gamma)\|>t} e^{-\psi(\mu_\theta(\gamma))}$$

where the implied constant depends only on N. Since $\sum_{\gamma\in\Gamma}e^{-\psi(\mu_\theta(\gamma))}<\infty$ implies that $\lim_{t\to\infty}\sum_{\gamma\in\Gamma,\|\mu_\theta(\gamma)\|>t}e^{-\psi(\mu_\theta(\gamma))}=0$, we have $\nu(\Lambda_\theta^N)=0$, finishing the proof. □

Comparing with ψ_Γ. Quint showed that for a Zariski dense discrete subgroup $\Gamma<G$, the existence of a (Γ,ψ)-conformal measure on $\mathcal{F}_\theta$ for $\psi\in\mathfrak{a}_\theta^*$ implies the inequality

$$\psi\circ p_\theta+2\rho_{\Pi-\theta}\ge\psi_\Gamma\quad\text{on } \mathfrak{a},\tag{8.7}$$

where $2\rho_{\Pi-\theta}$ is the sum of all positive roots which can be written as $\mathbb{Z}$-linear combinations of elements of $\Pi-\theta$ (counted with multiplicity) [36, Theorem 8.1]. For θ-transverse subgroups, Theorem 1.4 and (1.2) imply that the term $2\rho_{\Pi-\theta}$ turns out to be redundant:

Corollary 8.5. *Let $\Gamma<G$ be a Zariski dense θ-transverse subgroup and $\psi\in\mathfrak{a}_\theta^*$ be (Γ,θ)-proper. If there exists a (Γ,ψ)-conformal measure ν on $\mathcal{F}_\theta$, then*

$$\psi\circ p_\theta\ge\psi_\Gamma\quad\text{on } \mathfrak{a}.\tag{8.8}$$

Moreover, if $\nu(\Lambda_\theta^{\mathsf{con}})>0$, then $\psi\circ p_\theta$ is tangent to ψ_Γ.

Proof. The first statement follows from Theorem 7.1 and Lemma 3.13. For the second claim, if $\nu(\Lambda_\theta^{\mathsf{con}}) > 0$, then we have $\sum_{\gamma\in\Gamma} e^{-(\psi\circ p_\theta)(\mu(\gamma))} = \infty$ by Theorem 8.2. If $\psi \circ p_\theta$ were strictly bigger than ψ_Γ, then by [35, Lemma 3.1.3] we would have $\sum_{\gamma\in\Gamma} e^{-(\psi\circ p_\theta)(\mu(\gamma))} < \infty$. Therefore $\psi \circ p_\theta$ must be tangent to ψ_Γ. ☐

9. Properly discontinuous actions of Γ

Recall $\mathcal{F}_\theta^{(2)} = \{(\xi, \eta) \in \mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)} : \xi, \eta \text{ are in general position}\}$ and consider the action of G on the space $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ defined as

$$g.(\xi, \eta, u) = (g\xi, g\eta, u + \beta_\xi^\theta(g^{-1}, e)) \tag{9.1}$$

for all $g \in G$ and $(\xi, \eta, u) \in \mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$. A discrete subgroup $\Gamma < G$ preserves the subspace $\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ where

$$\Lambda_\theta^{(2)} = (\Lambda_\theta \times \Lambda_{\mathrm{i}(\theta)}) \cap \mathcal{F}_\theta^{(2)}.$$

When $\theta = \Pi$, the Hopf parametrization of G/M gives a G-equivariant homeomorphism between $\mathcal{F}^{(2)} \times \mathfrak{a}$ and G/M, and hence any discrete subgroup $\Gamma < G$ acts properly discontinuously on $\mathcal{F}^{(2)}\times\mathfrak{a}$ and hence the quotient space $\Gamma\backslash\Lambda_\Pi^{(2)} \times \mathfrak{a}$ is a locally compact Hausdorff space. For a general θ, this is not the case. The aim of this section is to establish the following two theorems on properly discontinuous actions of θ-transverse subgroups.

Theorem 9.1. *If Γ is a non-elementary θ-transverse subgroup, the Γ-action on $\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ is properly discontinuous and hence*

$$\Omega_\theta := \Gamma\backslash\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$$

is a locally compact Hausdorff space.

For a (Γ, θ)-proper form $\varphi \in \mathfrak{a}_\theta^*$, consider the Γ-action

$$\gamma.(\xi, \eta, s) = (\gamma\xi, \gamma\eta, s + \varphi(\beta_\xi^\theta(\gamma^{-1}, e))) \tag{9.2}$$

for all $\gamma \in \Gamma$ and $(\xi, \eta, s) \in \Lambda_\theta^{(2)} \times \mathbb{R}$.

Theorem 9.2. *Let Γ be a non-elementary θ-transverse subgroup of G and $\varphi \in \mathfrak{a}_\theta^*$ a (Γ, θ)-proper form. Then the action Γ on $\Lambda_\theta^{(2)} \times \mathbb{R}$ given by* (9.2) *is properly discontinuous and hence*

$$\Omega_\varphi := \Gamma\backslash\Lambda_\theta^{(2)} \times \mathbb{R}$$

is a locally compact Hausdorff space. Moreover, Ω_φ is compact if and only if Γ is θ-Anosov.

Definition 9.3. Let Z be a compact metrizable space with at least 3 points. An action of a countable group Γ on Z by homeomorphisms is called a *convergence group action* if for any sequence of distinct elements $\gamma_n \in \Gamma$, there exist a subsequence γ_{n_k} and $a, b \in Z$ such that as $k \to \infty$, $\gamma_{n_k}(z)$ converges to a for all $z \in Z - \{b\}$, uniformly on compact subsets.

We will use the following property of a θ-transverse subgroup:

Proposition 9.4. [22, Theorem 4.16] *For a θ-transverse subgroup Γ, the action of Γ on Λ_θ is a convergence group action.*

It is also proved in [22] that Λ_θ is same as the limit set as the convergence group action; this also follows from Lemma 2.5. In particular, if Γ is non-elementary, then the Γ-action on Λ_θ is minimal.

The following observation is useful to transfer statements from θ symmetric to general θ.

Lemma 9.5. *Suppose that Γ is θ-antipodal. For any $\theta_1 \subset \theta_2 \subset \theta \cup \mathrm{i}(\theta)$, the projection map $p : \Lambda_{\theta_2} \to \Lambda_{\theta_1}$ given by $gP_{\theta_2} \to gP_{\theta_1}$ is a Γ-equivariant homeomorphism. In particular, for any (Γ, ψ)-conformal measure ν supported on Λ_{θ_1} for $\psi \in \mathfrak{a}^*_{\theta_1} \subset \mathfrak{a}^*_{\theta_2}$, the pull back $p^*\nu$ is a (Γ, ψ)-conformal measure on Λ_{θ_2}.*

Proof. It suffices to show that p is injective when $\theta_2 = \theta \cup \mathrm{i}(\theta)$. Suppose that $\xi \neq \eta \in \Lambda_{\theta\cup \mathrm{i}(\theta)}$. By the θ-antipodality of Γ, $\xi = gP_{\theta\cup\mathrm{i}(\theta)}$ and $\eta = gw_0P_{\theta\cup\mathrm{i}(\theta)}$ for some $g \in G$. Then $p(\xi) = gP_{\theta_1}$ and $p(\eta) = gw_0P_{\theta_1}$, and hence $p(\xi) \neq p(\eta)$, showing that p is injective. ☐

The following observation will be useful:

Lemma 9.6. *Let Γ be a non-elementary θ-transverse subgroup and $\gamma_i \in \Gamma$ an infinite sequence. Let $(\xi_i, \eta_i) \in \Lambda^{(2)}_\theta$ be a convergent sequence in $\Lambda^{(2)}_\theta$. If the sequence $\gamma_i(\xi_i, \eta_i)$ converges in $\Lambda^{(2)}_\theta$, then there exists $R > 0$ so that either*

$$\xi_i \in O^\theta_R(o, \gamma_i^{-1}o) \quad \text{for all } i \geq 1; \text{ or}$$

$$\eta_i \in O^{\mathrm{i}(\theta)}_R(o, \gamma_i^{-1}o) \quad \text{for all } i \geq 1.$$

In particular, if the sequence $\gamma_i(\xi, \eta) \in \Lambda^{(2)}_\theta$ converges in $\Lambda^{(2)}_\theta$ for some $(\xi, \eta) \in \Lambda^{(2)}_\theta$, then γ_i^{-1} converges conically either to ξ or η.

Proof. Set $(\xi, \eta) = \lim_i(\xi_i, \eta_i) \in \Lambda^{(2)}_\theta$ and $(\xi_0, \eta_0) = \lim_i \gamma_i(\xi_i, \eta_i) \in \Lambda^{(2)}_\theta$. Since the projections $\Lambda_{\theta\cup\mathrm{i}(\theta)} \to \Lambda_\theta$ and $\Lambda_{\theta\cup\mathrm{i}(\theta)} \to \Lambda_{\mathrm{i}(\theta)}$ are Γ-equivariant homeomorphisms by Lemma 9.5, we also let $\xi', \xi'_0, \xi'_i \in \Lambda_{\theta\cup\mathrm{i}(\theta)}$ be the preimages of ξ, ξ_0, and ξ_i for all $i \geq 1$ under the projection $\Lambda_{\theta\cup\mathrm{i}(\theta)} \to \Lambda_\theta$ respectively, and similarly $\eta', \eta'_0, \eta'_i \in \Lambda_{\theta\cup\mathrm{i}(\theta)}$ the preimages of η, η_0, and η_i. Note that $\xi' \neq \eta'$, $\xi'_0 \neq \eta'_0$, and $\xi'_i \neq \eta'_i$ for all $i \geq 1$ and $\xi'_i \to \xi'$, $\eta'_i \to \eta'$, $\gamma_i\xi'_i \to \xi'_0$, and $\gamma_i\eta'_i \to \eta'_0$ as $i \to \infty$.

Since the action of Γ on $\Lambda_{\theta\cup\mathrm{i}(\theta)}$ is a convergence group action by Proposition 9.4, there exist $a, b \in \Lambda_{\theta\cup\mathrm{i}(\theta)}$ such that

$$\gamma_i|_{\Lambda_{\theta\cup\mathrm{i}(\theta)} - \{b\}} \to a \tag{9.3}$$

uniformly on compact subsets, after passing to a subsequence. That is, for any compact subsets $C_a \subset \Lambda_{\theta\cup\mathrm{i}(\theta)} - \{a\}$ and $C_b \subset \Lambda_{\theta\cup\mathrm{i}(\theta)} - \{b\}$,

$$\#\{\gamma_i : \gamma_i C_b \cap C_a \neq \emptyset\} < \infty,$$

or equivalently $\#\{\gamma_i^{-1} : \gamma_i^{-1} C_a \cap C_b \neq \emptyset\} < \infty$. Therefore we have, as $i \to \infty$,

$$\gamma_i^{-1}|_{\Lambda_{\theta \cup \mathrm{i}(\theta)} - \{a\}} \to b \tag{9.4}$$

uniformly on compact subsets.

We claim that

$$(a, b) = (\eta_0', \xi') \quad \text{or} \quad (a, b) = (\xi_0', \eta'). \tag{9.5}$$

Suppose $\xi' \neq b$. Excluding finitely many elements from $\{\xi_i' : i \geq 1\}$, we may assume that $\{\xi_i' : i \geq 1\} \cup \{\xi'\}$ is a compact subset of $\Lambda_{\theta \cup \mathrm{i}(\theta)} - \{b\}$. Hence (9.3) implies that $\xi_0' = \lim_i \gamma_i \xi_i' = a$. If η' were not equal to b, then we may also assume that $\{\eta_i' : i \geq 1\} \cup \{\eta'\}$ is a compact subset of $\Lambda_{\theta \cup \mathrm{i}(\theta)} - \{b\}$ and hence (9.3) implies $\eta_0' = \lim_i \gamma_i \eta_i' = a$. Since $\xi_0' \neq \eta_0'$, this is a contradiction. This implies $\eta' = b$. Now suppose that $\xi' = b$. Since $\eta' \neq \xi' = b$, we have $\eta_0' = \lim_i \gamma_i \eta_i' = a$ by the above argument. This proves the claim.

Now (9.4) and (9.5) imply that

$$\gamma_i^{-1}|_{\Lambda_{\theta \cup \mathrm{i}(\theta)} - \{\eta_0'\}} \to \xi' \quad \text{or} \quad \gamma_i^{-1}|_{\Lambda_{\theta \cup \mathrm{i}(\theta)} - \{\xi_0'\}} \to \eta' \tag{9.6}$$

uniformly on compact subsets.

Since Γ is $\theta \cup \mathrm{i}(\theta)$-regular, we may assume that by passing to a subsequence, the sequence γ_i^{-1} converges to some point, say, $z = \lim_i \gamma_i^{-1}$, in $\Lambda_{\theta \cup \mathrm{i}(\theta)}$ in the sense of Definition 2.2. We claim that z is either ξ' or η'. Write $\gamma_i^{-1} = k_i b_i \ell_i^{-1} \in KA^+K$ using the Cartan decomposition. By passing to a subsequence, we may assume that $k_i \to k_0 \in K$ and $\ell_i \to \ell_0 \in K$. Choose $x \in \Lambda_{\theta \cup \mathrm{i}(\theta)} - \{\eta_0', \xi_0', \}$ in general position with $\ell_0 w_0 P_{\theta \cup \mathrm{i}(\theta)} = \lim_i \gamma_i$, which is possible by the θ-antipodality and non-elementary hypothesis of Γ. Since Γ is $\theta \cup \mathrm{i}(\theta)$-regular, by Lemma 2.1, we have $\min_{\alpha \in \theta \cup \mathrm{i}(\theta)} \alpha(\log b_i) \to \infty$. Hence, by Lemma 2.5, we have

$$\gamma_i^{-1} x \to z = k_0 P_{\theta \cup \mathrm{i}(\theta)}.$$

Therefore, it follows from (9.6) that $z = \xi'$ or η'.

If $\lim_i \gamma_i^{-1} = \xi'$, then by Lemma 5.9, there exists $R_1 > 0$ such that $\xi_i' \in O_{R_1}^{\theta \cup \mathrm{i}(\theta)}(o, \gamma_i^{-1} o)$ for all $i \geq 1$. Otherwise, if $\lim_i \gamma_i^{-1} = \eta'$, then we apply Lemma 5.9 to the sequence (η_i', ξ_i') to obtain $R_2 > 0$ such that $\eta_i' \in O_{R_2}^{\theta \cup \mathrm{i}(\theta)}(o, \gamma_i^{-1} o)$ for all $i \geq 1$. Setting $R := \max(R_1, R_2)$ and taking the projections $\Lambda_{\theta \cup \mathrm{i}(\theta)} \to \Lambda_\theta$ and $\Lambda_{\theta \cup \mathrm{i}(\theta)} \to \Lambda_{\mathrm{i}(\theta)}$, we have either

$$\begin{aligned} \xi_i &\in O_R^\theta(o, \gamma_i^{-1} o) && \text{for all } i \geq 1; \text{ or} \\ \eta_i &\in O_R^{\mathrm{i}(\theta)}(o, \gamma_i^{-1} o) && \text{for all } i \geq 1, \end{aligned}$$

completing the proof. $\square$

Proposition 9.7. *Let Γ be a non-elementary θ-transverse subgroup and $\varphi \in \mathfrak{a}_\theta^*$ a (Γ, θ)-proper form. Let $\gamma_i \in \Gamma$ be an infinite sequence and $(\xi_i, \eta_i) \in \Lambda_\theta^{(2)}$ a convergent sequence in $\Lambda_\theta^{(2)}$. If the sequence $\gamma_i(\xi_i, \eta_i) \in \Lambda_\theta^{(2)}$*

converges in $\Lambda_\theta^{(2)}$*, then the sequence* $\varphi(\beta^\theta_{\xi_i}(\gamma_i^{-1}, e))$ *is unbounded. In particular,* $\beta^\theta_{\xi_i}(\gamma_i^{-1}, e)$ *is unbounded.*

Proof. By Lemma 9.6, there exists $R > 0$ so that either

$$\begin{aligned} \xi_i &\in O_R^\theta(o, \gamma_i^{-1}o) \quad \text{for all } i \geq 1; \text{ or} \\ \eta_i &\in O_R^{\mathrm{i}(\theta)}(o, \gamma_i^{-1}o) \quad \text{for all } i \geq 1. \end{aligned}$$

We consider these two cases separately.

Case A. Suppose that $\xi_i \in O_R^\theta(o, \gamma_i^{-1}o)$ for all $i \geq 1$. By Lemma 5.10, we have

$$\sup_i \|\beta^\theta_{\xi_i}(e, \gamma_i^{-1}) - \mu_\theta(\gamma_i^{-1})\| < \infty$$

and hence

$$\sup_i |\varphi(\beta^\theta_{\xi_i}(e, \gamma_i^{-1}) - \mu_\theta(\gamma_i^{-1}))| < \infty.$$

The θ-regularity of Γ implies $\mu_\theta(\gamma_i^{-1}) \to \infty$ as $i \to \infty$. Since φ is (Γ, θ)-proper, we have $\varphi(\mu_\theta(\gamma_i^{-1})) \to \infty$. Therefore

$$\varphi(\beta^\theta_{\xi_i}(\gamma_i^{-1}, e)) = -\varphi(\beta^\theta_{\xi_i}(e, \gamma_i^{-1})) \to -\infty,$$

as desired.

Case B. Now suppose that $\eta_i \in O_R^{\mathrm{i}(\theta)}(o, \gamma_i^{-1}o)$ for all $i \geq 1$. Then there exist a sequence $k_i \in K$ and a sequence $a_i \to \infty$ in A^+ such that $\eta_i = k_i P_{\mathrm{i}(\theta)}$ for all $i \geq 1$ and the sequence $\gamma_i k_i a_i$ is bounded. By the hypothesis that the sequence (ξ_i, η_i) converges in $\Lambda_\theta^{(2)}$, there exists a bounded sequence $h_i \in G$ such that $(\xi_i, \eta_i) = h_i L_\theta$, which means that $\xi_i = h_i P_\theta$ and $\eta_i = h_i w_0 P_{\mathrm{i}(\theta)}$. Since $\eta_i = h_i w_0 P_{\mathrm{i}(\theta)} = k_i P_{\mathrm{i}(\theta)}$ for each i, we have $h_i w_0 m_i' p_i = k_i$ for some $m_i' \in M_{\mathrm{i}(\theta)}$ and $p_i \in P$, using $P_{\mathrm{i}(\theta)} = M_{\mathrm{i}(\theta)} P$. Since the sequences h_i, k_i, and m_i' are bounded, the sequence $p_i \in P$ is bounded as well. This implies that the sequence $a_i^{-1} p_i a_i$ is bounded since $a_i \in A^+$ by Lemma 2.3. Hence it follows from the boundedness of the sequence $\gamma_i k_i a_i = \gamma_i h_i w_0 m_i' p_i a_i = \gamma_i h_i w_0 m_i' a_i (a_i^{-1} p_i a_i)$ that

$$\text{the sequence } g_i := \gamma_i h_i w_0 m_i' a_i \text{ is bounded.}$$

For each i, set $m_i = w_0 m_i' w_0^{-1} \in M_\theta$. Then

$$\eta_i = h_i w_0 P_{\mathrm{i}(\theta)} = h_i w_0 m_i' P_{\mathrm{i}(\theta)} = h_i m_i w_0 P_{\mathrm{i}(\theta)}, \quad \xi_i = h_i P_\theta = h_i m_i P_\theta$$

and

$$g_i = \gamma_i h_i w_0 m_i' a_i = \gamma_i h_i m_i w_0 a_i.$$

Using $\xi_i = h_i m_i P_\theta$, we have

$$\begin{aligned} \beta^\theta_{\xi_i}(\gamma_i^{-1}, e) &= \beta^\theta_{\gamma_i\xi_i}(e, \gamma_i) = \beta^\theta_{\gamma_i\xi_i}(e, g_i) + \beta^\theta_{\gamma_i\xi_i}(g_i, \gamma_i) \\ &= \beta^\theta_{\gamma_i\xi_i}(e, g_i) + \beta^\theta_{\xi_i}(h_i m_i w_0 a_i, e) \\ &= \beta^\theta_{\gamma_i\xi_i}(e, g_i) + \beta^\theta_{P_\theta}(w_0 a_i, e) + \beta^\theta_{P_\theta}(e, m_i^{-1} h_i^{-1}). \end{aligned}$$

Since g_i and $m_i^{-1}h_i^{-1}$ are bounded sequences, the sequences $\beta^\theta_{\gamma_i\xi_i}(e, g_i)$ and $\beta^\theta_{P_\theta}(e, m_i^{-1}h_i^{-1})$ are bounded by [29, Lemma 5.1].

Hence it suffices to show that as $i \to \infty$,

$$\varphi(\beta^\theta_{P_\theta}(w_0 a_i, e)) \to \infty. \tag{9.7}$$

Note that $\beta^\theta_{P_\theta}(w_0 a_i, e) = p_\theta(\beta_P(w_0 a_i, e))$ and

$$\beta_P(w_0 a_i, e) = \beta_P(w_0 a_i w_0^{-1}, e) = \mathrm{i}(\log a_i).$$

Since the sequences $g_i = \gamma_i h_i m_i w_0 a_i$ and $h_i m_i$ are bounded and $\gamma_i^{-1} g_i = h_i m_i w_0 a_i$, we have $\|\mu(\gamma_i^{-1}) - \log a_i\| = \|\mu(\gamma_i) - \mathrm{i}(\log a_i)\|$ is uniformly bounded by Lemma 2.1 and the identity (2.1). Therefore

$$\sup_i |\varphi(\mu_\theta(\gamma_i) - (p_\theta \circ \mathrm{i})(\log a_i))| < \infty.$$

It follows from the θ-regularity of Γ and the (Γ, θ)-properness of φ that $\varphi(\mu_\theta(\gamma_i)) \to \infty$ as $i \to \infty$, and hence $\varphi((p_\theta \circ \mathrm{i})(\log a_i)) \to \infty$, implying (9.7). Therefore, we have $\varphi(\beta^\theta_{\xi_i}(\gamma_i^{-1}, e)) \to \infty$. This finishes the proof. $\square$

Recall the definition of a θ-Anosov subgroup given in the introduction. Anosov subgroups are word hyperbolic. The notion of a θ-conical set in [22] is equal to the one we use here for θ-Anosov subgroups, by the Morse property of θ-Anosov subgroups obtained in [22].

Theorem 9.8. [22, Theorem 1.1] *For a θ-transverse subgroup Γ, Γ is θ-Anosov if and only if $\Lambda_\theta = \Lambda_\theta^{\mathsf{con}}$.*

Proof of Theorems 9.1 and 9.2. Suppose to the contrary that the Γ-action on $\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ is not properly discontinuous. Then there exists a compact subset $Q \subset \Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ such that $\gamma_i Q \cap Q \neq \emptyset$ for an infinite sequence $\gamma_i \in \Gamma$. In particular, there exists a sequence $(\xi_i, \eta_i, u_i) \in Q$ such that $\gamma_i(\xi_i, \eta_i, u_i) \in Q$ for all $i \geq 1$. By passing to a subsequence, we may assume that the sequences (ξ_i, η_i, u_i) and $\gamma_i(\xi_i, \eta_i, u_i)$ converge in $Q \subset \Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$. On the other hand,

$$\gamma_i(\xi_i, \eta_i, u_i) = (\gamma_i \xi_i, \gamma_i \eta_i, u_i + \beta^\theta_{\xi_i}(\gamma_i^{-1}, e)) \quad \text{for all } i \geq 1$$

which cannot converge by Proposition 9.7, yielding a contradiction. Hence Theorem 9.1 follows.

The first part of Theorem 9.2 follows from Proposition 9.7 as well. Now suppose that Ω_φ is compact. Fix a sequence $s_i \to +\infty$ and let $\xi \in \Lambda_\theta$. Choose any $\eta \in \Lambda_{\mathrm{i}(\theta)}$ so that $(\xi, \eta) \in \Lambda_\theta^{(2)}$. Then there exists a sequence $\gamma_i \in \Gamma$ such that the sequence $\gamma_i(\xi, \eta, s_i) = (\gamma_i \xi, \gamma_i \eta, s_i + \varphi(\beta^\theta_\xi(\gamma_i^{-1}, e)))$ converges by passing to a subsequence. Hence the sequence $\gamma_i(\xi, \eta)$ is convergent in $\Lambda_\theta^{(2)}$ and $\varphi(\beta^\theta_\xi(\gamma_i^{-1}, e)) \to -\infty$ as $i \to \infty$. By Lemma 9.6, the sequence γ_i^{-1} converges to ξ or η conically as $i \to \infty$. If $\gamma_i^{-1} \to \eta$ conically, then as in the Case B of the proof of Proposition 9.7, we must have $\varphi(\beta^\theta_\xi(\gamma_i^{-1}, e)) \to +\infty$, which is impossible. Therefore, $\gamma_i^{-1} \to \xi$ conically as $i \to \infty$, and hence

$\xi \in \Lambda_\theta^{\mathsf{con}}$. Since ξ is arbitrary, we have $\Lambda_\theta = \Lambda_\theta^{\mathsf{con}}$. By Theorem 9.8, Γ is θ-Anosov.

Suppose that Γ is Anosov. By [10, Theorem 10.1], we have $\varphi > 0$ on $\mathcal{L}_\theta - \{0\}$. Hence it is a consequence of the Hölder reparametrization theorem for Anosov subgroups ([5, Proposition 4.1], [11, Theorem 4.15]) that Ω_φ is compact (see also [11, Theorem 3.5]). This finishes the proof. $\square$

10. Bowen-Margulis-Sullivan measures on Ω_θ and Ω_φ

Let $\Gamma < G$ be a non-elementary θ-transverse subgroup and $\psi \in \mathfrak{a}_\theta^*$. As ψ can be considered as a linear form on $\mathfrak{a}$ which is p_θ-invariant and hence $\psi \circ \mathrm{i}$ is a linear form on $\mathfrak{a}$ which is $p_{\mathrm{i}(\theta)}$-invariant, we have $\psi \circ \mathrm{i} \in \mathfrak{a}_{\mathrm{i}(\theta)}^*$. For a pair of a (Γ, ψ)-conformal measure ν on Λ_θ and a $(\Gamma, \psi \circ \mathrm{i})$-conformal measure ν_{i} on $\Lambda_{\mathrm{i}(\theta)}$, we define a Radon measure $d\tilde{\mathsf{m}}_{\nu,\nu_{\mathrm{i}}}$ on $\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ as follows:

$$d\tilde{\mathsf{m}}_{\nu,\nu_{\mathrm{i}}}(\xi, \eta, u) = e^{\psi\left(\beta_\xi^\theta(e,g) + \mathrm{i}(\beta_\eta^{\mathrm{i}(\theta)}(e,g))\right)} d\nu(\xi) d\nu_{\mathrm{i}}(\eta) du \tag{10.1}$$

where $g \in G$ is chosen so that $(\xi, \eta) = (gP_\theta, gw_0P_{\mathrm{i}(\theta)})$ and du is the Lebesgue measure on $\mathfrak{a}_\theta$. This definition is independent of the choice of g by Lemma 10.1 below. The measure $d\tilde{\mathsf{m}}_{\nu,\nu_{\mathrm{i}}}$ is left Γ-invariant and right A_θ-invariant. We denote by

$$\mathsf{m}_{\nu,\nu_{\mathrm{i}}} \tag{10.2}$$

the A_θ-invariant Borel measure on Ω_θ induced by $\tilde{\mathsf{m}}_{\nu,\nu_{\mathrm{i}}}$, which we call the Bowen-Margulis-Sullivan measure associated to the pair (ν, ν_{i}).

Lemma 10.1. *If $g, g' \in G$ satisfy $(\xi, \eta) = (gP_\theta, gw_0P_{\mathrm{i}(\theta)}) = (g'P_\theta, g'w_0P_{\mathrm{i}(\theta)})$, then*

$$\beta_\xi^\theta(e,g) + \mathrm{i}(\beta_\eta^{\mathrm{i}(\theta)}(e,g)) = \beta_\xi^\theta(e,g') + \mathrm{i}(\beta_\eta^{\mathrm{i}(\theta)}(e,g')).$$

Proof. The hypothesis on g and g' means that $g' = gh$ for some $h \in L_\theta$.

Since

$$\begin{aligned}&\beta_\xi^\theta(e,g') + \mathrm{i}(\beta_\eta^{\mathrm{i}(\theta)}(e,g')) \\ &= (\beta_\xi^\theta(e,g) + \mathrm{i}(\beta_\eta^{\mathrm{i}(\theta)}(e,g))) + (\beta_{P_\theta}^\theta(e,h) + \mathrm{i}(\beta_{w_0P_{\mathrm{i}(\theta)}}^{\mathrm{i}(\theta)}(e,h)))\end{aligned}$$

it suffices to prove that

$$\beta_{P_\theta}^\theta(e,h) + \mathrm{i}(\beta_{w_0P_{\mathrm{i}(\theta)}}^{\mathrm{i}(\theta)}(e,h)) = 0.$$

Write $h = as$ where $a \in A_\theta$ and $s \in S_\theta$. Since $p_\theta(\log(A \cap S_\theta)) = 0$ and

$$\beta_{P_\theta}(e,s) + \mathrm{i}(\beta_{w_0P_{\mathrm{i}(\theta)}}(e,s)) \in \log(A \cap S_\theta),$$

we have

$$\beta_{P_\theta}^\theta(e,h) + \mathrm{i}(\beta_{w_0P_{\mathrm{i}(\theta)}}^{\mathrm{i}(\theta)}(e,h)) = \beta_{P_\theta}^\theta(e,a) + \mathrm{i}(\beta_{w_0P_{\mathrm{i}(\theta)}}^{\mathrm{i}(\theta)}(e,a)).$$

On the other hand, by the definition of the Busemann map, $\beta_P(e,a) = \log a$ and $\beta_{w_0P}(e,a) = \beta_P(e, w_0aw_0^{-1}) = \mathrm{Ad}_{w_0}(\log a) = -\mathrm{i}(\log a)$. Hence

$$\beta_P(e,a) + \mathrm{i}(\beta_{w_0P}(e,a)) = \log a - \mathrm{i}^2(\log a) = 0,$$

finishing the proof. $\square$

For a (Γ,θ)-proper form φ, consider the Γ-equivariant projection $\Lambda_\theta^{(2)}\times\mathfrak{a}_\theta\to\Lambda_\theta^{(2)}\times\mathbb{R}$ given by $(\xi,\eta,u)\to(\xi,\eta,\varphi(u))$. By Theorem 9.2, this induces an affine bundle with fiber $\ker\varphi$:

$$\Omega_\theta\to\Omega_\varphi; \tag{10.3}$$

it is a standard fact that such a bundle is indeed a trivial vector bundle and hence we have a homeomorphism

$$\Omega_\theta\simeq\Omega_\varphi\times\ker\varphi\simeq\Omega_\varphi\times\mathbb{R}^{\#\theta-1}. \tag{10.4}$$

We denote by the push-forward of the measure $\mathsf{m}_{\nu,\nu_{\mathrm{i}}}$ on Ω_φ by $\mathsf{m}^\varphi_{\nu,\nu_{\mathrm{i}}}$ which is an $\mathbb{R}$-invariant Radon measure on Ω_φ. Then

$$\mathsf{m}_{\nu,\nu_{\mathrm{i}}}=\mathsf{m}^\varphi_{\nu,\nu_{\mathrm{i}}}\otimes\mathrm{Leb}_{\ker\varphi}. \tag{10.5}$$

By the following proposition, the measures $\mathsf{m}_{\nu,\nu_{\mathrm{i}}}$ and $\mathsf{m}^\varphi_{\nu,\nu_{\mathrm{i}}}$ are non-zero.

Proposition 10.2. *Let $\Gamma<G$ be a discrete subgroup and let λ and λ_{i} be probability measures on $\mathcal{F}_\theta$ and $\mathcal{F}_{\mathrm{i}(\theta)}$ respectively. Suppose one of the following:*

1. *Γ is Zariski dense and λ is Γ-quasi-invariant.*
2. *Γ is non-elementary θ-transverse, λ is Γ-quasi-invariant, and λ and λ_{i} are supported on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively.*

Then

$$(\lambda\times\lambda_{\mathrm{i}})(\mathcal{F}_\theta^{(2)})>0. \tag{10.6}$$

Proof. Suppose $(\lambda\times\lambda_{\mathrm{i}})(\mathcal{F}_\theta^{(2)})=0$. Then by Fubini's theorem,

$$\lambda\left(\{\xi\in\mathcal{F}_\theta:(\xi,\eta)\in\mathcal{F}_\theta^{(2)}\}\right)=0\quad\text{for }\lambda_{\mathrm{i}}\text{-a.e. }\eta\in\mathcal{F}_{\mathrm{i}(\theta)}. \tag{10.7}$$

We now deduce a contradiction in each case. In the case of (1), let $\eta\in\mathcal{F}_{\mathrm{i}(\theta)}$. Since $\operatorname{supp}\nu\subset\mathcal{F}_\theta$ must be Zariski dense in this case by Lemma 5.2 and $\{\xi\in\mathcal{F}_\theta:(\xi,\eta)\in\mathcal{F}_\theta^{(2)}\}$ is a non-empty Zariski open subset, we have $\lambda\left(\{\xi\in\mathcal{F}_\theta:(\xi,\eta)\in\mathcal{F}_\theta^{(2)}\}\right)>0$, contradicting (10.7).

In the case (2), let $\eta\in\Lambda_{\mathrm{i}(\theta)}$. Since Γ is θ-transverse, there exists $\xi_0\in\Lambda_\theta$ such that $\Lambda_\theta-\{\xi_0\}\subset\{\xi\in\mathcal{F}_\theta:(\xi,\eta)\in\mathcal{F}_\theta^{(2)}\}$. Hence it suffices to note that $\lambda(\Lambda_\theta-\{\xi_0\})>0$. If not, λ is supported on $\{\xi_0\}$, which must be fixed by Γ due to the quasi-invariance of λ.

Since the Γ-action on Λ_θ is minimal (Proposition 9.4), $\Lambda_\theta=\{\xi_0\}$, contradicting the non-elementary hypothesis on Γ. $\square$

11. Conservativity and ergodicity of the $\mathfrak{a}_\theta$-action

In this section, we expand the dichotomies in Theorem 8.2 to a criterion on conservativity and ergodicity of $\mathfrak{a}_\theta$-action on the quotient space $\Omega_\theta=\Gamma\backslash\Lambda_\theta^{(2)}\times\mathfrak{a}_\theta$, or equivalently a criterion on conservativity and ergodicity of

$\mathbb{R}$-action on the quotient space $\Omega_\varphi = \Gamma\backslash\Lambda_\theta^{(2)} \times \mathbb{R}$, when Γ is a non-elementary θ-transverse subgroup and φ is a (Γ, θ)-proper linear form. First of all, this makes sense thanks to Theorems 9.1 and 9.2.

We recall the notion of complete conservativity and ergodicity. Let H be a locally compact unimodular group. We denote by dh the Haar measure on H. Consider the dynamical system (H, Ω, λ) where Ω is a separable, locally compact and σ-compact topological space on which H acts continuously and λ is a Radon measure which is quasi-invariant by H. A Borel subset $B \subset \Omega$ is called wandering if $\int_H \mathbb{1}_B(h.w)dh < \infty$ for μ-almost all $w \in B$. The Hopf decomposition theorem says that Ω can be written as the disjoint union $\Omega_C \cup \Omega_D$ of H-invariant subsets where Ω_D is a countable union of wandering subsets which is maximal in the sense that Ω_C does not contain any wandering subset of positive measure. If $\lambda(\Omega_D) = 0$, the system is called completely conservative. If $\lambda(\Omega_C) = 0$, the system is called completely dissipative. The dynamical system (H, Ω, λ) is ergodic if any H-invariant λ-measurable subset is either null or co-null. An ergodic system (H, Ω, λ) is either completely conservative or completely dissipative. If (H, Ω, λ) is ergodic, H is countable and λ is atomless, then it is completely conservative [20, Theorem 14]. The following is standard [30, Lemma 6.1]:

Lemma 11.1. *Suppose that λ is H-invariant. Then (H, Ω, λ) is completely conservative if and only if for λ-a.e. $x \in \Omega$, there exists a compact subset $B_x \subset \Omega$ such that $\int_{h\in H} \mathbb{1}_{B_x}(h.x)\, dh = \infty$.*

The following theorem implies Theorem 1.9 in the introduction. For a non-elementary θ-transverse subgroup $\Gamma < G$ and $\psi \in \mathfrak{a}_\theta^*$, we denote by

$$\mathcal{M}_\psi^\theta \subset \mathsf{M}_\psi^\theta$$

the space of all (Γ, ψ)-conformal measures *supported on* Λ_θ.

Theorem 11.2. *Let $\Gamma < G$ be a non-elementary θ-transverse subgroup. Let $\psi \in \mathfrak{a}_\theta^*$ be (Γ, θ)-proper such that $\mathcal{M}_\psi^\theta \neq \emptyset$. Then the following are equivalent to each other.*

1. $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$ *(resp.* $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} < \infty$*);*
2. *For any $\nu \in \mathcal{M}_\psi^\theta$, $\nu(\Lambda_\theta^{\mathsf{con}}) > 0$ (resp. $\nu(\Lambda_\theta^{\mathsf{con}}) = 0$);*
3. *For any $\nu \in \mathcal{M}_\psi^\theta$, $\nu(\Lambda_\theta^{\mathsf{con}}) = 1$ (resp. $\nu(\Lambda_\theta^{\mathsf{con}}) = 0$);*
4. *For any $(\nu, \nu_{\mathrm{i}}) \in \mathcal{M}_\psi^\theta \times \mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$, the Γ-action on $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ is completely conservative and ergodic (resp. completely dissipative and non-ergodic);*
5. *For any $(\nu, \nu_{\mathrm{i}}) \in \mathcal{M}_\psi^\theta \times \mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$, the $\mathfrak{a}_\theta$-action on $(\Omega_\theta, \mathsf{m}_{\nu,\nu_{\mathrm{i}}})$ is completely conservative and ergodic (resp. completely dissipative and non-ergodic);*
6. *For any $(\nu, \nu_{\mathrm{i}}) \in \mathcal{M}_\psi^\theta \times \mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$ and any (Γ, θ)-proper $\varphi \in \mathfrak{a}_\theta^*$, the $\mathbb{R}$-action on $(\Omega_\varphi, \mathsf{m}_{\nu,\nu_{\mathrm{i}}}^\varphi)$ is completely conservative and ergodic (resp. completely dissipative and non-ergodic).*

In the proof of Theorem 11.2, we will use the following observation.

Lemma 11.3. *Let $\Gamma < G$ be a non-elementary θ-transverse subgroup. Let λ and λ_{i} be Γ-quasi-invariant probability measures on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively. If the Γ-action on $(\Lambda_\theta^{(2)}, \lambda \times \lambda_{\mathrm{i}})$ is ergodic, then $\lambda \times \lambda_{\mathrm{i}}$ has no atom in $\Lambda_\theta^{(2)}$.*

Proof. By Proposition 10.2, we have $(\lambda \times \lambda_{\mathrm{i}})(\Lambda_\theta^{(2)}) > 0$. Suppose that $\lambda \times \lambda_{\mathrm{i}}$ has an atom, say $(\xi_0, \eta_0) \in \Lambda_\theta^{(2)}$. By the ergodicity hypothesis, $\lambda \times \lambda_{\mathrm{i}}$ is supported on a single Γ-orbit $\Gamma(\xi_0, \eta_0) \subset \Lambda_\theta^{(2)}$. Since $\lambda(\xi_0) > 0$ and $\lambda_{\mathrm{i}}(\eta_0) > 0$, we have

$$(\Gamma\xi_0 \times \Gamma\eta_0) \cap \Lambda_\theta^{(2)} \subset \Gamma(\xi_0, \eta_0).$$

Since Γ is θ-antipodal,

$$\Gamma\xi_0 \subset \Gamma_{\eta_0}\xi_0 \cup \{\eta_0'\}$$

where Γ_{η_0} is the stabilizer of η_0 in Γ and η_0' is the image of η_0 under the Γ-equivariant homeomorphism $\Lambda_{\mathrm{i}(\theta)} \to \Lambda_\theta$ obtained in Lemma 9.5. In addition, the Γ-equivariance of $\Lambda_{\mathrm{i}(\theta)} \to \Lambda_\theta$ implies that $\Gamma_{\eta_0} = \Gamma_{\eta_0'}$ and hence

$$\Gamma\xi_0 \subset \Gamma_{\eta_0'}\xi_0 \cup \{\eta_0'\}. \tag{11.1}$$

Since the Γ-action on Λ_θ is a convergence group action (Proposition 9.4), Λ_θ is perfect and equal to the set of all accumulation points of $\Gamma\xi_0$. On the other hand, $\Gamma_{\eta_0'}$ is an elementary subgroup and hence $\Gamma_{\eta_0'}\xi_0$ has at most two accumulation points in Λ_θ ([45], [4]). Therefore, we obtain a contradiction. $\square$

Proof of Theorem 11.2. Note that $\mathfrak{a}_\theta^*$ can be regarded as a subspace of $\mathfrak{a}_{\theta\cup\mathrm{i}(\theta)}^*$ and that $\psi \in \mathfrak{a}_\theta^*$ is (Γ, θ)-proper if and only if $\psi \circ \mathrm{i}$ is $(\Gamma, \mathrm{i}(\theta))$-proper. By Lemma 9.5, we have Γ-equivariant homeomorphisms $\Lambda_\theta \to \Lambda_{\theta\cup\mathrm{i}(\theta)} \to \Lambda_{\mathrm{i}(\theta)}$ and hence we can push-forward measures in $\mathcal{M}_\psi^\theta$ to $\mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$. In particular, $\mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)} \neq \emptyset$. Note that since Γ is non-elementary θ-transverse, the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows from Lemma 8.1 and Theorem 8.2.

The divergent case. We will show (3) $\Rightarrow$ (5) $\Rightarrow$ (4), (3) $\Rightarrow$ (6) $\Rightarrow$ (4), and (4) $\Rightarrow$ (1), which will then finish the proof of this case.

In order to show (3) $\Rightarrow$ (5), assume (3). Consider a pair $(\nu, \nu_{\mathrm{i}}) \in \mathcal{M}_\psi^\theta \times \mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$. Then for ν-a.e. $\xi \in \Lambda_\theta$, ξ belongs to $\Lambda_\theta^{\mathsf{con}}$, that is, there exist $g \in G$ and sequences $\gamma_i \in \Gamma$, $m_i \in M_\theta$ and $a_i \in A^+$ such that $\xi = gP_\theta$, the sequence $\gamma_i g m_i a_i$ is bounded, and the sequence γ_i is infinite. By the θ-regularity of Γ and Lemma 2.1, we have $\min_{\alpha\in\theta} \alpha(\log a_i) \to \infty$ as $i \to \infty$. For any $\eta \in \Lambda_{\mathrm{i}(\theta)}$ such that $(\xi, \eta) \in \Lambda_\theta^{(2)}$ and any $u \in \mathfrak{a}_\theta$, there exists $n \in N_\theta$ and $a \in A_\theta$ such that $gnaS_\theta \in G/S_\theta$ represents $(\xi, \eta, u) \in \Lambda_\theta^{(2)} \times \mathfrak{a}_\theta \subset G/S_\theta$. Since $a_i \in A^+$, the sequence

$$\gamma_i g n a m_i a_i = (\gamma_i g m_i a_i)(a_i^{-1} m_i^{-1} n m_i a_i a) \quad \text{is bounded.}$$

This implies that writing $u_i = p_\theta(\log a_i) \in \mathfrak{a}_\theta^+$,

$$\gamma_i(\xi, \eta, u + u_i) \in \Lambda_\theta^{(2)} \times \mathfrak{a}_\theta \quad \text{is precompact.}$$

Moreover, since $\alpha(\log a_i) \to \infty$ for all $\alpha \in \theta$, we also have $u_i \to \infty$ in $\mathfrak{a}_\theta$. Projecting to Ω_θ, this implies that there exists a compact subset $Q \subset \Omega_\theta$ so that

$$\int_{v \in \mathfrak{a}_\theta} \mathbb{1}_Q(\Gamma(\xi, \eta, u + v)) dv = \infty.$$

Since this holds for ν-a.e. $\xi \in \Lambda_\theta$, any $\eta \in \Lambda_{\mathrm{i}(\theta)}$ with $(\xi, \eta) \in \Lambda_\theta^{(2)}$, and any $u \in \mathfrak{a}_\theta$, the $\mathfrak{a}_\theta$-action on $(\Omega_\theta, \mathsf{m}_{\nu,\nu_{\mathrm{i}}})$ is completely conservative by Lemma 11.1. By [26, Lemma 8.7], the complete conservativity implies the ergodicity, showing (5).

To see the implication (5) $\Rightarrow$ (4), note that for $(\nu, \nu_{\mathrm{i}}) \in \mathcal{M}_\psi^\theta \times \mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$, the ergodicity of the $\mathfrak{a}_\theta$-action on $(\Omega_\theta, \mathsf{m}_{\nu,\nu_{\mathrm{i}}})$ is equivalent to the ergodicity of the Γ-action on $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ by the definition of $\mathsf{m}_{\nu,\nu_{\mathrm{i}}}$. Hence, if (5) holds, then $\nu \times \nu_{\mathrm{i}}$ has no atom in $\Lambda_\theta^{(2)}$ by Lemma 11.3. Consider the measure λ on $\Lambda_\theta^{(2)}$ defined by

$$d\lambda(\xi, \eta) := e^{\psi\left(\beta_\xi^\theta(e,g) + \mathrm{i}(\beta_\eta^{\mathrm{i}(\theta)}(e,g))\right)} d\nu(\xi) d\nu_{\mathrm{i}}(\eta)$$

where $g \in G$ is chosen so that $(\xi, \eta) = (gP_\theta, gw_0P_{\mathrm{i}(\theta)})$ as in (10.1). Then λ is Γ-invaraint, Γ-ergodic, and atomless. Since Γ is countable, this implies that the Γ-action on $(\Lambda_\theta^{(2)}, \lambda)$ is completely conservative [20, Theorem 14]. Therefore, (4) follows. This establishes (3) $\Rightarrow$ (5) $\Rightarrow$ (4). The implications (3) $\Rightarrow$ (6) $\Rightarrow$ (4) can be proved by a similar argument.

To show the implication (4) $\Rightarrow$ (1), fixing a pair $(\nu, \nu_{\mathrm{i}}) \in \mathcal{M}_\psi^\theta \times \mathcal{M}_{\psi\circ\mathrm{i}}^{\mathrm{i}(\theta)}$, we will show that the complete conservativity of the Γ-action on $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ implies (1). Since $(\Gamma, \Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ is completely conservative, it follows from Lemma 11.1 that for $\nu \times \nu_{\mathrm{i}}$-a.e. $(\xi, \eta) \in \Lambda_\theta^{(2)}$, there exists a compact subset $B_{(\xi,\eta)} \subset \Lambda_\theta^{(2)}$ and a sequence $\gamma_i \in \Gamma$ such that $\gamma_i(\xi, \eta) \in B_{(\xi,\eta)}$ for all i. In particular, after passing to a subsequence, we have that the sequence $\gamma_i(\xi, \eta)$ is convergent in $\Lambda_\theta^{(2)}$. By Lemma 9.6, we have $\gamma_i^{-1} \to \xi$ or $\gamma_i^{-1} \to \eta$ conically. In particular, either $\xi \in \Lambda_\theta^{\mathsf{con}}$ or $\eta \in \Lambda_{\mathrm{i}(\theta)}^{\mathsf{con}}$, and therefore

$$\max\{\nu(\Lambda_\theta^{\mathsf{con}}), \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathsf{con}})\} > 0.$$

In either case, it follows from Theorem 8.2 that

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \sum_{\gamma \in \Gamma} e^{-(\psi\circ\mathrm{i})(\mu_{\mathrm{i}(\theta)}(\gamma))} = \infty.$$

Now (1) follows.

The convergent case. From the divergent case, we have the following equivalences for the convergent case:

$$\begin{array}{ccccccc} & & & & & & (4) \\ & & & & & \swarrow\!\!\swarrow & \\ (1) & \Leftrightarrow & (2) & \Leftrightarrow & (3) & \Leftarrow & (5) \\ & & & & & \nwarrow\!\!\nwarrow & \\ & & & & & & (6) \end{array}$$

We first observe (4) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6). As mentioned in the proof of the divergent case, the ergodicity in (4), (5), and (6) are all equivalent to each other. Moreover, if $B \subset \Lambda_\theta^{(2)}$ is a wandering set for the Γ-action on $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ where $(\nu, \nu_{\mathrm{i}}) \in \mathcal{M}_\psi^\theta \times \mathcal{M}_{\psi \circ \mathrm{i}}^{\mathrm{i}(\theta)}$, then for any non-empty compact subset $V \subset \mathfrak{a}_\theta$, the set $\Gamma(B \times V) \subset \Omega_\theta$ is a wandering set for the $\mathfrak{a}_\theta$-action on $(\Omega_\theta, \mathsf{m}_{\nu,\nu_{\mathrm{i}}})$. Since $\mathfrak{a}_\theta$ is σ-compact, this implies that if $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ is a countable union of wandering subsets, then so is $(\Omega_\theta, \mathsf{m}_{\nu,\nu_{\mathrm{i}}})$, up to measure zero. Therefore, the complete dissipativity in (4) implies the one in (5), and hence (4) $\Rightarrow$ (5) follows. The implication (4) $\Rightarrow$ (6) can be shown similarly.

We finish the proof by showing (1) $\Rightarrow$ (4). Assume (1) and fix $(\nu, \nu_{\mathrm{i}}) \in \mathcal{M}_\psi^\theta \times \mathcal{M}_{\psi \circ \mathrm{i}}^{\mathrm{i}(\theta)}$. We first show that the Γ-action on $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ is completely dissipative. We write the Hopf decomposition $\Lambda_\theta^{(2)} = \Omega_C \cup \Omega_D$ and suppose to the contrary that $(\nu \times \nu_{\mathrm{i}})(\Omega_C) > 0$. By applying Lemma 11.1 to the restriction $(\nu \times \nu_{\mathrm{i}})|_{\Omega_C}$, we deduce that there exists a Borel subset $\Omega \subset \Lambda_\theta^{(2)}$ with $(\nu \times \nu_{\mathrm{i}})(\Omega) > 0$ such that for any $(\xi, \eta) \in \Omega$, there exist a compact subset $B_{(\xi,\eta)} \subset \Omega$ and a sequence $\gamma_i \in \Gamma$ such that $\gamma_i(\xi, \eta) \in B_{(\xi,\eta)}$ for all i. Hence after passing to a subsequence, the sequence $\gamma_i(\xi, \eta)$ is convergent in $\Omega \subset \Lambda_\theta^{(2)}$, and therefore it follows from Lemma 9.6 that $\gamma_i^{-1} \to \xi$ or $\gamma_i^{-1} \to \eta$ conically. Since $(\nu \times \nu_{\mathrm{i}})(\Omega) > 0$, it implies

$$\max\{\nu(\Lambda_\theta^{\mathsf{con}}), \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathsf{con}})\} > 0.$$

In either case, it follows from Theorem 8.2 that

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \sum_{\gamma \in \Gamma} e^{-(\psi \circ \mathrm{i})(\mu_{\mathrm{i}(\theta)}(\gamma))} = \infty,$$

which contradicts (1). Therefore, $(\nu \times \nu_{\mathrm{i}})(\Omega_C) = 0$ and hence the Γ-action on $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ is completely dissipative.

Now it remains to show that the Γ-action on $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ is non-ergodic. Suppose not. Then the Γ-action on $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ is ergodic, and hence $\nu \times \nu_{\mathrm{i}}$ has no atom in $\Lambda_\theta^{(2)}$ by Lemma 11.3. As before, since Γ is countable, this must imply that the Γ-action on $(\Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$ is completely conservative [20, Theorem 14]. This is a contradiction, and (4) follows. $\square$

Proof of Theorem 1.4. Theorem 7.1 implies Theorem 1.4(1). Theorem 1.4(2) follows from Theorem 8.2 and the following corollary. $\square$

Corollary 11.4. *Let Γ be a Zariski dense θ-transverse subgroup. If $\psi \in \mathfrak{a}_\theta^*$ is (Γ,θ)-proper with $\mathsf{M}_\psi^\theta \neq \emptyset$ and $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$, then $\#\mathsf{M}_\psi^\theta = 1$.*

Proof. By Theorem 7.1 and the hypothesis on ψ, we have $\delta_\psi = 1$. By Proposition 5.13, there exists a (Γ,ψ)-conformal measure on $\mathcal{F}_{\theta\cup\mathsf{i}(\theta)}$, and is supported on $\Lambda_{\theta\cup\mathsf{i}(\theta)}$. Moreover it is unique by [9, Theorem 1.4]. It then follows from Lemma 9.5 that there exists a unique (Γ,ψ)-conformal measure on $\mathcal{F}_\theta$ as well. $\square$

12. Lebesgue measures of conical sets and disjoint dimensions

In this section, we discuss some of consequences of Theorem 8.2.

Lebesgue measure of conical sets.

Theorem 12.1. *If $\Gamma < G$ is a Zariski dense θ-transverse subgroup, then*

$$\Lambda_\theta = \mathcal{F}_\theta \quad \text{or} \quad \mathrm{Leb}_\theta(\Lambda_\theta^{\mathsf{con}}) = 0.$$

Moreover, in the former case, θ is the simple root of a rank one factor of G.

We need the following proposition for the second claim of the above theorem.

Proposition 12.2. *Suppose that Γ is Zariski dense and θ-antipodal and that $\Lambda_\theta = \mathcal{F}_\theta$. Then θ consists of the simple root of a rank one factor of G.*

Proof. We write G as the almost direct product of simple real algebraic groups $G = G_1 \cdots G_m$. Let n be an index such that θ contains a simple root of G_n. Denoting by $\pi_n : G \to G_n$ the canonical projection, $\pi_n(P_\theta)$ is a proper parabolic subgroup of G_n and the limit set of $\overline{\pi_n(\Gamma)}$ in $G_n/\pi_n(P_\theta)$ is equal to all of $G_n/\pi_n(P_\theta)$, as the limit set of a Zariski dense subgroup is the unique minimal set (Lemma 5.2). Suppose that the rank of G_n is at least 2. Fix $kP_{\theta\cup\mathsf{i}(\theta)} \in \Lambda_{\theta\cup\mathsf{i}(\theta)}$ for some $k \in K$. Let w be a Weyl element given by Lemma 12.3 below such that $w \notin w_0 N_\theta^+ P_\theta \cup P_\theta$. Noting that $w_0 N_{\theta\cup\mathsf{i}(\theta)}^+ P_{\theta\cup\mathsf{i}(\theta)} M_\theta \subset w_0 P_\theta^+ P_\theta = w_0 N_\theta^+ P_\theta$, we have

$$w \notin w_0 N_{\theta\cup\mathsf{i}(\theta)}^+ P_{\theta\cup\mathsf{i}(\theta)} M_\theta \cup P_{\theta\cup\mathsf{i}(\theta)} M_\theta. \tag{12.1}$$

Note again that both Λ_θ and Λ_Π are unique Γ-minimal subsets of $\mathcal{F}_\theta$ and $\mathcal{F}$, and hence the canonical projection $\mathcal{F} \to \mathcal{F}_\theta$ maps Λ_Π onto Λ_θ. Since $\mathcal{F} = K/M$ and $kwM_\theta \in \mathcal{F}_\theta = K/M_\theta = \Lambda_\theta$, we may choose $m \in M_\theta$ such that $kwmP \in \Lambda_\Pi$, and hence $kwmP_{\theta\cup\mathsf{i}(\theta)} \in \Lambda_{\theta\cup\mathsf{i}(\theta)}$. Then by (12.1),

$$wm \notin w_0 N_{\theta\cup\mathsf{i}(\theta)}^+ P_{\theta\cup\mathsf{i}(\theta)} \cup P_{\theta\cup\mathsf{i}(\theta)}.$$

The condition that $wm \notin P_{\theta\cup\mathsf{i}(\theta)}$ implies that $kwmP_{\theta\cup\mathsf{i}(\theta)} \cap kP_{\theta\cup\mathsf{i}(\theta)} = \emptyset$. Also, by Corollary 2.6, the condition that $wm \notin w_0 N_{\theta\cup\mathsf{i}(\theta)}^+ P_{\theta\cup\mathsf{i}(\theta)}$ implies that $(kwmP_{\theta\cup\mathsf{i}(\theta)}, kP_{\theta\cup\mathsf{i}(\theta)}) \notin G.(P_{\theta\cup\mathsf{i}(\theta)}, w_0 P_{\theta\cup\mathsf{i}(\theta)})$, that is, $kwmP_{\theta\cup\mathsf{i}(\theta)}$ is not in general position with $P_{\theta\cup \mathrm{i}(\theta)}$. This yields a contradiction to the $\theta\cup\mathrm{i}(\theta)$-antipodality of Γ. Therefore for any n such that θ contains a simple root of G_n, the rank of G_n must be one. If there are $n\neq n'$ with this property, the map $\gamma\to(\pi_n(\gamma),\pi_{n'}(\gamma))$ must be a Zariski dense subgroup of $G_nG_{n'}$ with full limit set $G_n/\pi_n(P_\theta)\times G_{n'}/\pi_{n'}(P_\theta)$. However this yields a contradiction to the θ-antipodal property, because the product of two rank one geometric boundaries does not have the antipodal property. Therefore θ must be a singleton, proving the claim. $\square$

We now prove the following lemma which was used in the above proof.

Lemma 12.3. *If G has a connected normal subgroup G_n of rank at least 2 and $\theta\subset\Pi$ contains a simple root of G_n, then we can find a representative of a Weyl element $w\in N_K(A)$ such that $w\notin w_0N_\theta^+P_\theta\cup P_\theta$.*

Proof. By replacing θ with the intersection of θ and the set of simple roots of G_n, we may assume without loss of generality that $G=G_n$. Since the rank of G is at least 2, we can find a representative $w\in N_K(A)$ of a Weyl element such that $\mathrm{Ad}_w(\mathfrak{a}_\theta^+)$ is equal to neither $\mathfrak{a}_\theta^+$ nor $-\mathfrak{a}_{\mathrm{i}(\theta)}^+$. If w were contained in $P_\theta\cap K=M_\theta$, w would commute with $\mathfrak{a}_\theta$ and hence $\mathrm{Ad}_w(\mathfrak{a}_\theta^+)=\mathfrak{a}_\theta^+$. Therefore $w\notin P_\theta$. On the other hand, if $w\in w_0N_\theta^+P_\theta$, then $w_0^{-1}w\in M_\theta$ by Corollary 2.6, and hence $\mathrm{Ad}_w(\mathfrak{a}_\theta^+)=\mathrm{Ad}_{w_0}(\mathfrak{a}_\theta^+)=-\mathfrak{a}_{\mathrm{i}(\theta)}^+$, which contradicts our choice of w. Hence $w\notin w_0N_\theta^+P_\theta$. $\square$

Proof of Theorem 12.1. Note that Leb_θ is a $(\Gamma,2\rho\circ p_\theta)$-conformal measure where ρ is the half sum of all positive roots of $(\mathfrak{g},\mathfrak{a}^+)$ [36, Lemma 6.3]. If $\Lambda_\theta\neq\mathcal{F}_\theta$, $\mathrm{Leb}_\theta(\Lambda_\theta^{\mathsf{con}})\leq\mathrm{Leb}_\theta(\Lambda_\theta)<1$ as $\mathcal{F}_\theta-\Lambda_\theta$ is a non-empty open subset. Therefore $\mathrm{Leb}_\theta(\Lambda_\theta^{\mathsf{con}})=0$ by Theorem 8.2. The second claim follows from Proposition 12.2 above. $\square$

Disjoint dimensions and entropy drop. Recall from the introduction that

$$\mathcal{D}_\Gamma^\theta=\{\psi\in\mathfrak{a}_\theta^*:(\Gamma,\theta)\text{-proper},\delta_\psi=1,\mathcal{P}_\psi(1)=\infty\}.$$

Lemma 12.4. *For a Zariski dense θ-transverse Γ, we have*

$$\mathcal{D}_\Gamma^\theta=\{\psi\in\mathfrak{a}_\theta^*:(\Gamma,\theta)\textit{-proper},\exists\ \textit{a}\ (\Gamma,\psi)\textit{-conformal measure},\mathcal{P}_\psi(1)=\infty\}.$$

Proof. The inclusion $\subset$ follows from Proposition 5.13. If there exists a (Γ,ψ)-conformal measure on $\mathcal{F}_\theta$ for (Γ,θ)-proper ψ, then $\delta_\psi\leq1$ by Theorem 7.3. If $\delta_\psi<1$, $\mathcal{P}_\psi(1)<\infty$. Hence this implies the inclusion $\supset$. $\square$

Note that any subgroup of a θ-transverse subgroup of G is again a θ-transverse subgroup.

Theorem 12.5 (Disjoint dimensions)**.** *Let $\Gamma<G$ be a non-elementary θ-transverse subgroup. For any subgroup $\Gamma_0<\Gamma$ with $\Lambda_\theta(\Gamma_0)\neq\Lambda_\theta(\Gamma)$, we have*

$$\mathcal{D}_\Gamma^\theta\cap\mathcal{D}_{\Gamma_0}^\theta=\emptyset.$$

Proof. Let $\psi \in \mathcal{D}_\Gamma^\theta$. By Proposition 5.13, there exists a (Γ, ψ)-conformal measure ν on $\Lambda_\theta(\Gamma)$. By Theorem 8.2, $\nu(\Lambda_\theta^{\mathsf{con}}(\Gamma)) = 1$.

Since $\Lambda_\theta(\Gamma_0) \neq \Lambda_\theta(\Gamma)$, $\Lambda_\theta(\Gamma) - \Lambda_\theta(\Gamma_0)$ is a non-empty open subset of $\Lambda_\theta(\Gamma)$. Hence, it follows from the Γ-minimality on $\Lambda_\theta(\Gamma)$ and the compactness of $\Lambda_\theta(\Gamma)$ that $\Lambda_\theta(\Gamma)$ is covered by translates of $\Lambda_\theta(\Gamma) - \Lambda_\theta(\Gamma_0)$ under finitely many elements of Γ. Since ν is Γ-quasi-invariant, this implies $\nu(\Lambda_\theta(\Gamma) - \Lambda_\theta(\Gamma_0)) > 0$, and hence, $\nu(\Lambda_\theta^{\mathsf{con}}(\Gamma_0)) < 1$. Moreover, by the θ-antipodality of Γ, it also follows from $\nu(\Lambda_\theta(\Gamma) - \Lambda_\theta(\Gamma_0)) > 0$ that $\nu \in \mathsf{N}_{\Gamma_0,\psi}^\theta$ in Theorem 8.2. Again by Theorem 8.2, $\sum_{\gamma\in\Gamma_0} e^{-\psi(\mu_\theta(\gamma))} < \infty$. Hence $\psi \notin \mathcal{D}_{\Gamma_0}^\theta$, finishing the proof. ☐

This turns out to be equivalent to the entropy drop phenomenon which is proved by Canary-Zhang-Zimmer [9, Theorem 4.1] for $\theta = \mathrm{i}(\theta)$:

Corollary 12.6 (Entropy drop)**.** *Let $\Gamma < G$ be a non-elementary θ-transverse subgroup. Let $\Gamma_0 < \Gamma$ be a subgroup such that $\Lambda_\theta(\Gamma_0) \neq \Lambda_\theta(\Gamma)$. If $\psi \in \mathfrak{a}_\theta^*$ with $\delta_\psi(\Gamma) < \infty$ and $\sum_{\gamma\in\Gamma_0} e^{-\delta_\psi(\Gamma_0)\psi(\mu_\theta(\gamma))} = \infty$, then*

$$\delta_\psi(\Gamma_0) < \delta_\psi(\Gamma).$$

Proof. Suppose that $\delta_\psi(\Gamma) < \infty$; this implies that ψ is (Γ, θ)-proper. Let $\Gamma_0 < \Gamma$ be a subgroup such that $\sum_{\gamma\in\Gamma_0} e^{-\delta_\psi(\Gamma_0)\psi(\mu_\theta(\gamma))} = \infty$ and $\delta_\psi(\Gamma_0) = \delta_\psi(\Gamma)$. If we set $\phi = \delta_\psi(\Gamma) \cdot \psi = \delta_\psi(\Gamma_0) \cdot \psi$, then $\delta_\phi(\Gamma) = \delta_\phi(\Gamma_0) = 1$. Since $\infty = \sum_{\gamma\in\Gamma_0} e^{-\phi(\mu_\theta(\gamma))} \leq \sum_{\gamma\in\Gamma} e^{-\phi(\mu_\theta(\gamma))}$, we have $\phi \in \mathcal{D}_\Gamma^\theta \cap \mathcal{D}_{\Gamma_0}^\theta$. By Theorem 12.5, this implies that $\Lambda_\theta(\Gamma_0) = \Lambda_\theta(\Gamma)$, proving the corollary. ☐

13. Conformal measures for θ-Anosov subgroups

Note that Γ is θ-Anosov if and only if Γ is $\theta \cup \mathrm{i}(\theta)$-Anosov by (2.1).

Proposition 13.1 ([18], [22, Theorem 1.1])**.** *If Γ is θ-Anosov, then*

(1) *Γ is θ-regular;*
(2) *$\Lambda_\theta = \Lambda_\theta^{\mathsf{con}}$;*
(3) *$\mathcal{L}_\theta - \{0\} \subset \operatorname{int} \mathfrak{a}_\theta^+$;*
(4) *θ-antipodal.*

In particular, a θ-Anosov subgroup is θ-transverse.

Sambarino [41, Theorem A] showed that if Γ is θ-Anosov, then the set $\{\psi \in \mathfrak{a}_\theta^* : \delta_\psi = 1\}$ is analytic and is equal to the boundary of a strictly convex subset $\{\psi \in \mathfrak{a}_\theta^* : 0 < \delta_\psi < 1\}$. By the duality lemma ([37, Section 4], [40, Lemma 4.8]), we then deduce the following property of the θ-growth indicator:

Theorem 13.2. *If Γ is Zariski dense θ-Anosov, then ψ_Γ^θ is strictly concave, differentiable on $\operatorname{int} \mathcal{L}_\theta$, and vertically tangent.*

The vertical tangency means that if $\psi_\Gamma^\theta(u) = \psi(u)$ for some (Γ, θ)-critical form ψ and $u \neq 0$, then $u \in \operatorname{int} \mathcal{L}_\theta$. Recall

$$\mathcal{T}_\Gamma^\theta := \{\psi \in \mathfrak{a}_\theta^* : \psi \text{ is } (\Gamma, \theta)\text{-critical}\}.$$

Corollary 13.3. *Let $\Gamma < G$ be a Zariski dense θ-Anosov subgroup. For any subgroup $\Gamma_0 < \Gamma$,*

$$\mathcal{T}_\Gamma^\theta \cap \mathcal{T}_{\Gamma_0}^\theta = \emptyset \quad \iff \quad \psi_{\Gamma_0}^\theta < \psi_\Gamma^\theta \text{ on } \operatorname{int} \mathcal{L}_\theta(\Gamma).$$

Proof. Suppose that $\psi \in \mathcal{T}_\Gamma^\theta \cap \mathcal{T}_{\Gamma_0}^\theta$. Then there exists $u \in \mathcal{L}_\theta(\Gamma_0)$ such that

$$\psi_{\Gamma_0}^\theta(u) = \psi(u).$$

Since $\psi_{\Gamma_0}^\theta \leq \psi_\Gamma^\theta \leq \psi$, it follows that ψ is tangent to ψ_Γ^θ at u as well. By the vertical tangency of ψ_Γ^θ (Theorem 13.2), $u \in \operatorname{int} \mathcal{L}_\theta(\Gamma)$. Therefore, the implication ($\Leftarrow$) follows.

Conversely, suppose that $\psi_{\Gamma_0}^\theta(u) = \psi_\Gamma^\theta(u)$ for some $u \in \operatorname{int} \mathcal{L}_\theta(\Gamma)$. Then by the concavity of ψ_Γ^θ (Theorem 13.2), there exists $\psi \in \mathcal{T}_\Gamma^\theta$ such that $\psi(u) = \psi_\Gamma^\theta(u)$. Since $\psi_{\Gamma_0}^\theta \leq \psi_\Gamma^\theta \leq \psi$ and $\psi_{\Gamma_0}^\theta(u) = \psi_\Gamma^\theta(u)$, we have $\psi \in \mathcal{T}_\Gamma^\theta \cap \mathcal{T}_{\Gamma_0}^\theta$. This shows the implication ($\Rightarrow$). ☐

Lemma 13.4. *If Γ is Zariski dense θ-Anosov, then*

$$\mathcal{T}_\Gamma^\theta = \{\psi \in \mathfrak{a}_\theta^* : (\Gamma, \theta)\text{-proper}, \delta_\psi = 1\} = \mathcal{D}_\Gamma^\theta.$$

Proof. The second identity is proved in [41, Section 5.9]. It suffices to prove the inclusion $\subset$ in the first equality due to Corollary 4.6. Suppose that $\psi \in \mathfrak{a}_\theta^*$ is tangent to ψ_Γ^θ. Since ψ_Γ^θ is vertically tangent (Theorem 13.2), $\psi > \psi_\Gamma^\theta$ on $\partial \mathcal{L}_\theta$. It follows that $\psi > 0$ on $\mathcal{L}_\theta$. Hence by the second claim in Corollary 4.6, $\delta_\psi = 1$. ☐

Lemma 13.5. *If Γ is a non-elementary θ-Anosov subgroup and there exists a (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$ for $\psi \in \mathfrak{a}_\theta^*$, then ψ is (Γ, θ)-proper.*

Proof. If $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} < \infty$, then it implies that $\#\{\gamma \in \Gamma : \psi(\mu_\theta(\gamma)) < T\}$ is finite for any $T > 0$. Therefore ψ is (Γ, θ)-proper. If $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$, then $\nu(\Lambda_\theta) = 1$ by Theorem 8.2. This implies that $\limsup \frac{1}{T} \log \#\{\gamma \in \Gamma : \psi(\mu_\theta(\gamma)) < T\} < \infty$ by [41, Theorem A]. Therefore, ψ is (Γ, θ)-proper in either case. ☐

Proof of Theorem 1.11. Let Γ be Zariski dense θ-Anosov. Note that a θ-Anosov group is θ-transverse. Hence (1) follows from Theorem 7.1 since ψ is (Γ, θ)-proper by Lemma 13.5.

Since $\Lambda_\theta = \Lambda_\theta^{\mathsf{con}}$ (Proposition 13.1), $(a) \Leftrightarrow (b)$ in (2) follows from Theorem 8.2. The equivalence $(b) \Leftrightarrow (c)$ follows from Lemma 13.4 and Sambarino's parametrization of the space of all conformal measures on Λ_θ as $\{\delta_\psi = 1\}$ [41, Theorem A], together with (1) shown above. For (3), let ψ be a (Γ, θ)-critical form. By Lemma 13.4 and Proposition 5.13, there exists a (Γ, ψ)-conformal measure ν_ψ on Λ_θ, which is the unique (Γ, ψ)-conformal measure on Λ_θ by [41, Theorem A] (see also Corollary 11.4). Since $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$, by Theorem 8.2, any (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$ is supported on Λ_θ. Moreover, by Theorem 11.2, the $\mathfrak{a}_\theta$-action on $(\Omega_\theta, \mathsf{m}_{\nu_\psi, \nu_{\psi \circ \mathrm{i}}})$ is completely conservative and ergodic. This finishes the proof. ☐

Proof of Corollary 1.12. Since a θ-Anosov subgroup is θ-transverse and $\Lambda_\theta = \Lambda_\theta^{\mathsf{con}}$ (Theorem 9.8), we deduce from Theorem 12.1 that either $\Lambda_\theta = \mathcal{F}_\theta$ or $\mathrm{Leb}_\theta(\Lambda_\theta) = 0$. In the former case, θ is the simple root of a rank one factor G_0 of G with $\mathcal{F}_\theta = \Lambda_\theta$ by Proposition 12.2, the projection of Γ to G_0 is a convex cocompact subgroup with full limit set, and hence a cocompact lattice of G_0. $\square$

Proof of Corollary 1.13. Consider the map $\mathcal{T}_\Gamma^\theta \to \{u \in \operatorname{int} \mathcal{L}_\theta : \|u\| = 1\}$ given by $\psi \mapsto u_\psi$, where u_ψ satisfies $\psi(u_\psi) = \psi_\Gamma^\theta(u_\psi)$. By Theorem 13.2, ψ_Γ^θ is strictly concave and vertically tangent, and hence such a map is well-defined and also surjective. Moreover, since ψ_Γ^θ is differentiable on $\operatorname{int} \mathcal{L}_\theta$ (Theorem 13.2), this map is injective as well, and therefore bijective. This gives the one-to-one correspondence between (1) and (2).

By Lemma 13.4 and [41, Theorem A], for each $\psi \in \mathcal{T}_\Gamma^\theta$, there exists a unique (Γ, ψ)-conformal measure ν_ψ supported on Λ_θ, and vice versa. Hence the map $\psi \mapsto \nu_\psi$ is the one-to-one correspondence between (1) and (3).

Finally, by Theorem 1.11, the sets (3) and (4) are in fact identical, which finishes the proof. $\square$

Proof of Corollary 1.14. By Theorem 12.5 and Lemma 13.4, it remains to prove the second part. Since $\Gamma_0 < \Gamma$, we have $\psi_{\Gamma_0}^\theta \le \psi_\Gamma^\theta$. Suppose that $\psi_{\Gamma_0}^\theta(u) = \psi_\Gamma^\theta(u)$ for some u in the interior of $\mathcal{L}_\theta(\Gamma)$. Then there exists a tangent form ψ to ψ_Γ^θ at u by Corollary 3.12. Since $\psi_{\Gamma_0}^\theta \le \psi_\Gamma^\theta$ and $\psi_{\Gamma_0}^\theta(u) = \psi_\Gamma^\theta(u)$, ψ is also tangent to $\psi_{\Gamma_0}^\theta$ at u. Hence $\psi \in \mathcal{T}_\Gamma^\theta \cap \mathcal{T}_{\Gamma_0}^\theta$, contradicting the first part. $\square$

Declarations

Data availability. This manuscript has no associated data.

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References

[1] J. Aaronson and D. Sullivan. Rational ergodicity of geodesic flows. *Ergodic Theory Dynam. Systems*, 4(2):165–178, 1984.
[2] L. Ahlfors. Finitely generated Kleinian groups. *Amer. J. Math.*, 86:413–429, 1964.
[3] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.*, 7(1):1–47, 1997.
[4] B. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.
[5] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino. The pressure metric for Anosov representations. *Geom. Funct. Anal.*, 25(4):1089–1179, 2015.
[6] M. Burger, O. Landesberg, M. Lee, and H. Oh. The Hopf–Tsuji–Sullivan dichotomy in higher rank and applications to Anosov subgroups. *J. Mod. Dyn.*, 19:301–330, 2023.

[7] R. Canary, T. Zhang, and A. Zimmer. Cusped Hitchin representations and Anosov representations of geometrically finite Fuchsian groups. *Adv. Math.*, 404(part B):Paper No. 108439, 67, 2022.
[8] R. Canary, T. Zhang, and A. Zimmer. Entropy rigidity for cusped hitchin representations. *Preprint, arXiv:2201.04859*, 2022.
[9] R. Canary, T. Zhang, and A. Zimmer. Patterson-Sullivan measures for transverse subgroups. *J. Mod. Dyn.*, 20:319–377, 2024.
[10] R. Canary, T. Zhang, and A. Zimmer. Patterson-Sullivan measures for relatively Anosov groups. *Math. Ann.*, 392(2):2309–2363, 2025.
[11] M. Chow and P. Sarkar. Local mixing of one-parameter diagonal flows on Anosov homogeneous spaces. *Int. Math. Res. Not. IMRN*, (18):15834–15895, 2023.
[12] K. Corlette and A. Iozzi. Limit sets of discrete groups of isometries of exotic hyperbolic spaces. *Trans. Amer. Math. Soc.*, 351(4):1507–1530, 1999.
[13] T. Das, D. Simmons, and M. Urbański. Tukia's isomorphism theorem in CAT(-1) spaces. *Ann. Acad. Sci. Fenn. Math.*, 41(2):659–680, 2016.
[14] S. Dey and M. Kapovich. Patterson-Sullivan theory for Anosov subgroups. *Trans. Amer. Math. Soc.*, 375(12):8687–8737, 2022.
[15] S. Edwards, M. Lee, and H. Oh. Uniqueness of conformal measures and local mixing for Anosov groups. *Michigan Math. J.*, 72:243–259, 2022.
[16] R. Grigorchuk and J.-F. Quint. Directional counting for automatic languages. *in preparation*.
[17] F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard. Anosov representations and proper actions. *Geom. Topol.*, 21(1):485–584, 2017.
[18] O. Guichard and A. Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.*, 190(2):357–438, 2012.
[19] E. Hopf. Ergodic theory and the geodesic flow on surfaces of constant negative curvature. *Bull. Amer. Math. Soc.*, 77:863–877, 1971.
[20] V. Kaimanovich. Hopf decomposition and horospheric limit sets. *Ann. Acad. Sci. Fenn. Math.*, 35(2):335–350, 2010.
[21] M. Kapovich and B. Leeb. Discrete isometry groups of symmetric spaces. In *Handbook of group actions. Vol. IV*, volume 41 of *Adv. Lect. Math. (ALM)*, pages 191–290. Int. Press, Somerville, MA, 2018.
[22] M. Kapovich, B. Leeb, and J. Porti. Anosov subgroups: dynamical and geometric characterizations. *Eur. J. Math.*, 3(4):808–898, 2017.
[23] M. Kapovich, B. Leeb, and J. Porti. A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings. *Geom. Topol.*, 22(7):3827–3923, 2018.
[24] D. M. Kim, Y. Minsky, and H. Oh. Hausdorff dimension of directional limit sets for self-joinings of hyperbolic manifolds. *J. Mod. Dyn.*, 19:433–453, 2023.
[25] D. M. Kim, Y. Minsky, and H. Oh. Tent property of the growth indicator functions and applications. *Geom. Dedicata*, 218(1):Paper No.14, 18, 2024.
[26] D. M. Kim, H. Oh, and Y. Wang. Ergodic dichotomy for subspace flows in higher rank. *Commun. Am. Math. Soc.*, 5:1–47, 2025.
[27] S. Kochen and C. Stone. A note on the Borel-Cantelli lemma. *Illinois J. Math.*, 8:248–251, 1964.
[28] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165(1):51–114, 2006.
[29] M. Lee and H. Oh. Invariant Measures for Horospherical Actions and Anosov Groups. *Int. Math. Res. Not. IMRN*, (19):16226–16295, 2023.
[30] M. Lee and H. Oh. Dichotomy and Measures on Limit Sets of Anosov Groups. *Int. Math. Res. Not. IMRN*, (7):5658–5688, 2024.
[31] H. Oh. Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants. *Duke Math. J. 113 (2002), no. 1, 133–192.*
[32] S. Patterson. The limit set of a Fuchsian group. *Acta Math.*, 136(3-4):241–273, 1976.

[33] R. Potrie and A. Sambarino. Eigenvalues and entropy of a Hitchin representation. *Invent. Math.*, 209(3):885–925, 2017.
[34] B. Pozzetti, A. Sambarino, and A. Wienhard. Anosov representations with Lipschitz limit set. *Geom. Topol.*, 27(8):3303–3360, 2023.
[35] J.-F. Quint. Divergence exponentielle des sous-groupes discrets en rang supérieur. *Comment. Math. Helv.*, 77(3):563–608, 2002.
[36] J.-F. Quint. Mesures de Patterson-Sullivan en rang supérieur. *Geom. Funct. Anal.*, 12(4):776–809, 2002.
[37] J.-F. Quint. L'indicateur de croissance des groupes de Schottky. *Ergodic Theory Dynam. Systems*, 23(1):249–272, 2003.
[38] J.-F. Quint. Propriété de Kazhdan et sous-groupes discrets de covolume infini. In *Travaux mathématiques. Fasc. XIV*, volume 14 of *Trav. Math.*, pages 143–151. Univ. Luxemb., Luxembourg, 2003.
[39] T. Roblin. Ergodicité et équidistribution en courbure négative. *Mém. Soc. Math. Fr. (N.S.)*, (95):vi+96, 2003.
[40] A. Sambarino. Hyperconvex representations and exponential growth. *Ergodic Theory Dynam. Systems*, 34(3):986–1010, 2014.
[41] A. Sambarino. A report on an ergodic dichotomy. *Ergodic Theory Dynam. Systems*, 44(1):236–289, 2024.
[42] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.
[43] M. Tsuji. *Potential theory in modern function theory.* Maruzen Co. Ltd., Tokyo, 1959.
[44] P. Tukia. On isomorphisms of geometrically finite Möbius groups. *Inst. Hautes Études Sci. Publ. Math.*, (61):171–214, 1985.
[45] P. Tukia. Convergence groups and Gromov's metric hyperbolic spaces. *New Zealand J. Math.*, 23(2):157–187, 1994.
[46] A. Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.

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