ERGODIC DICHOTOMY FOR SUBSPACE FLOWS IN HIGHER RANK

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Abstract. In this paper, we study the ergodicity of a one-parameter diagonalizable subgroup of a connected semisimple real algebraic group G acting on a homogeneous space or, more generally, a homogeneous-like space, equipped with a Bowen-Margulis-Sullivan type measure. These flow spaces are associated with Anosov subgroups of G, or more generally, with transverse subgroups of G.

We obtain an ergodicity criterion similar to the Hopf-Tsuji-Sullivan dichotomy for the ergodicity of the geodesic flow on hyperbolic manifolds. In addition, we extend this criterion to the action of any connected diagonal subgroup of arbitrary dimension. Our criterion provides a codimension dichotomy on the ergodicity of a connected diagonalizable subgroup for general Anosov subgroups. This generalizes an earlier work by Burger-Landesberg-Lee-Oh for *Borel* Anosov subgroups.

Contents

1. Introduction

A continuous flow ϕ_t on a phase space X with an invariant measure m is called *ergodic* if any invariant measurable subset has measure zero or comeasure zero.

If m is a probability measure on X, the Birkhoff ergodic theorem states that the ergodicity of the dynamical system (X, ϕ_t, m) is equivalent to the

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condition that for any $f \in L^1(X, \mathsf{m})$, the time average along the trajectory of almost every point $x \in X$ is equal to the space average with respect to m:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\phi_t(x))dt = \int_X f\ d\mathsf{m} \quad \text{ for } \mathsf{m}\text{-almost all } x \in X.$$

Hence, ergodicity ensures that the system's behavior, when observed over a long time, reflects the statistical properties of the entire phase space. This property is crucial in understanding the long-term behavior of dynamical systems.

The main dynamical system of interest in homogeneous dynamics arises from the quotient $\Gamma\backslash G$ of a connected semisimple real algebraic group G by a discrete subgroup Γ of G. Any one-parameter subgroup $H = \{\phi_t : t \in \mathbb{R}\}$ of G acts on $\Gamma\backslash G$ by translations on the right, giving rise to a continuous dynamical system $(\Gamma\backslash G, H)$. The ergodicity in *finite-volume* homogeneous dynamics is well understood thanks to the Moore ergodicity theorem: if Γ is an irreducible lattice, any non-compact closed subgroup H acts ergodically on $(\Gamma\backslash G, \mathsf{m}_G)$ where m_G denotes a G-invariant (finite) measure on $\Gamma\backslash G$.

The concept of ergodicity becomes much more delicate and challenging to prove for an *infinite* measure system. Firstly, the Birkhoff ergodic theorem no longer holds, but we have the Hopf ratio ergodic theorem: for a conservative and ergodic action of a continuous flow ϕ_t on a σ-finite measure space (X, m), for any $f, g \in L^1(X, \mathsf{m})$ with $g > 0$, we have

$$\lim_{T\to\infty} \frac{\int_0^T f(\phi_t(x))dt}{\int_0^T g(\phi_t(x))dt} = \frac{\int_X f\ d\mathsf{m}}{\int_X g\ d\mathsf{m}} \quad \text{ for } \mathsf{m}\text{-almost all } x \in X.$$

When $\mathsf{m}(X) = \infty$, while the denominator $\int_0^T g(\phi_t(x))dt$ depends on the initial position x, unlike in the finite measure case, the ratio of time averages still converges to the ratio of space averages. In particular, almost all trajectories are dense in the phase space.

One of the first significant results on ergodicity in infinite measure systems is the Hopf-Tsuji-Sullivan dichotomy for the geodesic flow on hyperbolic manifolds with respect to Bowen-Margulis-Sullivan measures. The hyperbolic nature of the geodesic flow and the quasi-product structure of Bowen-Margulis-Sullivan measures, with respect to the stable and unstable foliations for the geodesic flow, have been crucial in their approach. This methodology was successfully extended to a one-parameter diagonalizable flow on higher rank homogeneous spaces, which are quotients of a connected semisimple real algebraic group of higher rank by Borel Anosov subgroups, by Burger-Landesberg-Lee-Oh [9]. The main aim of this paper is to generalize this result for general Anosov subgroups that are not necessarily Borel Anosov, as well as to address the action of any connected diagonalizable subgroup.

Background and Motivation. To give some background, let G be a connected semisimple real algebraic group. Fix a Cartan decomposition

$$G = KA^+K$$

where K is a maximal compact subgroup of G and $A^+ = \exp \mathfrak{a}^+$ is a positive Weyl chamber of a maximal split torus A of G. Let M be the centralizer of A in K. For any Zariski dense discrete subgroup Γ of G, we have a natural locally compact Hausdorff space $\Gamma\backslash G/M$ on which A acts by translations on the right. For any non-zero vector $u \in \mathfrak{a}^+$, consider the one-parameter subgroup $A_u = \{\exp tu : t \in \mathbb{R}\}$. The ergodicity criterion for the A_u-action on $\Gamma\backslash G/M$ with respect to a Bowen-Margulis-Sullivan measure was obtained by Burger-Landesberg-Lee-Oh [9] in terms of the divergence of an appropriate directional Poincaré series. In particular, it was shown that for Borel Anosov subgroups of G (in other words, Anosov subgroups with respect to a minimal parabolic subgroup), the ergodicity of the A_u-action is completely determined by the rank of G. Soon after [9] appeared, Sambarino [34] gave a different proof for this rank dichotomy, but it applied only when $\operatorname{rank} G \neq 3$.

On the other hand, the ergodicity criterion of [9] is not very useful in practice for Anosov subgroups that are not Borel Anosov, since there seems to be no way to decide whether the relevant directional Poincaré series diverges or not.

In this paper, we consider a different dynamical space than $\Gamma\backslash G/M$ depending on the Anosov type of Γ. Let θ be a non-empty subset of simple roots. Consider the standard parabolic subgroup $P_\theta = A_\theta S_\theta N_\theta$ where $A_\theta S_\theta$ is a Levi-subgroup with A_θ being the central real split torus and N_θ is the unipotent radical. The double quotient space $\Gamma\backslash G/S_\theta$ is precisely $\Gamma\backslash G/M$ when P_θ is a minimal parabolic subgroup, but it is not Hausdorff for a general P_θ. If Γ is a θ-Anosov subgroup, there exists a locally compact Hausdorff subspace $\Omega_\theta \subset \Gamma\backslash G/S_\theta$ on which A_θ acts by translations. We will obtain the ergodicity criterion for the action of a one-parameter subgroup of A_θ on Ω_θ in terms of the associated directional Poincaré series.

In fact, this viewpoint and our criterion can be applied to a much more general class of discrete subgroups, called θ-transverse subgroups. For θ-Anosov subgroups, our criterion provides the dichotomy for the ergodicity of A_u on Ω_θ with respect to a Bowen-Margulis-Sullivan measure in terms of the cardinality of θ. When $\#\theta = 3$, this was an open question while the other cases were obtained by Sambarino [34]. Although our proof closely follows the general strategy of [9], a major difficulty arises from the non-compactness of S_θ which requires new ideas and new technical arguments to overcome.

Flow space. To discuss θ-transverse subgroups and the associated flow space Ω_θ, we need to introduce some notation and definitions. We denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection defined by the condition $g \in K \exp \mu(g) K$

for $g \in G$. Let Π be the set of all simple roots for $(\operatorname{Lie} G, \mathfrak{a}^+)$. Let $\mathrm{i} : \Pi \to \Pi$ denote the opposition involution (see (2.1)). Fix a non-empty subset

$$\theta \subset \Pi.$$

Consider the θ-boundary:

$$\mathcal{F}_\theta = G/P_\theta,$$

where P_θ is the standard parabolic subgroup associated with θ. We say that two points $\xi \in \mathcal{F}_\theta$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ are in general position if the pair (ξ, η) belongs to the unique open G-orbit in $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)}$ under the diagonal action of G.

Let $\Gamma < G$ be a Zariski dense discrete subgroup. Let Λ_θ denote the θ-limit set of Γ, which is the unique Γ-minimal subset of $\mathcal{F}_\theta$ (Definition 2.4). We say that Γ is *θ-transverse* if it satisfies

- (θ-*regularity*): $\liminf_{\gamma \in \Gamma} \alpha(\mu(\gamma)) = \infty$ for all $\alpha \in \theta$;
- (θ-*antipodality*): any distinct $\xi, \eta \in \Lambda_{\theta \cup \mathrm{i}(\theta)}$ are in general position.

The class of θ-transverse subgroups includes all discrete subgroups of rank one Lie groups, θ-Anosov subgroups and their relative versions. Note also that every subgroup of a θ-transverse subgroup is again θ-transverse. The class of transverse subgroups is regarded as a generalization of all rank one discrete subgroups, while the class of Anosov subgroups is regarded as a generalization of rank one convex cocompact subgroups.

In the rest of the introduction, we assume that

$$\Gamma \text{ is a Zariski dense } \theta\text{-transverse subgroup of } G.$$

In order to introduce an appropriate substitute of $\Gamma \backslash G/M$ for a θ-transverse subgroup Γ, recall the Langlands decomposition $P_\theta = A_\theta S_\theta N_\theta$ where A_θ is the maximal split central torus, S_θ is an almost direct product of a semisimple algebraic subgroup and a compact central torus and N_θ is the unipotent radical of P_θ. The diagonalizable subgroup A_θ acts on the quotient space G/S_θ by translations on the right. The left translation action of Γ on G/S_θ is in general not properly discontinuous (cf. [2], [23]) unless $\theta = \Pi$, in which case S_θ is compact. However the action of Γ is properly discontinuous on the following closed A_θ-invariant subspace ([22, Thm. 9.1]):

$$\tilde{\Omega}_\theta := \{[g] \in G/S_\theta : gP_\theta \in \Lambda_\theta, g w_0 P_{\mathrm{i}(\theta)} \in \Lambda_{\mathrm{i}(\theta)}\}$$

where w_0 is the longest Weyl element. Therefore the quotient space

$$\Omega_\theta := \Gamma \backslash \tilde{\Omega}_\theta$$

is a second countable locally compact Hausdorff space equipped with the right translation action of A_θ which is non-wandering. Denoting by $\Lambda_\theta^{(2)}$ the set of all pairs $(\xi, \eta) \in \Lambda_\theta \times \Lambda_{\mathrm{i}(\theta)}$ in general position, we have (see (5.2)):

$$\Omega_\theta \simeq \Gamma \backslash \left(\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta \right).$$

By a subspace flow on Ω_θ, we mean the action of the subgroup $A_W = \exp W$ for a non-zero linear subspace $W < \mathfrak{a}_\theta$.

The main goal of this paper is to study the ergodic properties of the subspace flows on Ω_θ with respect to Bowen-Margulis-Sullivan measures. The most essential case turns out to be the action of one-parameter subgroups of A_θ which we call directional flows. We first present the ergodic dichotomy for directional flows.

Directional flows. Fixing a non-zero vector $u \in \mathfrak{a}_\theta^+$, we are interested in ergodic properties of the action of the one-parameter subgroup

$$A_u = \{a_{tu} = \exp tu : t \in \mathbb{R}\}$$

on the space Ω_θ. We say that $\xi \in \Lambda_\theta$ is a u-directional conical point if there exists $g \in G$ such that $\xi = gP_\theta$ and $[g]a_{t_i u} \in \Omega_\theta$ belongs to a compact subset for some sequence $t_i \to +\infty$. We denote by Λ_θ^u the set of all u-directional conical points, that is[1],

$$\Lambda_\theta^u := \{gP_\theta \in \Lambda_\theta : [g] \in \Omega_\theta, \limsup_{t\to+\infty}[g]a_{tu} \neq \emptyset\}.$$

See Definition 5.4 and Lemma 5.5 for an equivalent definition of Λ_θ^u given in terms of shadows. It is clear from the definition that Λ_θ^u is an important object in the study of the recurrence of A_u-orbits. Another important player in our ergodic dichotomy is the directional ψ-Poincaré series for a linear form $\psi \in \mathfrak{a}_\theta^*$. To define it, we set $\mu_\theta := p_\theta \circ \mu$ to be the $\mathfrak{a}_\theta$-valued Cartan projection where $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ is the unique projection, invariant under all Weyl elements fixing $\mathfrak{a}_\theta$ pointwise. The u-directional ψ-Poincaré series is of the form

$$\sum_{\gamma \in \Gamma_{u,R}} e^{-\psi(\mu_\theta(\gamma))} \tag{1.1}$$

where $\Gamma_{u,R} := \{\gamma \in \Gamma : \|\mu_\theta(\gamma) - \mathbb{R}u\| < R\}$ for a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_\theta$ and $R > 0$. In considering these objects, it is natural to restrict to those linear forms ψ such that $\psi \circ \mu_\theta : \Gamma \to [-\varepsilon, \infty)$ is a proper map for some $\varepsilon > 0$, which we call (Γ, θ)-proper linear forms. A Borel probability measure ν on $\mathcal{F}_\theta$ is called a (Γ, ψ)-conformal measure if

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi^\theta(e,\gamma))} \quad \text{for all } \gamma \in \Gamma \text{ and } \xi \in \mathcal{F}_\theta$$

where $\gamma_*\nu(D) = \nu(\gamma^{-1}D)$ for any Borel subset $D \subset \mathcal{F}_\theta$ and β_ξ^θ denotes the $\mathfrak{a}_\theta$-valued Busemann map defined in (2.4). For a (Γ, θ)-proper $\psi \in \mathfrak{a}_\theta^*$, a (Γ, ψ)-conformal measure can exist only when $\psi \geq \psi_\Gamma^\theta$ where ψ_Γ^θ is the θ-growth indicator of Γ [22, Thm. 7.1].

Here is our main theorem for directional flows, relating the ergodicity of A_u, the divergence of the u-directional Poincaré series, and the size of conformal measures on u-directional conical sets:

Theorem 1.1 (Ergodic dichotomy for directional flows)**.** *Let Γ be a Zariski dense θ-transverse subgroup of G. Fix a non-zero vector $u \in \mathfrak{a}_\theta^+$ and a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$. Suppose that there exists a pair (ν, ν_{i}) of*

[1] The set $\limsup_{t\to+\infty}[g]a_{tu}$ consists of all limits $\lim_{t_i\to+\infty}[g]a_{t_i u}$.

(Γ,ψ) and $(\Gamma,\psi\circ \mathrm{i})$-conformal measures on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively. Let $\mathsf{m}=\mathsf{m}(\nu,\nu_{\mathrm{i}})$ denote the associated Bowen-Margulis-Sullivan measure on Ω_θ (see (6.2)).

In each of the following complementary cases, claims (1)-(3) are equivalent to each other. If m is u-balanced (Definition 7.1), then (1)-(5) are all equivalent.

The first case:

1. $\max\left(\nu(\Lambda_\theta^u),\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})\right)>0$;
2. $(\Omega_\theta, A_u, \mathsf{m})$ *is completely conservative;*
3. $(\Omega_\theta, A_u, \mathsf{m})$ *is ergodic;*
4. $\sum_{\gamma\in\Gamma_{u,R}} e^{-\psi(\mu_\theta(\gamma))}=\infty$ *for some* $R>0$;
5. $\nu(\Lambda_\theta^u)=1=\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})$.

The second case:

1. $\nu(\Lambda_\theta^u)=0=\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})$;
2. $(\Omega_\theta, A_u, \mathsf{m})$ *is completely dissipative;*
3. $(\Omega_\theta, A_u, \mathsf{m})$ *is non-ergodic;*
4. $\sum_{\gamma\in\Gamma_{u,R}} e^{-\psi(\mu_\theta(\gamma))}<\infty$ *for all* $R>0$;
5. $\nu(\Lambda_\theta^u)=0=\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})$.

We remark that in the first case, (1) is again equivalent to the condition $\max\left(\nu(\Lambda_\theta^u),\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})\right)=1$.

Remark 1.2.
1. When $\theta=\Pi$, or equivalently when S_θ is compact, Theorem 1.1 was obtained for a general Zariski dense discrete subgroup $\Gamma<G$ by Burger-Landesberg-Lee-Oh [9, Thm. 1.4].
2. The u-balanced condition is required only for the implication (4) $\Rightarrow$ (5) in the first case, which takes up the most significant portion of our proof. This condition can be verified for Anosov subgroups, as we will discuss later (Theorem 1.6, Corollary 1.7).
3. By a recent work [6, Prop. 10.1], the existence of a (Γ,ψ)-conformal measure on Λ_θ implies that ψ is (Γ,θ)-proper. Therefore the hypothesis that ψ is (Γ,θ)-proper is unnecessary.
4. When G is of rank one, this is precisely the classical Hopf-Tsuji-Sullivan dichotomy (see [35], [17], [36], [32, Thm. 1.7], etc.).

Our proof of Theorem 1.1 is a generalization of the approach of [9] to a general θ. The main difficulties arise from the non-compactness of S_θ which we overcome using special properties of θ-transverse subgroups such as regularity, anitipodality and the convergence group actions on the limit sets.

Subspace flows. We now turn to the ergodic dichotomy for general subspace flows. Let W be a non-zero linear subspace of $\mathfrak{a}_\theta$ and set $A_W =$

$\{\exp w : w \in W\}$. The W-conical set of Γ is defined as

$$\Lambda_\theta^W = \{gP_\theta \in \mathcal{F}_\theta : [g] \in \Omega_\theta,\ \limsup [g](A_W \cap A^+) \neq \emptyset\}; \tag{1.2}$$

see Definition 9.1 and Lemma 9.5 for an equivalent definition of Λ_θ^W given in terms of shadows. For $R > 0$, we set

$$\Gamma_{W,R} = \{\gamma \in \Gamma : \|\mu_\theta(\gamma) - W\| < R\}. \tag{1.3}$$

Theorem 1.3 (Ergodic dichotomy for subspace flows)**.** *Let $\psi, \nu, \nu_{\mathrm{i}}$ and m be as in Theorem 1.1. Let $W < \mathfrak{a}_\theta$ be a non-zero linear subspace. In the following complementary cases, claims (1)-(3) are equivalent to each other. If m is W-balanced as in Definition 9.6, then (1)-(5) are all equivalent.*

The first case:

1. $\max\left(\nu(\Lambda_\theta^W), \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(W)})\right) > 0$*;*
2. $(\Omega_\theta, A_W, \mathsf{m})$ *is completely conservative;*
3. $(\Omega_\theta, A_W, \mathsf{m})$ *is ergodic;*
4. $\sum_{\gamma \in \Gamma_{W,R}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ *for some* $R > 0$*;*
5. $\nu(\Lambda_\theta^W) = 1 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(W)})$.

The second case:

1. $\nu(\Lambda_\theta^W) = 0 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(W)})$*;*
2. $(\Omega_\theta, A_W, \mathsf{m})$ *is completely dissipative;*
3. $(\Omega_\theta, A_W, \mathsf{m})$ *is non-ergodic;*
4. $\sum_{\gamma \in \Gamma_{W,R}} e^{-\psi(\mu_\theta(\gamma))} < \infty$ *for all* $R > 0$*;*
5. $\nu(\Lambda_\theta^W) = 0 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(W)})$.

Remark 1.4. When W is all of $\mathfrak{a}_\theta$, a similar dichotomy was obtained in ([25], [10], [22]). In this case, the W-balanced condition of m is not required in our proof; see Remark 9.9. Hence we give a different proof of the ergodicity criterion for the A_θ-action [22, Thm. 1.8].

A special feature of a transverse subgroup is that for any (Γ, θ)-proper form ψ, the projection $\tilde{\Omega}_\theta \to \Lambda_\theta^{(2)} \times \mathbb{R}$ given by $(\xi, \eta, v) \mapsto (\xi, \eta, \psi(v))$ induces a $\ker\psi$-bundle structure of Ω_θ over the base space $\Omega_\psi := \Gamma\backslash\Lambda_\theta^{(2)} \times \mathbb{R}$ with the Γ-action given in (6.3). In particular, we have a vector bundle isomorphism

$$\Omega_\theta \simeq \Omega_\psi \times \ker\psi.$$

The $\ker\psi$-bundle $\Omega_\theta \to \Omega_\psi$ plays an important role in our proof of Theorem 1.3. Indeed, such a vector bundle $\Omega_\theta \to \Omega_\psi$ factors through the space $\Omega_{W^\diamond} := \Gamma\backslash\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta/(W \cap \ker\psi)$. Denoting by m' the Radon measure on $\Omega_{W^\diamond}$ such that $\mathsf{m} = \mathsf{m}' \otimes \mathrm{Leb}_{W\cap\ker\psi}$, the $W \cap \ker\psi$-bundle $(\Omega_\theta, \mathsf{m}) \to (\Omega_{W^\diamond}, \mathsf{m}')$ enables us to adapt arguments of Pozzetti-Sambarino [28] in obtaining Theorem 1.3 from the ergodic dichotomy of the directional flow A_u on $\Omega_{W^\diamond}$ for any $u \in W$ such that $\psi(u) > 0$.

Remark 1.5. We remark that the Zariski dense hypothesis on Γ is used to ensure the non-arithmeticity of the Jordan projection of Γ, which implies that the subgroup generated by $p_\theta(\lambda(\Gamma))$ is dense in $\mathfrak{a}_\theta$ [4]. This is a key ingredient in the discussion of transitivity subgroup (Proposition 8.3). In fact, Theorem 1.3 (and hence Theorem 1.1) works for a non-Zariski dense θ-transverse subgroup Γ as well, provided that $p_\theta(\lambda(\Gamma))$ generates a dense subgroup of $\mathfrak{a}_\theta$.

The case of θ-Anosov subgroups. A finitely generated subgroup $\Gamma < G$ is called θ-Anosov if there exist constants $C, C' > 0$ such that for all $\alpha \in \theta$ and $\gamma \in \Gamma$,

$$\alpha(\mu(\gamma)) \geq C|\gamma| - C'$$

where $|\gamma|$ is the word length of γ with respect to a fixed finite generating set of Γ ([24], [16], [19], [15], [7]). By the work of Kapovich-Leeb-Porti [19], a θ-transverse subgroup $\Gamma < G$ is θ-Anosov if Λ_θ is equal to the θ-conical set $\Lambda_\theta^{\mathsf{con}}$ of Γ (see (5.3) for definition). If Γ is a θ-Anosov subgroup, then for each unit vector u in the interior of the limit cone $\mathcal{L}_\theta$, there exists a unique linear form $\psi_u \in \mathfrak{a}_\theta^*$ tangent to the growth indicator ψ_Γ^θ at u and a unique (Γ, ψ_u)-conformal measure ν_u on Λ_θ. Moreover $u \mapsto \psi_u$ and $u \mapsto \nu_u$ give bijections among the directions in $\operatorname{int}\mathcal{L}_\theta$, the space of tangent linear forms to ψ_Γ^θ, and the space of Γ-conformal measures supported on Λ_θ ([26], [34], [22]). Let

$$\mathsf{m}_u = \mathsf{m}(\nu_u, \nu_{\mathrm{i}(u)}) \tag{1.4}$$

denote the Bowen-Margulis-Sullivan measure on Ω_θ associated with the pair $(\nu_u, \nu_{\mathrm{i}(u)})$. We deduce the following codimension dichotomy from Theorem 1.3:

Theorem 1.6 (Codimension dichotomy)**.** *Let $\Gamma < G$ be a Zariski dense θ-Anosov subgroup. Let $u \in \operatorname{int}\mathcal{L}_\theta$ and $W < \mathfrak{a}_\theta$ be a linear subspace containing u. The following are equivalent:*

1. $\operatorname{codim} W \leq 2$ *(resp.* $\operatorname{codim} W \geq 3$*);*
2. $\nu_u(\Lambda_\theta^W) = 1$ *(resp.* $\nu_u(\Lambda_\theta^W) = 0$*);*
3. $(\Omega_\theta, A_W, \mathsf{m}_u)$ *is ergodic and completely conservative (resp. non-ergodic and completely dissipative);*
4. $\sum_{\gamma \in \Gamma_{W,R}} e^{-\psi_u(\mu_\theta(\gamma))} = \infty$ *for some* $R > 0$ *(resp.* $\sum_{\gamma \in \Gamma_{W,R}} e^{-\psi_u(\mu_\theta(\gamma))} < \infty$ *for all* $R > 0$*).*

We can view this dichotomy phenomenon depending on $\operatorname{codim} W$ as consistent with a classical theorem about random walks in $\mathbb{Z}^d$ (or Brownian motions in $\mathbb{R}^d$), which are transient if and only if $d \geq 3$. Since $\operatorname{codim} W = \#\theta - \dim W$, we have the following corollary:

Corollary 1.7 (θ-rank dichotomy)**.** *Let $\Gamma < G$ be a Zariski dense θ-Anosov subgroup and let $u \in \operatorname{int}\mathcal{L}_\theta$. Then $\#\theta \leq 3$ if and only if the directional flow A_u on $(\Omega_\theta, \mathsf{m}_u)$ is ergodic.*

For a θ-Anosov subgroup Γ, Ω_{ψ_u} is a *compact* metric space ([33] and [11, Appendix]), and hence $\Omega_{W^\diamond}$ is a vector bundle over a *compact* space Ω_{ψ_u} with fiber $\mathbb{R}^{\operatorname{codim} W}$. Moreover, we have the following local mixing result due to Sambarino [34, Thm. 2.5.2] (see also [12]) that for any $f_1, f_2 \in C_c(\Omega_{W^\diamond})$,[2]

$$\lim_{t\to\infty} t^{\frac{\operatorname{codim} W}{2}} \int_{\Omega_{W^\diamond}} f_1(x) f_2(x a_{tu}) d\mathsf{m}'_u(x) = \kappa_u \mathsf{m}'_u(f_1)\mathsf{m}'_u(f_2) \tag{1.5}$$

where $\kappa_u > 0$ is a constant depending only on u. In particular, m'_u satisfies the u-balanced hypothesis. The key part of our proof lies in establishing the inequalities (Propositions 10.3 and 10.6) that for all large enough $R > 0$,[3]

$$\left(\int_1^T t^{-\frac{\operatorname{codim} W}{2}} dt\right)^{1/2} \ll \sum_{\substack{\gamma\in\Gamma_{W,R} \\ \psi_u(\mu_\theta(\gamma))\le \delta T}} e^{-\psi_u(\mu_\theta(\gamma))} \ll \int_1^T t^{-\frac{\operatorname{codim} W}{2}} dt$$

for $T > 2$ where $\delta = \psi_u(u) > 0$. Therefore, $\sum_{\gamma\in\Gamma_{W,R}} e^{-\psi_u(\mu_\theta(\gamma))} = \infty$ if and only if $\operatorname{codim} W \le 2$.

Remark 1.8.
(1) When $\theta = \Pi$ and $\dim W = 1$, Theorem 1.6 and hence Corollary 1.7 were obtained in [9]; in this case, $\operatorname{codim} W \le 2$ translates into $\operatorname{rank} G \le 3$.
(2) For a general θ, when $\dim W = 1$ and $\operatorname{codim} W \ne 2$, Sambarino proved the equivalence (1)-(3) of Theorem 1.6 using a different approach [34]; for instance, the directional Poincaré series was not discussed in his work. This was extended by Pozzetti-Sambarino [28] for subspace flows, but still under the hypothesis $\operatorname{codim} W \ne 2$, using an approach similar to [34]. Thus, Theorem 1.6 settles the open case of $\operatorname{codim} W = 2$.
(3) We mention that in ([20], [21], [28]), the sizes of directional/subspace conical limit sets were used as a key input in estimating Hausdorff dimensions of certain subsets of the limit sets.
(4) Theorem 1.6 and Corollary 1.7 are not true for a general θ-transverse subgroup, e.g., there are discrete subgroups in a rank one Lie group which are not of divergence type. Consider a normal subgroup Γ of a non-elementary convex cocompact subgroup Γ_0 of a rank one Lie group G with $\Gamma_0/\Gamma \simeq \mathbb{Z}^d$ for $d \ge 0$. In this case, by a theorem of Rees [31, Thm. 4.7], $d \le 2$ if and only if Γ is of divergence type, i.e., its Poincaré series diverges at the critical exponent of Γ. Using the local mixing result [27, Thm. 4.7] which is of the form as (1.5) with $t^{\operatorname{codim} W/2}$ replaced by $t^{d/2}$ and Corollary 7.14, the approach of our paper gives an alternative proof of Rees' theorem.

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[2]The notation $C_c(X)$ for a topological space X means the space of all continuous functions on X with compact supports.

[3]The notation $f(T) \ll g(T)$ means that there is a constant $c > 0$ such that $f(T) \le cg(T)$ for all T in a given range.

(5) Corollaries 7.14 and 9.10 reduce the divergence of the u-directional Poincaré series to the local mixing rate for the A_u-flow. For example, we expect the local mixing rate of relatively θ-Anosov subgroups to be same as that of Anosov subgroups, which would then imply Theorem 1.6 and Corollary 1.7 for those subgroups.

Examples of ergodic actions on $\Gamma\backslash G/S_\theta$. By the work of Guéritaud-Guichard-Kassel-Wienhard [15], there are examples of Borel Anosov subgroups which act properly discontinuously on G/S_θ for some $\theta \neq \Pi$ ([15, Coro. 1.10, Coro. 1.11]), in which case our rank dichotomy theorem can be stated for the one-parameter subgroup action on $\Gamma\backslash G/S_\theta$. We discuss one example where $G = \mathrm{SL}_d(\mathbb{R})$, $d \geq 3$. For $2 \leq k \leq d-2$, let H_k be the block diagonal subgroup $\begin{pmatrix} I_k & \\ & \mathrm{SL}_{d-k}(\mathbb{R}) \end{pmatrix} \simeq \mathrm{SL}_{d-k}(\mathbb{R})$ where I_k denotes the $(k \times k)$-identity matrix. Set $\alpha_i(\mathrm{diag}(v_1, \cdots, v_d)) = v_i - v_{i+1}$ for $1 \leq i \leq d-1$; so $\Pi = \{\alpha_i : 1 \leq i \leq d-1\}$ is the set of all simple roots for G. For $\theta = \{\alpha_1, \cdots, \alpha_k\}$, we have $S_\theta = H_k$. Let $\Gamma < G$ be a Π-Anosov subgroup. Then Γ acts properly discontinuously on $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_{d-k}(\mathbb{R})$ by [15, Coro. 1.9, Coro. 1.10]. Hence any Radon measure on Ω_θ can be considered as a Radon measure on $\Gamma\backslash\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_{d-k}(\mathbb{R})$. Then Theorem 1.6 implies the following:

Corollary 1.9. *Let $\Gamma < \mathrm{SL}_d(\mathbb{R})$ be a Zariski dense Π-Anosov subgroup (e.g., Hitchin subgroups), $2 \leq k \leq d-2$ and $\theta = \{\alpha_1, \cdots, \alpha_k\}$. Let $u \in \mathrm{int}\,\mathcal{L}_\theta$ and m_u be as in (1.4). We have $k = 2, 3$ if and only if the A_u-action on $(\Gamma\backslash\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_{d-k}(\mathbb{R}), \mathsf{m}_u)$ is ergodic.*

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2. Preliminaries

Throughout the paper, let G be a connected semisimple real algebraic group. In this section, we review some basic facts about the Lie group structure of G and the notion of convergence of elements of G to boundaries, following [22, Sec. 2] to which we refer for more details.

Let $P < G$ be a minimal parabolic subgroup with a fixed Langlands decomposition $P = MAN$ where A is a maximal real split torus of G, M is the maximal compact subgroup of P commuting with A and N is the unipotent radical of P. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebras of G and A. Fix a positive Weyl chamber $\mathfrak{a}^+ < \mathfrak{a}$ so that $\log N$ consists of positive root subspaces and set $A^+ = \exp \mathfrak{a}^+$. We fix a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. We denote by

$$\mu : G \to \mathfrak{a}^+$$

the Cartan projection defined by the condition $g \in K \exp \mu(g) K$ for $g \in G$. Let $X = G/K$ be the associated Riemannian symmetric space, and set $o = [K] \in X$. Fix a K-invariant norm $\| \cdot \|$ on $\mathfrak{g}$ and a Riemanian metric d on X, induced from the Killing form on $\mathfrak{g}$. The Weyl group $\mathcal{W}$ is given by $N_K(A)/C_K(A)$, where $N_K(A)$ and $C_K(A)$ denote the normalizer and the centralizer of A in K respectively. Oftentimes,we will identify $\mathcal{W}$ with the chosen set of representatives from $N_K(A)$, and hence treat $\mathcal{W}$ as a subset of G.

Lemma 2.1. [3, Lem. 4.6] *For any compact subset $Q \subset G$, there exists $C = C(Q) > 0$ such that for all $g \in G$,*

$$\sup_{q_1, q_2 \in Q} \|\mu(q_1 g q_2) - \mu(g)\| \leq C.$$

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$ denote the set of all roots, $\Phi^+ \subset \Phi$ the set of all positive roots, and $\Pi \subset \Phi^+$ the set of all simple roots. Fix an element $w_0 \in K$ of order 2 in the normalizer of A representing the longest Weyl element so that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The map

$$\mathrm{i} = -\mathrm{Ad}_{w_0} : \mathfrak{a} \to \mathfrak{a} \tag{2.1}$$

is called the opposition involution. It induces an involution $\Phi \to \Phi$ preserving Π, for which we use the same notation i, such that $\mathrm{i}(\alpha) \circ \mathrm{Ad}_{w_0} = -\alpha$ for all $\alpha \in \Phi$. We have $\mu(g^{-1}) = \mathrm{i}(\mu(g))$ for all $g \in G$.

Henceforth, we fix a non-empty subset $\theta \subset \Pi$. Let P_θ denote a standard parabolic subgroup of G corresponding to θ; that is, P_θ is generated by MA and all root subgroups U_α, where α ranges over all positive roots and any negative root which is a $\mathbb{Z}$-linear combination of $\Pi - \theta$. Hence $P_\Pi = P$. Let

$$\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha, \qquad \mathfrak{a}_\theta^+ = \mathfrak{a}_\theta \cap \mathfrak{a}^+,$$

$$A_\theta = \exp \mathfrak{a}_\theta, \quad \text{and} \quad A_\theta^+ = \exp \mathfrak{a}_\theta^+.$$

Let $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ denote the projection invariant under $w \in \mathcal{W}$ fixing $\mathfrak{a}_\theta$ pointwise. We also write

$$\mu_\theta := p_\theta \circ \mu : G \to \mathfrak{a}_\theta^+.$$

Definition 2.2. For a discrete subgroup $\Gamma < G$, its θ-limit cone $\mathcal{L}_\theta = \mathcal{L}_\theta(\Gamma)$ is defined as the the asymptotic cone of $\mu_\theta(\Gamma)$ in $\mathfrak{a}_\theta$, that is, $u \in \mathcal{L}_\theta$ if and only if $u = \lim_{i\to\infty} t_i \mu_\theta(\gamma_i)$ for some sequences $t_i \to 0$ and $\gamma_i \in \Gamma$. If Γ is Zariski dense, $\mathcal{L}_\theta$ is a convex cone with non-empty interior by [3]. Setting $\mathcal{L} = \mathcal{L}_\Pi$, we have $p_\theta(\mathcal{L}) = \mathcal{L}_\theta$.

We have the Levi-decomposition $P_\theta = L_\theta N_\theta$ where L_θ is the centralizer of A_θ and $N_\theta = R_u(P_\theta)$ is the unipotent radical of P_θ. We set $M_\theta = K \cap P_\theta \subset L_\theta$. We may then write $L_\theta = A_\theta S_\theta$ where S_θ is an almost direct product of a connected semisimple real algebraic subgroup and a compact center.

Letting $B_\theta = S_\theta \cap A$ and $B_\theta^+ = \{b \in B_\theta : \alpha(\log b) \geq 0 \text{ for all } \alpha \in \Pi - \theta\}$, we have the Cartan decomposition of S_θ:

$$S_\theta = M_\theta B_\theta^+ M_\theta.$$

Note that $A = A_\theta B_\theta$ and $A^+ \subset A_\theta^+ B_\theta^+$. The space $\mathfrak{a}_\theta^* = \mathrm{Hom}(\mathfrak{a}_\theta, \mathbb{R})$ can be identified with the subspace of $\mathfrak{a}^*$ which is p_θ-invariant: $\mathfrak{a}_\theta^* = \{\psi \in \mathfrak{a}^* : \psi \circ p_\theta = \psi\}$; so for $\theta_1 \subset \theta_2$, we have $\mathfrak{a}_{\theta_1}^* \subset \mathfrak{a}_{\theta_2}^*$.

The θ-boundary $\mathcal{F}_\theta$ and convergence to $\mathcal{F}_\theta$. We set

$$\mathcal{F}_\theta = G/P_\theta \quad \text{and} \quad \mathcal{F} = G/P.$$

Let

$$\pi_\theta : \mathcal{F} \to \mathcal{F}_\theta$$

denote the canonical projection map given by $gP \mapsto gP_\theta$, $g \in G$. We set

$$\xi_\theta = [P_\theta] \in \mathcal{F}_\theta. \tag{2.2}$$

By the Iwasawa decomposition $G = KP = KAN$, the subgroup K acts transitively on $\mathcal{F}_\theta$, and hence $\mathcal{F}_\theta \simeq K/M_\theta$.

We consider the following notion of convergence of a sequence in G to an element of $\mathcal{F}_\theta$. For a sequence $g_i \in G$, we say $g_i \to \infty$ θ-regularly if $\min_{\alpha\in\theta} \alpha(\mu(g_i)) \to \infty$ as $i \to \infty$.

Definition 2.3. For a sequence $g_i \in G$ and $\xi \in \mathcal{F}_\theta$, we write $\lim_{i\to\infty} g_i = \lim_{i\to\infty} g_i o = \xi$ and say g_i (or $g_i o \in X$) converges to ξ if

- $g_i \to \infty$ θ-regularly; and
- $\lim_{i\to\infty} \kappa_i \xi_\theta = \xi$ in $\mathcal{F}_\theta$ for some $\kappa_i \in K$ such that $g_i \in \kappa_i A^+ K$.

Definition 2.4. The θ-limit set of a discrete subgroup Γ can be defined as follows:

$$\Lambda_\theta = \Lambda_\theta(\Gamma) := \{\xi \in \mathcal{F}_\theta : \xi = \lim_{i\to\infty} \gamma_i,\ \gamma_i \in \Gamma\}$$

where $\lim_{i\to\infty} \gamma_i$ is defined as in Definition 2.3. If Γ is Zariski dense, this is the unique Γ-minimal subset of $\mathcal{F}_\theta$ ([3], [30]). If we set $\Lambda = \Lambda_\Pi$, then $\pi_\theta(\Lambda) = \Lambda_\theta$.

Lemma 2.5 ([22, Lem. 2.6-7], see also [26] for $\theta = \Pi$)**.** *Let $g_i \in G$ be an infinite sequence.*

(1) *If g_i converges to $\xi \in \mathcal{F}_\theta$ and $p_i \in X$ is a bounded sequence, then*

$$\lim_{i\to\infty} g_i p_i = \xi.$$

(2) *If a sequence $a_i \to \infty$ in A^+ θ-regularly, and $g_i \to g \in G$, then for any $p \in X$, we have*

$$\lim_{i\to\infty} g_i a_i p = g\xi_\theta.$$

Jordan projections. A loxodromic element $g \in G$ is of the form $g = ha_g mh^{-1}$ for $h \in G$, $a_g \in \operatorname{int} A^+$ and $m \in M$; moreover $a_g \in \operatorname{int} A^+$ is uniquely determined. We set

$$\lambda(g) := \log a_g \in \mathfrak{a}^+ \quad \text{and} \quad y_g := hP \in \mathcal{F}, \tag{2.3}$$

called the Jordan projection and the attracting fixed point of g respectively.

Theorem 2.6. [4] *For any Zariski dense subgroup $\Gamma < G$, the subgroup generated by $\{\lambda(\gamma) : \gamma$ is a loxodromic element of $\Gamma\}$ is dense in $\mathfrak{a}$.*

Busemann maps. The $\mathfrak{a}$-valued Busemann map $\beta : \mathcal{F} \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g,h) := \sigma(g^{-1},\xi) - \sigma(h^{-1},\xi)$$

where $\sigma(g^{-1},\xi) \in \mathfrak{a}$ is the unique element such that $g^{-1}k \in K \exp(\sigma(g^{-1},\xi))N$ for any $k \in K$ with $\xi = kP$. For $(\xi, g, h) \in \mathcal{F}_\theta \times G \times G$, we define

$$\beta^\theta_\xi(g,h) := p_\theta(\beta_{\xi_0}(g,h)) \quad \text{for } \xi_0 \in \pi_\theta^{-1}(\xi); \tag{2.4}$$

this is well-defined independent of the choice of ξ_0 [30, Lem. 6.1]. For $p, q \in X$ and $\xi \in \mathcal{F}_\theta$, we set $\beta^\theta_\xi(p,q) := \beta^\theta_\xi(g,h)$ where $g, h \in G$ satisfies $go = p$ and $ho = q$. It is easy to check this is well-defined.

Points in general position. Let $\check{P}_\theta$ be the standard parabolic subgroup of G opposite to P_θ such that $P_\theta \cap \check{P}_\theta = L_\theta$. We have $\check{P}_\theta = w_0 P_{\mathrm{i}(\theta)} w_0^{-1}$ and hence

$$\mathcal{F}_{\mathrm{i}(\theta)} = G/\check{P}_\theta.$$

For $g \in G$, we set

$$g^+_\theta := gP_\theta \quad \text{and} \quad g^-_\theta := gw_0P_{\mathrm{i}(\theta)};$$

as we fix θ in the entire paper, we write $g^\pm = g^\pm_\theta$ for simplicity when there is no room for confusion. Hence for the identity $e \in G$, $(e^+, e^-) = (P_\theta, \check{P}_\theta) = (\xi_\theta, w_0\xi_{\mathrm{i}(\theta)})$, where ξ_θ is as in (2.2). The G-orbit of (e^+, e^-) is the unique open G-orbit in $G/P_\theta \times G/\check{P}_\theta$ under the diagonal G-action. We set

$$\mathcal{F}^{(2)}_\theta = \{(g^+_\theta, g^-_\theta) : g \in G\}. \tag{2.5}$$

Two elements $\xi \in \mathcal{F}_\theta$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ are said to be in general position if $(\xi, \eta) \in \mathcal{F}^{(2)}_\theta$. Since $\check{P}_\theta = L_\theta \check{N}_\theta$ where $\check{N}_\theta$ is the unipotent radical of $\check{P}_\theta$, we have

$$(g^+_\theta, e^-_\theta) \in \mathcal{F}^{(2)}_\theta \quad \text{if and only if} \quad g \in \check{N}_\theta P_\theta. \tag{2.6}$$

The following lemma will be useful:

Lemma 2.7. [22, Coro. 2.5] *If $w \in \mathcal{W}$ is such that $mw \in \check{N}_\theta P_\theta$ for some $m \in M_\theta$, then $w \in M_\theta$. In particular, if $(w\xi_\theta, w_0\xi_{\mathrm{i}(\theta)}) = (w^+_\theta, e^-_\theta) \in \mathcal{F}^{(2)}_\theta$, then $w \in M_\theta$.*

Gromov products. The map $g \mapsto (g^+, g^-)$ for $g \in G$ induces a homeomorphism $G/L_\theta \simeq \mathcal{F}_\theta^{(2)}$. For $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$, we define the θ-Gromov product as

$$\mathcal{G}^\theta(\xi, \eta) = \beta_\xi^\theta(e, g) + \mathrm{i}(\beta_\eta^{\mathrm{i}(\theta)}(e, g))$$

where $g \in G$ satisfies $(g^+, g^-) = (\xi, \eta)$. This does not depend on the choice of g [22, Lem. 9.13].

Although the Gromov product is defined differently in [7], it coincides with ours (see [26, Lem. 3.11, Rmk. 3.13]); hence we have:

Proposition 2.8. [7, Prop. 8.12] *There exists $c > 1$ and $c' > 0$ such that for all $g \in G$,*

$$c^{-1}\|\mathcal{G}^\theta(g^+, g^-)\| \leq d(o, gL_\theta o) \leq c\|\mathcal{G}^\theta(g^+, g^-)\| + c'.$$

3. Continuity of shadows

The notion of shadows plays a crucial role in studying recurrence of diagonal flows. In this section, we recall the definition of θ-shadows and prove some basic properties. In particular, we prove that shadows vary continuously on viewpoints, which are of independent interests.

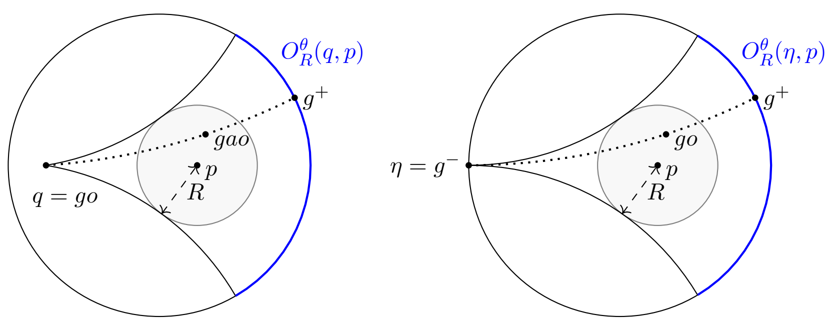

Figure 1. Shadows

For $p \in X$ and $R > 0$, let $B(p, R)$ denote the metric ball $\{x \in X : d(x, p) < R\}$. For $q \in X$, the θ-shadow $O_R^\theta(q, p) \subset \mathcal{F}_\theta$ of $B(p, R)$ viewed from q is defined as

$$O_R^\theta(q, p) = \{gP_\theta \in \mathcal{F}_\theta : g \in G,\ go = q,\ gA^+o \cap B(p, R) \neq \emptyset\} \tag{3.1}$$

We also define the θ-shadow $O_R^\theta(\eta, p) \subset \mathcal{F}_\theta$ viewed from $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ as follows:

$$O_R^\theta(\eta, p) = \{gP_\theta \in \mathcal{F}_\theta : g \in G,\ gw_0P_{\mathrm{i}(\theta)} = \eta,\ go \in B(p, R)\}.$$

For any $\tilde{\eta} \in \pi_{\mathrm{i}(\theta)}^{-1}(\eta)$, we have

$$O_R^\theta(q, p) = \pi_\theta(O_R^\Pi(q, p)) \quad \text{and} \quad O_R^\theta(\eta, p) = \pi_\theta(O_R^\Pi(\tilde{\eta}, p)). \tag{3.2}$$

Note that for all $g \in G$ and $\eta \in X \cup \mathcal{F}_{\mathrm{i}(\theta)}$,

$$gO_R^\theta(\eta, p) = O_R^\theta(g\eta, gp). \tag{3.3}$$

We define the $\mathfrak{a}_\theta$-valued distance $\underline{a}_\theta : X \times X \to \mathfrak{a}_\theta$ by

$$\underline{a}_\theta(q,p) := \mu_\theta(g^{-1}h)$$

where $q = go$ and $p = ho$ for $g, h \in G$. The following was shown for $\theta = \Pi$ in [26, Lem. 5.7] which directly implies the statement for general θ by (3.2).

Lemma 3.1. *There exists $\kappa > 0$ such that for any $q, p \in X$ and $R > 0$, we have*

$$\sup_{\xi \in O_R^\theta(q,p)} \|\beta_\xi^\theta(q,p) - \underline{a}_\theta(q,p)\| \le \kappa R.$$

Lemma 3.2. *For any compact subset $Q \subset G$ and $R > 0$, we have that for any $h \in G$ and $g \in Q$,*

$$O_R^\theta(go, ho) \subset O_{R+D_Q}^\theta(o, ho) \quad \textit{and} \quad O_R^\theta(hgo, o) \subset O_{R+D_Q}^\theta(ho, o)$$

where $D_Q := \max_{g \in Q} d(go, o)$.

Proof. Note that $d(ao, pao) \le d(o, po)$ for all $a \in A^+$ and $p \in P$. Let $g \in Q$ and $\xi \in O_R^\theta(go, ho)$. Then for some $k \in K$ and $a \in A^+$, we have $\xi = gkP_\theta$ and $d(gkao, ho) < R$. Write $gk = \ell p \in KP$ for $\ell \in K$ and $p \in P$ by the Iwasawa decomposition $G = KP$. Since $d(\ell ao, \ell pao) \le D_Q$, we have $d(\ell ao, ho) \le d(\ell ao, \ell pao) + d(gkao, ho) < D_Q + R$. Therefore $\xi \in O_{R+D_Q}^\theta(o, ho)$, proving the first claim. The second claim follows from the first by (3.3). $\square$

Lemma 3.3. *Let $p \in X$, $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ and $r > 0$. If a sequence $\eta_i \in \mathcal{F}_{\mathrm{i}(\theta)}$ converges to $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$, then for any $0 < \varepsilon < r$, we have*

$$O_{r-\varepsilon}^\theta(\eta_i, p) \subset O_r^\theta(\eta, p) \subset O_{r+\varepsilon}^\theta(\eta_i, p) \quad \textit{for all large } i \ge 1. \tag{3.4}$$

Proof. We first prove the second inclusion. Let $g \in G$ be such that $g^+ \in O_r^\theta(\eta, p)$, $g^- = \eta$ and $d(go, p) < r$. Since $\eta_i \to \eta$, we have $(g^+, \eta_i) \in \mathcal{F}_\theta^{(2)}$ for all large $i \ge 1$, and hence $(g^+, \eta_i) = (h_i^+, h_i^-)$ for some $h_i \in G$. In particular, $g = h_i q_i n_i$ for $q_i n_i \in L_\theta N_\theta = P_\theta$. By replacing h_i with $h_i q_i$, we may assume that $g = h_i n_i$. Since $h_i^- \to g^-$, we have $n_i^- \to e^-$, and hence $n_i \to e$ as $i \to \infty$. Therefore for all $i \ge 1$ large enough so that $d(n_i o, o) \le \varepsilon$, we have $d(h_i o, p) \le d(h_i o, h_i n_i o) + d(go, p) < \varepsilon + r$, and hence $g^+ = h_i^+ \in O_{r+\varepsilon}^\theta(\eta_i, p)$.

To prove the first inclusion, fix $k_i \in \mathrm{Stab}_G(p)$ such that $k_i \eta_i = \eta$ for each $i \ge 1$. After passing to a subsequence, we may assume that the sequence k_i converges to some $k \in \mathrm{Stab}_G(p)$ as $i \to \infty$. Since $\eta_i \to \eta$, we have $k\eta = \eta$. In particular, the sequence $k_i \eta$ converges to η. Applying the second inclusion of (3.4) to a sequence $k_i \eta$, we have

$$O_{r-\varepsilon}^\theta(k_i \eta_i, p) = O_{r-\varepsilon}^\theta(\eta, p) \subset O_r^\theta(k_i \eta, p) \quad \text{for all large } i \ge 1.$$

Since $k_i \in \mathrm{Stab}_G(p)$, it follows from (3.3) that $O_{r-\varepsilon}^\theta(k_i \eta_i, p) = k_i O_{r-\varepsilon}^\theta(\eta_i, p)$ and $O_r^\theta(k_i \eta, p) = k_i O_r^\theta(\eta, p)$. This proves the first inclusion. $\square$

We show that for a fixed $p \in X$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$, shadows $O^\theta_r(\eta, p)$ vary continuously on a small neighborhood of η in $G \cup \mathcal{F}_{\mathrm{i}(\theta)}$ (see [26, Lem. 5.6] for $\theta = \Pi$):

Proposition 3.4 (Continuity of shadows on viewpoints)**.** *Let $p \in X$, $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ and $r > 0$. If a sequence $q_i \in X$ converges to η as $i \to \infty$, then for any $0 < \varepsilon < r$, we have*

$$O^\theta_{r-\varepsilon}(q_i, p) \subset O^\theta_r(\eta, p) \subset O^\theta_{r+\varepsilon}(q_i, p) \quad \text{for all large } i \geq 1. \tag{3.5}$$

Proof. We first prove the second inclusion which requires more delicate arguments. By (3.3) and the fact that K acts transtively on $\mathcal{F}_{\mathrm{i}(\theta)}$, we may assume without loss of generality that $\eta = P_{\mathrm{i}(\theta)} = w_0^-$ and $p = o$. Write $q_i = k_i' a_i o$ with $k_i' \in K$ and $a_i \in A^+$ using Cartan decomposition. Since $q_i \to w_0^-$, we have $k_i' w_0^- \to w_0^-$ and $a_i \to \infty$ $\mathrm{i}(\theta)$-regularly.

By Lemma 3.3, we may assume $k_i' = e$ without loss of generality. By (3.2), the claim follows if we replace θ by any subset containing θ. Therefore we may assume without loss of generality that $\alpha(\log a_i)$ is uniformly bounded for all $\alpha \in \Pi - \mathrm{i}(\theta)$.

Let $\xi \in O^\theta_r(P_{\mathrm{i}(\theta)}, o)$, i.e., $\xi = hP_\theta$ for some $h \in G$ such that $d(ho, o) < r$ and $hw_0 P_{\mathrm{i}(\theta)} = P_{\mathrm{i}(\theta)}$. Since $P_{\mathrm{i}(\theta)} = PM_{\mathrm{i}(\theta)}$ and $w_0^{-1} M_{i(\theta)} w_0 = M_\theta$, we may assume $hw_0 \in P$ by replacing h with hm for some $m \in M_\theta$. We need to show that for some $p_i \in P_\theta$ such that $hp_i o = a_i o$, $d(p_i A^+ o, o) < \varepsilon$; this then implies $d(hp_i A^+ o, o) < r + \varepsilon$, and hence $\xi \in O^\theta_{r+\varepsilon}(a_i o, o)$.

We start by writing

$$a_i^{-1} h = k_i \tilde{a}_i n_i \in KAN, \;\; \tilde{a}_i = c_i d_i \in A_\theta B_\theta \text{ and } n_i = u_i v_i \in (S_\theta \cap N) N_\theta.$$

As $hw_0 \in P$ and $a_i \in A^+$, the sequence $a_i^{-1} h w_0 a_i$ is bounded. Since

$$a_i^{-1} h w_0 a_i = (k_i w_0)(w_0^{-1} \tilde{a}_i w_0 a_i)(a_i^{-1} w_0^{-1} n_i w_0 a_i) \in KAN^+,$$

it follows that both sequences $w_0^{-1} \tilde{a}_i w_0 a_i$ and $a_i^{-1} w_0^{-1} n_i w_0 a_i$ are bounded.

Since $w_0^{-1} n_i w_0 = (w_0^{-1} u_i w_0)(w_0^{-1} v_i w_0) \in S_{\mathrm{i}(\theta)} N^+_{\mathrm{i}(\theta)}$ and $a_i \in A^+$ with $a_i \to \infty$ $\mathrm{i}(\theta)$-regularly, the boundedness of $a_i^{-1} w_0^{-1} n_i w_0 a_i$ implies that $v_i \to e$ as $i \to \infty$ and u_i is bounded. On the other hand, the boundedness of $w_0^{-1} \tilde{a}_i w_0 a_i$ implies that $\tilde{a}_i \in w_0 a_i^{-1} w_0^{-1} A_C$ for some $C > 0$. As $a_i \to \infty$ $\mathrm{i}(\theta)$-regularly, it follows that $c_i \in A^+_\theta$ and $c_i \to \infty$ θ-regularly. Moreover, since $\max_{\alpha \in \Pi - \mathrm{i}(\theta)} \alpha(\log a_i)$ is uniformly bounded, the sequence d_i is bounded.

As $d_i u_i \in S_\theta$, we may write its Cartan decomposition $d_i u_i = m_i b_i m_i' \in M_\theta B^+_\theta M_\theta$. Since $c_i \to \infty$ θ-regularly and $d_i u_i$, and hence $b_i \in B^+_\theta$, is uniformly bounded, we have $c_i b_i \in A^+$ for all large $i \geq 1$. Set $p_i = (m_i^{-1} \tilde{a}_i n_i)^{-1} \in P_\theta$. Recalling $a_i^{-1} h = k_i \tilde{a}_i n_i$, we have $hp_i o = h n_i^{-1} \tilde{a}_i^{-1} o = a_i o$. Moreover, we have

$$p_i (c_i b_i) o = n_i^{-1} \tilde{a}_i^{-1} m_i c_i b_i m_i' o = n_i^{-1} \tilde{a}_i^{-1} c_i d_i u_i o = v_i^{-1} o$$

using the commutativity of M_θ and A_θ as well as the identity $m_i b_i m_i' = d_i u_i$. Since $v_i \to e$, we have $d(p_i (c_i b_i) o, o) \to 0$. This proves the second inclusion.

We now prove the first inclusion. Similarly, as in the previous case, we may assume that $q_i = a_i o$ for $a_i \in A^+$ and $\eta = P_{\mathrm{i}(\theta)}$. Let $\eta_i \in O^\theta_{r-\varepsilon}(a_i o, o)$, i.e., $\eta_i = a_i k_i P_\theta$ and $d(a_i k_i b_i o, o) < r - \varepsilon$ for some $k_i \in K$ and $b_i \in A^+$. Set $g_i = a_i k_i b_i$, which is a bounded sequence. We will find $n_i \in N_\theta$ such that $(g_i n_i)^- = P_{\mathrm{i}(\theta)}$ and $d(g_i n_i o, o) < r$ from which $\eta_i \in O^\theta_r(\eta, o)$ follows.

We may assume that g_i converges to some $g \in G$. Since $a_i \to \infty$ $\mathrm{i}(\theta)$-regularly, the boundedness of $g_i = a_i k_i b_i$ together with Lemma 2.1 implies that $b_i \to \infty$ θ-regularly. Since $a_i k_i \to P_{\mathrm{i}(\theta)}$ and $a_i k_i = g_i w_0 (w_0^{-1} b_i^{-1} w_0) w_0^{-1} \to g w_0 P_{\mathrm{i}(\theta)}$ as $i \to \infty$ by Lemma 2.5, we have

$$g w_0 P_{\mathrm{i}(\theta)} = P_{\mathrm{i}(\theta)}.$$

On the other hand, as $i \to \infty$, we have

$$g_i (P_\theta, w_0 P_{\mathrm{i}(\theta)}) \to g(P_\theta, w_0 P_{\mathrm{i}(\theta)}) = (g P_\theta, P_{\mathrm{i}(\theta)}).$$

Hence for all large $i \geq 1$, $g_i P_\theta$ is in general position with $P_{\mathrm{i}(\theta)}$ and thus we have a sequence $h_i \in G$ such that

$$(g_i P_\theta, P_{\mathrm{i}(\theta)}) = h_i (P_\theta, w_0 P_{\mathrm{i}(\theta)}).$$

As $g_i P_\theta = h_i P_\theta$, we write $h_i = g_i n_i \ell_i$ for some $n_i \in N_\theta$ and $\ell_i \in L_\theta$. Note that $(g_i n_i)^- = h_i^- = P_{\mathrm{i}(\theta)}$. We now claim that $n_i \to e$, from which $d(g_i n_i o, o) \leq d(g_i n_i o, g_i o) + d(g_i o, o) < r$ follows for all large i.

Since $h_i (P_\theta, w_0 P_{\mathrm{i}(\theta)}) = (g_i P_\theta, P_{\mathrm{i}(\theta)}) \to (g P_\theta, P_{\mathrm{i}(\theta)}) = g(P_\theta, w_0 P_{\mathrm{i}(\theta)})$, we have $h_i L_\theta = g_i n_i L_\theta \to g L_\theta$. Since $g_i \to g$ and $n_i \in N_\theta$, we have $n_i \to e$ as $i \to \infty$. This finishes the proof. $\square$

Lemma 3.5. *Let $S > 0$. For any sequence $g_i \to \infty$ in G θ-regularly, the product $O^\theta_S(o, g_i o) \times O^{\mathrm{i}(\theta)}_S(g_i o, o)$ is precompact in $\mathcal{F}^{(2)}_\theta$ for all sufficiently large $i \geq 1$.*

Proof. Consider an infinite sequence $(\xi_i, \eta_i) \in O^\theta_S(o, g_i o) \times O^{\mathrm{i}(\theta)}_S(g_i o, o)$. By the θ-regularity of $g_i \to \infty$, we have $g_i o \to \xi$ as $i \to \infty$ for some $\xi \in \mathcal{F}_\theta$, after passing to a subsequence. For each i, we write $\xi_i = k_i P_\theta$ for $k_i \in K$ such that $d(k_i a_i o, g_i o) < S$ for some $a_i \in A^+$. In particular, $a_i \to \infty$ θ-regularly. After passing to a subsequence, we may assume that $k_i \to k \in K$ so that $k_i a_i o \to k P_\theta$ as $i \to \infty$. On the other hand, the boundedness of $d(k_i a_i o, g_i o) < S$ implies that $k_i a_i o \to \xi$ by Lemma 2.5. Therefore, $\xi = k P_\theta = \lim_i \xi_i$. By passing to a subsequence, we may assume that $\eta_i \to \eta$ for some $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$. Since $g_i o \to \xi$, and $\eta_i \in O^{\mathrm{i}(\theta)}_S(g_i o, o)$, it follows from Proposition 3.4 that $\eta \in O^{\mathrm{i}(\theta)}_{2S}(\xi, o)$. In particular, $(\xi, \eta) \in \mathcal{F}^{(2)}_\theta$. $\square$

4. Growth indicators and conformal measures on $\mathcal{F}_\theta$

In this section, we review the notion of θ-growth indicators and discuss their influence on conformal measures on the θ-boundary.

Let $\Gamma < G$ be a Zariski dense discrete subgroup. We say that Γ is θ-discrete if the restriction $\mu_\theta|_\Gamma : \Gamma \to \mathfrak{a}^+_\theta$ is a proper map. Observe that

Γ is θ-discrete if and only if the counting measure on $\mu_\theta(\Gamma)$ weighted with multiplicity is locally finite i.e., finite on compact subsets. Following Quint's notion of growth indicators [29], we have introduced the following in [22]:

Definition 4.1 (θ-growth indicator)**.** For a θ-discrete subgroup $\Gamma < G$, we define the θ-growth indicator $\psi_\Gamma^\theta : \mathfrak{a}_\theta \to [-\infty, \infty]$ as follows: if $u \in \mathfrak{a}_\theta$ is non-zero,

$$\psi_\Gamma^\theta(u) = \|u\| \inf_{u \in \mathcal{C}} \tau_{\mathcal{C}}^\theta \tag{4.1}$$

where $\mathcal{C} \subset \mathfrak{a}_\theta$ ranges over all open cones containing u, and $\psi_\Gamma^\theta(0) = 0$. Here $-\infty \le \tau_{\mathcal{C}}^\theta \le \infty$ is the abscissa of convergence of $s \mapsto \sum_{\gamma \in \Gamma, \mu_\theta(\gamma) \in \mathcal{C}} e^{-s\|\mu_\theta(\gamma)\|}$.

We showed ([22, Thm. 3.3]):

- $\psi_\Gamma^\theta < \infty$;
- ψ_Γ^θ is upper semi-continuous and concave;
- $\mathcal{L}_\theta = \{\psi_\Gamma^\theta \ge 0\} = \{\psi_\Gamma^\theta > -\infty\}$, and $\psi_\Gamma^\theta > 0$ on $\operatorname{int} \mathcal{L}_\theta$.

Let $\psi \in \mathfrak{a}_\theta^*$. Recall that a (Γ, ψ)-conformal measure ν is a Borel probability measure on $\mathcal{F}_\theta$ such that

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi^\theta(e,\gamma))} \quad \text{for all } \gamma \in \Gamma \text{ and } \xi \in \mathcal{F}_\theta.$$

A linear form $\psi \in \mathfrak{a}_\theta^*$ is said to be tangent to ψ_Γ^θ at $v \in \mathfrak{a}_\theta - \{0\}$ if $\psi \ge \psi_\Gamma^\theta$ and $\psi(v) = \psi_\Gamma^\theta(v)$.

Proposition 4.2 ([30, Thm. 8.4], [22, Prop. 5.8])**.** *For any $\psi \in \mathfrak{a}_\theta^*$ which is tangent to ψ_Γ^θ at an interior direction of $\mathfrak{a}_\theta^+$, there exists a (Γ, ψ)-conformal measure supported on Λ_θ.*

Recall that Γ is called *θ-transverse*, if

- Γ is *θ-regular*, i.e., $\liminf_{\gamma \in \Gamma} \alpha(\mu(\gamma)) = \infty$ for all $\alpha \in \theta$; and
- Γ is *θ-antipodal*, i.e., any distinct $\xi, \eta \in \Lambda_{\theta \cup \mathrm{i}(\theta)}$ are in general position.

Recall also that $\psi \in \mathfrak{a}_\theta^*$ is (Γ, θ)-proper if $\psi \circ \mu_\theta|_\Gamma$ is a proper map into $[-\varepsilon, \infty)$ for some $\varepsilon > 0$.

Theorem 4.3 ([30, Thm. 8.1] for $\theta = \Pi$, [22, Thm. 7.1] in general)**.** *Let Γ be a Zariski dense θ-transverse subgroup of G. If there exists a (Γ, ψ)-conformal measure ν on $\mathcal{F}_\theta$ for a (Γ, θ)-proper $\psi \in \mathfrak{a}_\theta^*$, then*

$$\psi \ge \psi_\Gamma^\theta.$$

Moreover, if $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$ in addition, then the abscissa of convergence of $s \mapsto \sum_{\gamma \in \Gamma} e^{-s\psi(\mu_\theta(\gamma))}$ is equal to one.

Shadow lemma. The following is an analog of Sullivan's shadow lemma for Γ-conformal measures on $\mathcal{F}_\theta$ which was proved in [22, Lem. 7.2].

Lemma 4.4 (Shadow lemma)**.** *Let ν be a (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$. We have the following:*

(1) *for some $R = R(\nu) > 0$, we have $c := \inf_{\gamma\in\Gamma} \nu(O^\theta_R(\gamma o, o)) > 0$; and*
(2) *for all $r \geq R$ and for all $\gamma \in \Gamma$,*

$$ce^{-\|\psi\|\kappa r}e^{-\psi(\mu_\theta(\gamma))} \leq \nu(O^\theta_r(o, \gamma o)) \leq e^{\|\psi\|\kappa r}e^{-\psi(\mu_\theta(\gamma))} \tag{4.2}$$

where $\kappa > 0$ is the constant given in Lemma 3.1.

If Γ is a θ-transverse subgroup with $\#\Lambda_\theta \geq 3$ (which is not necessarily Zariski dense), then (4.2) *holds for any (Γ, ψ)-conformal measure supported on Λ_θ.*

5. Directional recurrence for transverse subgroups

In this section, we recall the flow space Ω_θ for each θ-transverse subgroup Γ. We then define the directional conical set of Γ and give a characterization in terms of the recurrence set for directional flows on Ω_θ.

We suppose that Γ is a Zariski dense θ-transverse subgroup unless mentioned otherwise. The Γ-action on G/S_θ by left translations is not properly discontinuous in general, but there is a closed subspace $\tilde{\Omega}_\theta \subset G/S_\theta$ on which Γ acts properly discontinuously.

We first describe a parametrization of G/S_θ as $\mathcal{F}^{(2)}_\theta \times \mathfrak{a}_\theta$, which can be thought as a generalized Hopf-parametrization. For $g \in G$, let

$$[g] := (g^+, g^-, \beta^\theta_{g^+}(e, g)) \in \mathcal{F}^{(2)}_\theta \times \mathfrak{a}_\theta.$$

Consider the action of G on the space $\mathcal{F}^{(2)}_\theta \times \mathfrak{a}_\theta$ by

$$g.(\xi, \eta, b) = (g\xi, g\eta, b + \beta^\theta_\xi(g^{-1}, e)) \tag{5.1}$$

where $g \in G$ and $(\xi, \eta, b) \in \mathcal{F}^{(2)}_\theta \times \mathfrak{a}_\theta$. Then the map $G \to \mathcal{F}^{(2)}_\theta \times \mathfrak{a}_\theta$ given by $g \mapsto [g]$ factors through G/S_θ and defines a G-equivariant homeomorphism

$$G/S_\theta \simeq \mathcal{F}^{(2)}_\theta \times \mathfrak{a}_\theta.$$

The subgroup A_θ acts on G/S_θ on the right by $[g]a := [ga]$ for $g \in G$ and $a \in A_\theta$; this is well-defined as A_θ commutes with S_θ. The corresponding A_θ-action on $\mathcal{F}^{(2)}_\theta \times \mathfrak{a}_\theta$ is given by

$$(\xi, \eta, b).a = (\xi, \eta, b + \log a)$$

for $a \in A_\theta$ and $(\xi, \eta, b) \in \mathcal{F}^{(2)}_\theta \times \mathfrak{a}_\theta$. For $\theta = \Pi$, this homeomorphism is called the Hopf parametrization of G/M.

Set $\Lambda^{(2)}_\theta := (\Lambda_\theta \times \Lambda_{\mathrm{i}(\theta)}) \cap \mathcal{F}^{(2)}_\theta$, and define

$$\tilde{\Omega}_\theta = \Lambda^{(2)}_\theta \times \mathfrak{a}_\theta \tag{5.2}$$

which is a closed left Γ-invariant and right A_θ-invariant subspace of $\mathcal{F}^{(2)}_\theta \times \mathfrak{a}_\theta$.

Theorem 5.1. [22, Thm. 9.1] *If Γ is θ-transverse, then Γ acts properly discontinuously on $\tilde{\Omega}_\theta$ and hence*

$$\Omega_\theta := \Gamma\backslash\tilde{\Omega}_\theta$$

is a second countable locally compact Hausdorff space.

By [3], the set $\{(y_\gamma, y_{\gamma^{-1}}) \in \Lambda^{(2)} : \gamma \in \Gamma \text{ loxodromic}\}$ is dense in $\Lambda^{(2)}$ (see (2.3) for the notation y_γ). Hence the projection $\{(\pi_\theta(y_\gamma), \pi_{\mathrm{i}(\theta)}(y_{\gamma^{-1}}) \in \Lambda_\theta^{(2)} : \gamma \in \Gamma \text{ loxodromic}\}$ is dense in $\Lambda_\theta^{(2)}$. This implies that Ω_θ is a non-wandering set for A_θ, that is, for any open subset $O \subset \Omega_\theta$, the intersection $O \cap Oa_i$ is non-empty for some sequence $a_i \in A_\theta$ going to ∞.

Fix $u \in \mathfrak{a}_\theta^+ - \{0\}$ and set

$$a_{tu} = \exp tu \quad \text{for } t \in \mathbb{R}.$$

We describe the recurrent dynamics of the one-parameter subgroup $A_u = \{a_{tu} : t \in \mathbb{R}\}$ on Ω_θ. That is, for a given compact subset $Q_0 \subset \Omega_\theta$, we describe when the translate $Q_0 a_{tu}$ comes back to Q_0 and what the intersection $Q_0 a_{tu} \cap Q_0$ looks like for t large enough. This is equivalent to studying $Qa_{tu} \cap \Gamma Q$ for a compact subset $Q \subset \tilde{\Omega}_\theta \subset G/S_\theta$. Difficulties arise because S_θ is not compact, and the θ-transverse hypothesis on Γ is crucial in the following discussions.

We will repeatedly use the following lemma: note that the product $A_\theta^+ B_\theta^+$ is not contained in A^+ in general.

Lemma 5.2. *Suppose that $\gamma_i \in \Gamma$ and $d_i \in A_\theta^+ B_\theta^+$ are sequences such that the sequence $\gamma_i h_i m_i d_i$ is uniformly bounded for some bounded sequences $h_i \in G$ with $h_i P \in \Lambda$ and $m_i \in M_\theta$. Then there exists $w \in \mathcal{W} \cap M_\theta$ such that after passing to a subsequence,*

$$d_i \in wA^+w^{-1} \quad \textit{for all } i \geq 1.$$

Proof. Since $d_i \in A$, by passing to a subsequence, there exists $w \in \mathcal{W}$ such that $d_i = wc_iw^{-1}$ for some $c_i \in A^+$. We will show that $w \in M_\theta$. We may assume without loss of generality that as $i \to \infty$, h_i and m_i converge to some $h \in G$ and $m \in M_\theta$ respectively. The θ-regularity of Γ implies that $\gamma_i^{-1} \to \infty$ $\theta \cup \mathrm{i}(\theta)$-regularly. Since $h_i' := \gamma_i h_i m_i w c_i w^{-1}$ is bounded, it follows that $c_i \to \infty$ in A^+ $\theta \cup \mathrm{i}(\theta)$-regularly as well by Lemma 2.1.

By Lemma 2.5(1)-(2), we have that $\gamma_i^{-1} h_i'$ converges to a point in $\Lambda_{\theta\cup\mathrm{i}(\theta)}$ and $h_i m_i w c_i w^{-1} \to hmwP_{\theta\cup\mathrm{i}(\theta)}$ as $i \to \infty$. Therefore, we have $hmwP_{\theta\cup\mathrm{i}(\theta)} \in \Lambda_{\theta\cup\mathrm{i}(\theta)}$. Since $hP_{\theta\cup\mathrm{i}(\theta)} \in \Lambda_{\theta\cup\mathrm{i}(\theta)}$ by the hypothesis, it follows from the $\theta \cup \mathrm{i}(\theta)$-antipodality of Γ that either $wP_{\theta\cup\mathrm{i}(\theta)} = m^{-1}P_{\theta\cup\mathrm{i}(\theta)}$ or $wP_{\theta\cup\mathrm{i}(\theta)}$ is in general position with $m^{-1}P_{\theta\cup\mathrm{i}(\theta)}$. In the former case, by considering the projection to $\mathcal{F}_\theta$, we get $wP_\theta = m^{-1}P_\theta$ and hence $w \in M_\theta$ as desired. It remains to show that the latter case does not happen. The latter case would mean that $wP_{\mathrm{i}(\theta)}$ is in general position with $m^{-1}P_\theta = P_\theta$. By Lemma 2.7, this implies $w \in w_0 M_{\mathrm{i}(\theta)} = M_\theta w_0$. Writing $d_i = a_i b_i \in$

$A_\theta^+ B_\theta^+$ and $w = m_0 w_0$ with $m_0 \in M_\theta \cap N_K(A)$, we get $c_i = w^{-1} d_i w = w_0^{-1} a_i w_0 (w_0^{-1} m_0^{-1} b_i m_0 w_0) \in A_{\mathrm{i}(\theta)}(S_{\mathrm{i}(\theta)} \cap A) = A_{\mathrm{i}(\theta)} B_{\mathrm{i}(\theta)}$. As $c_i \in A^+ \subset A_{\mathrm{i}(\theta)}^+ B_{\mathrm{i}(\theta)}^+$, we must have $w_0^{-1} a_i w_0 \in A_{\mathrm{i}(\theta)}^+$, which is a contradiction since $a_i \in A_\theta^+$. This finishes the proof. $\square$

Proposition 5.3. *Let $Q \subset \tilde{\Omega}_\theta$ be a compact subset and $u \in \mathfrak{a}_\theta^+ - \{0\}$. There are positive constants $C_1 = C_1(Q), C_2 = C_2(Q)$ and $R = R(Q)$ such that if $[h] \in Q \cap \gamma Q a_{-tu}$ for some $h \in G$, $\gamma \in \Gamma$ and $t > 0$, then the following hold:*

(1) $\|\mu_\theta(\gamma) - tu\| < C_1$;
(2) $(h^+, h^-) \in O_R^\theta(o, \gamma o) \times O_R^{\mathrm{i}(\theta)}(\gamma o, o)$;
(3) $\|\mathcal{G}^\theta(h^+, h^-)\| < C_2$.

Proof. Let $Q' \subset G$ be a compact subset such that $Q' M_\theta = Q'$ and $Q \subset Q' S_\theta / S_\theta$.

To prove (1), suppose not. Then there exist sequences $\gamma_i \in \Gamma$, $h_i \in G$ and a sequence $t_i \to +\infty$ such that $\|\mu_\theta(\gamma_i) - t_i u\| \geq i$ and $[h_i] \in Q \cap \gamma_i Q a_{-t_i u}$ for all $i \geq 1$. By replacing h_i by an element in $h_i S_\theta$, we may assume that $h_i \in Q'$ and there exist $h_i' \in Q'$ and $s_i \in S_\theta$ such that $h_i s_i a_{t_i u} = \gamma_i h_i'$. Since $Q \subset \tilde{\Omega}_\theta$, we have $h_i P_\theta \in \Lambda_\theta$. By replacing h_i with an element of $h_i M_\theta$, we may assume that $h_i P \in \Lambda$ as well. Since $t_i \to +\infty$, $\gamma_i \to \infty$ in Γ. Writing $s_i = m_i b_i m_i' \in M_\theta B_\theta^+ M_\theta$ in the Cartan decomposition of S_θ, we have $h_i m_i a_{t_i u} b_i m_i' = \gamma_i h_i'$. By Lemma 5.2, by passing to a subsequence, there exists $w \in \mathcal{W} \cap M_\theta$ such that $a_{t_i u} b_i = w c_i w^{-1}$ for some $c_i \in A^+$. Since $c_i = a_{t_i u}(w^{-1} b_i w) \in A^+ \cap A_\theta B_\theta$, It follows that

$$\mu_\theta(c_i) = p_\theta(\log c_i) = t_i u.$$

Since $h_i m_i w c_i w^{-1} m_i' = \gamma_i h_i'$, we get that the sequence $\|\mu_\theta(\gamma_i) - \mu_\theta(c_i)\|$ is uniformly bounded by Lemma 2.1. Hence $\|\mu_\theta(\gamma_i) - t_i u\|$ is uniformly bounded, yielding a contradiction.

To prove (2), suppose not. Then there exist sequences $h_i \in Q$, $\gamma_i \in \Gamma$ and $t_i > 0$ such that $[h_i] \in Q \cap \gamma_i Q a_{-t_i u}$ and $h_i^+ \notin O_i^\theta(o, \gamma_i o)$ or $h_i^- \notin O_i^{\mathrm{i}(\theta)}(\gamma_i o, o)$ for all $i \geq 1$. As before, we may assume $h_i \in Q'$, $h_i P \in \Lambda$ and for some $h_i' \in Q'$ and $s_i \in S_\theta$, we have $h_i s_i a_{t_i u} = \gamma_i h_i'$. If γ_i were a bounded sequence, $O_i^\theta(o, \gamma_i o) \to \mathcal{F}_\theta$ and $O_i^{\mathrm{i}(\theta)}(o, \gamma_i o) \to \mathcal{F}_{\mathrm{i}(\theta)}$ as $i \to \infty$, which cannot be the case by the hypothesis on $h_i^\pm$. Hence $\gamma_i \to \infty$ in Γ. As in the proof of Item (1), there exist $w \in \mathcal{W} \cap M_\theta$, $b_i \in B_\theta^+$, $m_i, m_i' \in M_\theta$ and $c_i \in A^+$ such that

$$h_i m_i w c_i w^{-1} m_i' = \gamma_i h_i'$$

and $a_{t_i u} b_i = w c_i w^{-1}$. Then we have $h_i m_i w P_\theta = h_i P_\theta$ and $h_i m_i w c_i = \gamma_i h_i' m_i'^{-1} w$. Since $h_i' m_i'^{-1} w \in Q'$, it follows that

$$h_i^+ \in O_{R_0}^\theta(h_i o, \gamma_i o) \quad \text{for all } i \geq 1$$

where $R_0 = 1 + \max_{q \in Q' \cup Q' w_0} d(qo, o) > 0$. On the other hand, we have

$$h_i m_i w w_0^{-1} = \gamma_i h_i' m_i'^{-1} w w_0^{-1} (w_0 c_i^{-1} w_0^{-1}),$$

which is a bounded sequence. Since $\gamma_i h_i' m_i'^{-1} w w_0^{-1} P_{\mathrm{i}(\theta)} = h_i m_i w w_0^{-1} P_{\mathrm{i}(\theta)} = h_i w_0 P_{\mathrm{i}(\theta)}$, we have

$$h_i^- \in O_{R_0}^{\mathrm{i}(\theta)}(\gamma_i h_i' o, o) \quad \text{for all } i \geq 1.$$

Therefore, by Lemma 3.2, we have

$$(h_i^+, h_i^-) \in O_{2R_0}^{\theta}(o, \gamma_i o) \times O_{2R_0}^{\mathrm{i}(\theta)}(\gamma_i o, o) \quad \text{for all } i \geq 1,$$

yielding a contradiction.

To prove (3), as before, we may assume $h \in Q'$ and $h = \gamma h_1 a_{-tu} s$ for some $h_1 \in Q'$ and $s \in S_\theta$. Then we have

$$\begin{aligned} \beta_{h^+}^{\theta}(e, h) &= \beta_{h^+}^{\theta}(e, \gamma) + \beta_{e^+}^{\theta}(h_1^{-1}, e) + \beta_{e^+}^{\theta}(e, a_{-tu} s) \\ \beta_{h^-}^{\mathrm{i}(\theta)}(e, h) &= \beta_{h^-}^{\mathrm{i}(\theta)}(e, \gamma) + \beta_{e^-}^{\mathrm{i}(\theta)}(h_1^{-1}, e) + \beta_{e^-}^{\mathrm{i}(\theta)}(e, a_{-tu} s). \end{aligned}$$

Since $\beta_{e^+}^{\theta}(e, a_{-tu} s) + \mathrm{i}(\beta_{e^-}^{\mathrm{i}(\theta)}(e, a_{-tu} s)) = \mathcal{G}^\theta(e^+, e^-) = 0$, we deduce that

$$\mathcal{G}^\theta(h^+, h^-) = \beta_{h^+}^{\theta}(e, \gamma) + \mathrm{i}(\beta_{h^-}^{\mathrm{i}(\theta)}(e, \gamma)) + \beta_{e^+}^{\theta}(h_1^{-1}, e) + \mathrm{i}(\beta_{e^-}^{\mathrm{i}(\theta)}(h_1^{-1}, e)).$$

Observe that $\|\beta_{e^+}^{\theta}(h_1^{-1}, e) + \mathrm{i}(\beta_{e^-}^{\mathrm{i}(\theta)}(h_1^{-1}, e))\| \leq 2 \max_{q \in Q'} d(qo, o)$. Since $(h^+, h^-) \in O_R^\theta(o, \gamma o) \times O_R^{\mathrm{i}(\theta)}(\gamma o, o)$ by Item (2), it follows from Lemma 3.1 that

$$\|\beta_{h^+}^{\theta}(e, \gamma) - \mu_\theta(\gamma)\| \leq \kappa R \text{ and } \|\,\mathrm{i}(\beta_{h^-}^{\mathrm{i}(\theta)}(\gamma, e))) - \mathrm{i}(\mu_{\mathrm{i}(\theta)}(\gamma^{-1}))\| < \kappa R.$$

Since $\mu_\theta(\gamma) = \mathrm{i}(\mu_{\mathrm{i}(\theta)}(\gamma^{-1}))$, we get $\|\beta_{h^+}^{\theta}(e, \gamma) + \mathrm{i}(\beta_{h^-}^{\mathrm{i}(\theta)}(e, \gamma))\| \leq 2\kappa R$, and hence

$$\|\mathcal{G}^\theta(h^+, h^-)\| \leq 2\kappa R + 2 \max_{q \in Q'} d(qo, o).$$

This finishes the proof. $\square$

Directional conical sets. A point $\xi \in \mathcal{F}_\theta$ is called a θ-conical point of Γ if and only if there exist $R > 0$ and a sequence $\gamma_i \to \infty$ in Γ such that $\xi \in O_R^\theta(o, \gamma_i o)$, that is, $\xi = k_i P_\theta$ for some $k_i \in K$ such that $d(k_i A^+ o, \gamma_i o) < R$, for all $i \geq 1$. Using the identification $\mathcal{F}_\theta = K/M_\theta$, the θ-conical set of Γ is equal to

$$\Lambda_\theta^{\mathsf{con}} = \left\{ k M_\theta \in \mathcal{F}_\theta : k \in K \text{ and } \limsup \Gamma k M_\theta A^+ \neq \emptyset \right\}. \tag{5.3}$$

For $r > 0$, we set

$$\Gamma_{u,r} := \{\gamma \in \Gamma : \|\mu_\theta(\gamma) - \mathbb{R}u\| < r\}.$$

Definition 5.4 (Directional conical sets)**.** For $u \in \mathfrak{a}_\theta^+ - \{0\}$, we say $\xi \in \mathcal{F}_\theta$ is a u-directional conical point of Γ if there exist $R, r > 0$ and a sequence $\gamma_i \to \infty$ in $\Gamma_{u,r}$ such that for all $i \geq 1$,

$$\xi \in O_R^\theta(o, \gamma_i o),$$

that is, $\xi = k_i P_\theta$ for some $k_i \in K$ such that $d(k_i A^+ o, \gamma_i o) < R$. In other words, the u-directional conical set is given by
(5.4)

$$\Lambda_\theta^u = \{kM_\theta \in \mathcal{F}_\theta : k \in K \text{ and } \limsup \Gamma_{u,r}^{-1} kM_\theta A^+ \neq \emptyset \text{ for some } r > 0\}.$$

We note that $\Gamma_{u,r}^{-1} = \{\gamma \in \Gamma : \|\mu_{\mathrm{i}(\theta)}(\gamma) - \mathbb{R}\,\mathrm{i}(u)\| < r\}$.

Clearly, $\Lambda_\theta^u \subset \Lambda_\theta^{\mathsf{con}}$ for all $u \in \mathfrak{a}_\theta^+ - \{0\}$ and $\Lambda_\theta^u = \emptyset$ if $u \notin \mathcal{L}_\theta$. These notions of conical and directional conical sets can be defined for any discrete subgroup. On the other hand, for θ-transverse subgroups, these notions can also be defined in terms of recurrence of A_θ and A_u-actions on Ω_θ respectively: we emphasize that for a sequence $g_i \in G$, the sequence $[g_i] \in \tilde{\Omega}_\theta$ is precompact if and only if there exists $s_i \in S_\theta$ (which is not necessarily bounded) such that the sequence $g_i s_i$ is bounded in G.

Lemma 5.5 (Conical points and recurrence)**.** *Let Γ be θ-transverse. Then*

(1) *$\xi \in \Lambda_\theta^{\mathsf{con}}$ if and only if $\xi = gP_\theta$ for some $g \in G$ such that $[g] \in \tilde{\Omega}_\theta$ and $\gamma_i[g]a_i$ is precompact in $\tilde{\Omega}_\theta$ for infinite sequences $\gamma_i \in \Gamma$ and $a_i \in A_\theta^+$.*
(2) *$\xi \in \Lambda_\theta^u$ if and only if $\xi = gP_\theta$ for some $g \in G$ such that $[g] \in \tilde{\Omega}_\theta$ and $\gamma_i[g]a_{t_i u}$ is precompact in $\tilde{\Omega}_\theta$ for infinite sequences $\gamma_i \in \Gamma$ and $t_i > 0$.*

Proof. **Item (1):** Let $\xi \in \Lambda_\theta^{\mathsf{con}}$; so there exist $k \in K$, $\gamma_i \in \Gamma$, $m_i \in M_\theta$ and $c_i \in A^+$ so that $\xi = kP_\theta$ and $\gamma_i k m_i c_i$ is a bounded sequence in G. By the θ-regularity of Γ, we have $\Lambda_\theta^{\mathsf{con}} \subset \Lambda_\theta$ [22, Prop. 5.6(1)], and hence $k^+ = kP_\theta \in \Lambda_\theta$. Since $\Lambda_{\mathrm{i}(\theta)}$ is Zariski dense and $kN_\theta w_0 P_{\mathrm{i}(\theta)}$ is a Zariski open subset of $\mathcal{F}_{\mathrm{i}(\theta)}$, we have $(kn)^- \in \Lambda_{\mathrm{i}(\theta)}$ for some $n \in N_\theta$. Since $(kn)^+ = k^+ = \xi$, we have $[kn] \in \tilde{\Omega}_\theta$. Note that $\gamma_i k n m_i c_i = (\gamma_i k m_i c_i)(c_i^{-1} n_i' c_i)$ where $n_i' := m_i^{-1} n m_i \in N_\theta$ is a bounded sequence. Since $c_i \in A^+$, the sequence $c_i^{-1} n_i' c_i$ is bounded as well and hence $\gamma_i k n m_i c_i$ is bounded. Write $c_i = b_i a_i \in B_\theta^+ A_\theta^+$; so the sequence $\gamma_i (k n m_i b_i) a_i$ is contained in some compact subset of G and $m_i b_i \in S_\theta$. Since the map $g \mapsto [g] \in \tilde{\Omega}_\theta$ is continuous, and hence the image of a compact subset is compact, the sequence $\gamma_i [kn] a_i = [\gamma_i k n m_i b_i a_i]$ is precompact in $\tilde{\Omega}_\theta$, as desired.

Conversely, suppose that $\xi = gP_\theta$ for some $g \in G$ such that $[g] \in \tilde{\Omega}_\theta$ and $\gamma_i[g]a_i$ is precompact for infinite sequences $\gamma_i \in \Gamma$ and $a_i \in A_\theta^+$. We can replace g with an element in gM_θ so that $gP \in \Lambda$. Since the sequence $\gamma_i[g]a_i = [\gamma_i g a_i]$ is precompact, there exists a bounded sequence $h_i \in G$ such that for all $i \geq 1$, $[h_i] = \gamma_i[g]a_i \in \tilde{\Omega}_\theta$, that is, $g a_i s_i = \gamma_i^{-1} h_i$ for some $s_i \in S_\theta$. Writing the Cartan decomposition $s_i = m_i b_i m_i' \in M_\theta B_\theta^+ M_\theta$, we have $g m_i a_i b_i m_i' = \gamma_i^{-1} h_i$. Since the sequence $\gamma_i g m_i a_i b_i = h_i m_i'^{-1}$ is bounded, it follows from Lemma 5.2 that $a_i b_i = w c_i w^{-1}$ for some $w \in \mathcal{W} \cap M_\theta$ and $c_i \in A^+$, after passing to a subsequence. Hence we have $g m_i w c_i = \gamma_i^{-1} h_i m_i'^{-1} w$, which implies that $\xi = gP_\theta \in O_R^\theta(go, \gamma_i^{-1} o)$ for all i where

$R = 1 + \max_i d(h_i o, o)$. By Lemma 3.2, we have $\xi \in O^\theta_{R+d(go,o)}(o, \gamma_i^{-1} o)$ for all $i \ge 1$, completing the proof.

Item (2): Let $\xi \in \Lambda_\theta^u$. Then $\xi = kP_\theta$ for some $k \in K$ and $\gamma_i k m_i a_i$ is a bounded sequence in G for some infinite sequences $\gamma_i \in \Gamma_{u,r}^{-1}$, $m_i \in M_\theta$ and $a_i \in A^+$. Since $\xi = kP_\theta \in \Lambda_\theta^u$ and $\Lambda_\theta^u \subset \Lambda_\theta^{\mathsf{con}} \subset \Lambda_\theta$ by the θ-regularity of Γ [22, Prop. 5.6(1)], we have $k^+ \in \Lambda_\theta$. As in the proof of Item (1) above, there exists $n \in N_\theta$ so that $(kn)^- \in \Lambda_{\mathrm{i}(\theta)}$ and $\gamma_i k n m_i a_i$ is bounded. In particular, $[kn] \in \tilde{\Omega}_\theta$.

Since $\gamma_i k n m_i a_i$ is a bounded sequence in G and $\gamma_i^{-1} \in \Gamma_{u,r}$, we have $a_i = a_{t_i u} b_i$ for some $t_i > 0$ and a bounded sequence $b_i \in A$ by Lemma 2.1. Hence the sequence $\gamma_i k n m_i a_{t_i u}$ is bounded as well. Therefore, $\gamma_i [kn] a_{t_i u} = [\gamma_i k n m_i a_{t_i u}]$ is precompact in $\tilde{\Omega}_\theta$. Since $(kn)^+ = k^+ = \xi$, this shows the only if direction in (2).

To show the converse implication, suppose that the sequence $\gamma_i [g] a_{t_i u}$ is contained in some compact subset Q of $\tilde{\Omega}_\theta$ which we also assume contains $[g]$. Since $[g] \in Q \cap \gamma_i^{-1} Q a_{-t_i u}$, it follows from Proposition 5.3 that $\gamma_i^{-1} \in \Gamma_{u, C_1}$ and $g^+ = gP_\theta \in O_R^\theta(o, \gamma_i^{-1} o)$ for all $i \ge 1$ where $C_1 = C_1(Q)$ and $R = R(Q)$ are given in Proposition 5.3. Therefore, $g^+ \in \Lambda_\theta^u$. $\square$

Theorem 5.6. *Let $\Gamma < G$ be a Zariski dense discrete subgroup. Let $u \in \mathfrak{a}_\theta^+ - \{0\}$ and $\psi \in \mathfrak{a}_\theta^*$ be (Γ, θ)-proper. Suppose that $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} < \infty$ for all $r > 0$. For any (Γ, ψ)-conformal measure ν on $\mathcal{F}_\theta$, we have*

$$\nu(\Lambda_\theta^u) = 0.$$

Proof. For each $r > 0$, we set $\Lambda_{\theta,r}^u = \limsup_{\gamma \in \Gamma_{u,r}} O_r^\theta(o, \gamma o)$. In other words, $\xi \in \Lambda_{\theta,r}^u$ if and only if there exists a sequence $\gamma_i \to \infty$ in $\Gamma_{u,r}$ such that $\xi \in O_r^\theta(o, \gamma_i o)$ for all $i \ge 1$. Then $\Lambda_\theta^u = \bigcup_{r>0} \Lambda_{\theta,r}^u$. Let ν be a (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$. Since

$$\Lambda_{\theta,r}^u \subset \bigcup_{\gamma \in \Gamma_{u,r}, \|\mu_\theta(\gamma)\| > t} O_r^\theta(o, \gamma o) \quad \text{for all } t > 0,$$

it follows from Lemma 4.4 that

$$\nu(\Lambda_{\theta,r}^u) \ll \sum_{\gamma \in \Gamma_{u,r}, \|\mu_\theta(\gamma)\| > t} e^{-\psi(\mu_\theta(\gamma))} \quad \text{for all } t > 0. \tag{5.5}$$

Since $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} < \infty$, taking $t \to \infty$ in (5.5) implies $\nu(\Lambda_{\theta,r}^u) = 0$. Therefore, $\nu(\Lambda_\theta^u) = \limsup_{r \to \infty} \nu(\Lambda_{\theta,r}^u) = 0$. $\square$

Lemma 5.7. *If $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ for some $r > 0$, then $\psi(u) > 0$. If there exists a (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$ in addition, then*

$$\psi(u) = \psi_\Gamma^\theta(u).$$

Moreover the abscissa of convergence of the series $s \mapsto \sum_{\gamma \in \Gamma_{u,r}} e^{-s\psi(\mu_\theta(\gamma))}$ is equal to one.

Proof. Suppose that $\sum_{\gamma\in\Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$. Then $\#\Gamma_{u,r} = \infty$. If $\psi(u)$ were not positive, then $(\psi\circ\mu_\theta)(\Gamma_{u,r})$ is contained in the interval $(-\infty, \|\psi\|r]$. Therefore it contradicts the (Γ,θ)-proper hypothesis on ψ. Hence $\psi(u) > 0$.

Now suppose that there exists a (Γ,ψ)-conformal measure on $\mathcal{F}_\theta$. We then have $\psi \geq \psi_\Gamma^\theta$ by Theorem 4.3. Now suppose that $\psi(u) > \psi_\Gamma^\theta(u)$. We may assume that u is a unit vector as both ψ and ψ_Γ^θ are homogeneous of degree one. By the definition of ψ_Γ^θ, there exists an open cone $\mathcal{C}$ containing u so that $\sum_{\gamma\in\Gamma,\mu_\theta(\gamma)\in\mathcal{C}} e^{-\psi(u)\|\mu_\theta(\gamma)\|} < \infty$. Since $\mu_\theta(\Gamma_{u,r})$ is contained in $\mathcal{C}$ possibly except for finitely many elements, we have

$$\sum_{\gamma\in\Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} \ll \sum_{\gamma\in\Gamma_{u,r}} e^{-\psi(u)\|\mu_\theta(\gamma)\|} < \infty,$$

which is a contradiction. Therefore, $\psi(u) = \psi_\Gamma^\theta(u)$.

We now show the last claim. Since $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} \geq \sum_{\gamma\in\Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$, the abscissa of convergence of $s \mapsto \sum_{\gamma\in\Gamma} e^{-s\psi(\mu_\theta(\gamma))}$ is equal to one by Theorem 4.3. Hence the abscissa of convergence of $s \mapsto \sum_{\gamma\in\Gamma_{u,r}} e^{-s\psi(\mu_\theta(\gamma))}$ is at most one. Since $\sum_{\gamma\in\Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$, it must be exactly one. $\square$

6. Bowen-Margulis-Sullivan measures

In this short section, we recall the definition of Bowen-Margulis-Sullivan measures on Ω_θ. We also recall one-dimensional flow space Ω_ψ and the corresponding Bowen-Margulis-Sullivan measures on it.

Let $\Gamma < G$ be a Zariski dense θ-transverse subgroup. Recall our flow space from the previous section:

$$\Omega_\theta = \Gamma\backslash\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$$

where the action is given by (5.1).

Bowen-Margulis-Sullivan measures on Ω_θ. We may identify $\mathfrak{a}_\theta^*$ with $\{\psi \in \mathfrak{a}^* : \psi\circ p_\theta = \psi\}$. Hence for $\psi \in \mathfrak{a}_\theta^*$, we have $\psi\circ \mathrm{i} \in \mathfrak{a}_{\mathrm{i}(\theta)}^*$. For a pair of a (Γ,ψ)-conformal measure ν on Λ_θ and a $(\Gamma,\psi\circ\mathrm{i})$-conformal measure ν_{i} on $\Lambda_{\mathrm{i}(\theta)}$, we define a Radon measure $d\tilde{\mathsf{m}}_{\nu,\nu_{\mathrm{i}}}$ on $\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ as follows:

$$d\tilde{\mathsf{m}}_{\nu,\nu_{\mathrm{i}}}(\xi,\eta,b) = e^{\psi(\mathcal{G}^\theta(\xi,\eta))} d\nu(\xi) d\nu_{\mathrm{i}}(\eta) db \tag{6.1}$$

where db is the Lebesgue measure on $\mathfrak{a}_\theta$. It is easy to check that $\tilde{\mathsf{m}}_{\nu,\nu_{\mathrm{i}}}$ is left Γ-invariant, and hence induces a A_θ-invariant Radon measure on Ω_θ which we denote by

$$\mathsf{m}_{\nu,\nu_{\mathrm{i}}}. \tag{6.2}$$

We call it the Bowen-Margulis-Sullivan measure (or simply BMS measure) associated with the pair (ν,ν_{i}).

Bowen-Margulis-Sullivan measures on Ω_ψ. Let $\psi \in \mathfrak{a}_\theta^*$ be a (Γ,θ)-proper form. We remark that this implies that $\psi \geq 0$ on $\mathcal{L}_\theta$ and $\psi > 0$ on $\operatorname{int}\mathcal{L}_\theta$ [22, Lem. 4.3]. Consider the Γ-action on $\tilde{\Omega}_\psi := \Lambda_\theta^{(2)} \times \mathbb{R}$ given by

$$\gamma.(\xi.\eta, s) = (\gamma\xi, \gamma\eta, s + \psi(\beta_\xi^\theta(\gamma^{-1}, e))) \tag{6.3}$$

for $\gamma \in \Gamma$ and $(\xi, \eta, s) \in \Lambda_\theta^{(2)} \times \mathbb{R}$.

Theorem 6.1. [22, Thm. 9.2] *If Γ is Zariski dense θ-transverse and $\psi \in \mathfrak{a}_\theta^*$ is (Γ,θ)-proper, then Γ acts properly discontinuously on $\tilde{\Omega}_\psi$ and hence*

$$\Omega_\psi := \Gamma\backslash\tilde{\Omega}_\psi \tag{6.4}$$

is a second countable locally compact Hausdorff space.

The map $\Lambda_\theta^{(2)} \times \mathfrak{a}_\theta \to \Lambda_\theta^{(2)} \times \mathbb{R}$ given by $(\xi, \eta, v) \mapsto (\xi, \eta, \psi(v))$ is a principal $\ker\psi$-bundle which is trivial since $\ker\psi$ is a vector space. Therefore it induces a $\ker\psi$-equivariant homeomorphism between

$$\Omega_\theta \simeq \Omega_\psi \times \ker\psi. \tag{6.5}$$

Let

$$\mathsf{m}_{\nu,\nu_{\mathrm{i}}}^{\psi} \tag{6.6}$$

be the Radon measure on Ω_ψ induced from the Γ-invariant measure on $\tilde{\Omega}_\psi$:

$$d\tilde{\mathsf{m}}_{\nu,\nu_{\mathrm{i}}}^{\psi}(\xi,\eta,s) := e^{\psi(\mathcal{G}^\theta(\xi,\eta))} d\nu(\xi) d\nu(\eta) ds.$$

We then have

$$\mathsf{m}_{\nu,\nu_{\mathrm{i}}} = \mathsf{m}_{\nu,\nu_{\mathrm{i}}}^{\psi} \otimes \operatorname{Leb}_{\ker\psi}.$$

7. Directional conical sets and Poincaré series

In this section, we relate the divergence of the directional ψ-Poincaré series with the size of the directional conical set with respect to a (Γ,ψ)-conformal measure on $\mathcal{F}_\theta$. The main theorem of this section (Theorem 7.2) is the most significant part of Theorem 1.1.

Let $\Gamma < G$ be a Zariski dense θ-transverse subgroup. We fix

$$u \in \mathfrak{a}_\theta^+ - \{0\} \text{ and a } (\Gamma,\theta)\text{-proper } \psi \in \mathfrak{a}_\theta^*.$$

We also fix a pair ν, ν_{i} of (Γ,ψ) and $(\Gamma,\psi\circ\mathrm{i})$-conformal measures on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively. Denote by $\tilde{\mathsf{m}} = \tilde{\mathsf{m}}_{\nu,\nu_{\mathrm{i}}}$ and $\mathsf{m} = \mathsf{m}_{\nu,\nu_{\mathrm{i}}}$ the associated BMS measures on $\tilde{\Omega}_\theta$ and Ω_θ respectively.

Our dichotomy theorem is stated under a hypothesis on m which we call a u-balanced condition:

Definition 7.1 (u-balanced condition)**.** A Borel measure space (X, m) with $\{a_{tu}\}$-action is called u-balanced or simply m is u-balanced, if for any bounded Borel subset $O_i \subset X$ with $m(O_i) > 0$ for $i = 1, 2$, for all $T > 1$,

$$\int_0^T m(O_1 \cap O_1 a_{tu})dt \asymp^4 \int_0^T m(O_2 \cap O_2 a_{tu})dt.$$

The main goal of this section is to prove the following:

Theorem 7.2. *Suppose that* m *is* u*-balanced. If* $\sum_{\gamma\in\Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ *for some* $r > 0$*, then*

$$\nu(\Lambda_\theta^u) > 0 \quad \textit{and} \quad \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)}) > 0.$$

Remark 7.3. When $\sum_{\gamma\in\Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$, there exists at most one (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$ ([22, Thm. 1.5]). Furthermore, the existence of a (Γ, ψ)-conformal measure on Λ_θ implies the existence of $(\Gamma, \psi \circ \mathrm{i})$-conformal measure on $\Lambda_{\mathrm{i}(\theta)}$ as well. Indeed, it follows from [22, Thm. 7.1] that $\delta_\psi = 1$ where δ_ψ is the abscissa of the convergence of the Poincaré series $s \mapsto \sum_{\gamma\in\Gamma} e^{-s\psi(\mu_\theta(\gamma))}$. In particular, $\delta_{\psi\circ\mathrm{i}} = \delta_\psi = 1$. By [10] and [22, Lem. 9.5], there exists a $(\Gamma, \psi \circ \mathrm{i})$-conformal measure ν_{i} on $\Lambda_{\mathrm{i}(\theta)}$ which is the unique $(\Gamma, \psi\circ\mathrm{i})$-conformal measure on $\mathcal{F}_{\mathrm{i}(\theta)}$, since $\sum_{\gamma\in\Gamma} e^{-(\psi\circ\mathrm{i})(\mu_{\mathrm{i}(\theta)}(\gamma))} = \infty$ as well.

For simplicity, we set for all $t \in \mathbb{R}$

$$a_t = a_{tu} = \exp tu.$$

The following proposition is the key ingredient of the proof of Theorem 7.2:

Proposition 7.4. *Set* $\delta = \psi(u)$*, which is positive by Lemma 5.7.*

(1) *For any compact subset* $Q \subset \tilde{\Omega}_\theta$*, there exists* $r = r(Q) > 0$ *such that for any* $T > 1$*, we have*

$$\int_0^T \int_0^T \sum_{\gamma,\gamma'\in\Gamma} \tilde{\mathsf{m}}(Q \cap \gamma Q a_{-t} \cap \gamma' Q a_{-t-s})dtds \ll \left(\sum_{\substack{\gamma\in\Gamma_{u,r} \\ \psi(\mu_\theta(\gamma))\le\delta T}} e^{-\psi(\mu_\theta(\gamma))} \right)^2.$$

(2) *For any* $r > 0$*, there exists a compact subset* $Q' = Q'(r) \subset \tilde{\Omega}_\theta$ *such that for any* $T > 1$*,*

$$\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q' \cap \gamma Q' a_{-t})dt \gg \sum_{\substack{\gamma\in\Gamma_{u,r} \\ \psi(\mu_\theta(\gamma))\le\delta T}} e^{-\psi(\mu_\theta(\gamma))}.$$

To prove this proposition, we relate the integrals on the left hand sides to shadows and apply the shadow lemma. Together with results obtained in Section 5, the following proposition on the multiplicity bound on shadows for transverse subgroups is crucial.

[4] The notation $f(T) \asymp g(T)$ means that $f(T), g(T) \to \infty$ as $T \to \infty$ and $f(T) \ll g(T)$ and $g(T) \ll f(T)$.

Proposition 7.5. [22, Prop. 6.2] *For any $R, D > 0$, there exists $q = q(\psi, R, D) > 0$ such that for any $T > 0$, the collection of shadows*

$$\left\{ O_R^\theta(o, \gamma o) \subset \mathcal{F}_\theta : T \le \psi(\mu_\theta(\gamma)) \le T + D \right\}$$

has multiplicity at most q.

Lemma 7.6. *Let $Q \subset \tilde{\Omega}_\theta$ be a compact subset. For any $t > 1$, we have*

$$\tilde{\mathsf{m}}(Q \cap \gamma Q a_{-t}) \ll e^{-\psi(\mu_\theta(\gamma))}$$

where the implied constant is independent of t.

Proof. There exists $c_0 = c_0(Q) > 0$ such that if $Q \cap Qa \neq \emptyset$ for some $a \in A_\theta$, then $\|\log a\| < c_0$. By Proposition 5.3(2) and the compactness of Q, we have for large enough $R > 0$ that

$$\begin{aligned}
&\tilde{\mathsf{m}}(Q \cap \gamma Q a_{-t}) \\
&\ll \int_{O_R^\theta(o,\gamma o) \times O_R^{\mathrm{i}(\theta)}(\gamma o, o)} \int_{\mathfrak{a}_\theta} \mathbb{1}_{Q \cap \gamma Q a_{-t}}(\xi, \eta, b) e^{\psi(\mathcal{G}^\theta(\xi,\eta))} d\nu(\xi) d\nu_{\mathrm{i}}(\eta) db \\
&\ll \int_{O_R^\theta(o,\gamma o) \times O_R^{\mathrm{i}(\theta)}(\gamma o, o)} \mathbb{1}_{Q \cap \gamma Q a_{-t}}(\xi, \eta) e^{\psi(\mathcal{G}^\theta(\xi,\eta))} d\nu(\xi) d\nu_{\mathrm{i}}(\eta).
\end{aligned}$$

Since $\mathcal{G}^\theta(\xi, \eta)$ in the above integrand is uniformly bounded by Proposition 5.3(3), we obtain

$$\tilde{\mathsf{m}}(Q \cap \gamma Q a_{-t}) \ll \nu(O_R^\theta(o, \gamma o)) \nu_{\mathrm{i}}(O_R^{\mathrm{i}(\theta)}(\gamma o, o)).$$

By Lemma 4.4, we have

$$\tilde{\mathsf{m}}(Q \cap \gamma Q a_{-t}) \ll \nu(O_R^\theta(o, \gamma o)) \ll e^{-\psi(\mu_\theta(\gamma))}.$$

$\square$

The following is immediate from Proposition 5.3(1).

Lemma 7.7. *Let $Q \subset \tilde{\Omega}_\theta$ be a compact subset. If $Q \cap \gamma Q a_{-t} \cap \gamma' Q a_{-t-s} \neq \emptyset$ for some $\gamma, \gamma' \in \Gamma$ and $t, s > 0$, then we have*

1. $\|\mu_\theta(\gamma) - tu\|$, $\|\mu_\theta(\gamma^{-1}\gamma') - su\|$, $\|\mu_\theta(\gamma') - (t+s)u\| < C_1$;
2. $\psi(\mu_\theta(\gamma)) + \psi(\mu_\theta(\gamma^{-1}\gamma')) < \psi(\mu_\theta(\gamma')) + 3C_1\|\psi\|$

where $C_1 = C_1(Q)$ is given as in Proposition 5.3.

Proof of Proposition 7.4(1). Let $Q \subset \tilde{\Omega}_\theta$ be a compact subset. Fix $s, t > 0$. For $\gamma, \gamma' \in \Gamma$ such that $Q \cap \gamma Q a_{-t} \cap \gamma' Q a_{-t-s} \neq \emptyset$, it follows from Lemma 7.6 that

$$\tilde{\mathsf{m}}(Q \cap \gamma Q a_{-t} \cap \gamma' Q a_{-t-s}) \ll e^{-\psi(\mu_\theta(\gamma'))}.$$

By Lemma 7.7(2), we have $\psi(\mu_\theta(\gamma)) + \psi(\mu_\theta(\gamma^{-1}\gamma')) < \psi(\mu_\theta(\gamma')) + 3C_1\|\psi\|$ and hence

$$\tilde{\mathsf{m}}(Q \cap \gamma Q a_{-t} \cap \gamma' Q a_{-t-s}) \ll e^{-\psi(\mu_\theta(\gamma))} e^{-\psi(\mu_\theta(\gamma^{-1}\gamma'))}.$$

Since we also have $\|\mu_\theta(\gamma)-tu\|$, $\|\mu_\theta(\gamma^{-1}\gamma')-su\| < C_1$ by Lemma 7.7 where C_1 is given in Proposition 5.3(1), we deduce by replacing $\gamma^{-1}\gamma'$ with $\hat\gamma$ that

$$\sum_{\gamma,\gamma'\in\Gamma} \tilde{\mathsf{m}}(Q\cap\gamma Q a_{-t}\cap\gamma' Q a_{-t-s})$$
$$\ll \left(\sum_{\substack{\gamma\in\Gamma_{u,C_1}\\ \psi(\mu_\theta(\gamma))\in(\delta t-c,\delta t+c)}} e^{-\psi(\mu_\theta(\gamma))}\right)\left(\sum_{\substack{\hat\gamma\in\Gamma_{u,C_1}\\ \psi(\mu_\theta(\hat\gamma))\in(\delta s-c,\delta s+c)}} e^{-\psi(\mu_\theta(\hat\gamma))}\right)$$

where $c := C_1\|\psi\|$.

We observe that if $\psi(\mu_\theta(\gamma)) \in (\delta t - c, \delta t + c)$ for some $t \in [0,T]$, then $\psi(\mu_\theta(\gamma)) \le \delta T + c$. Hence we have

$$\int_0^T \left(\sum_{\substack{\gamma\in\Gamma_{u,C_1}\\ \psi(\mu_\theta(\gamma))\in(\delta t-c,\delta t+c)}} e^{-\psi(\mu_\theta(\gamma))}\right) dt \ll \sum_{\substack{\gamma\in\Gamma_{u,C_1}\\ \psi(\mu_\theta(\gamma))\le\delta T+c}} e^{-\psi(\mu_\theta(\gamma))}.$$

Similarly we also have

$$\int_0^T \left(\sum_{\substack{\hat\gamma\in\Gamma_{u,C_1}\\ \psi(\mu_\theta(\hat\gamma))\in(\delta s-c,\delta s+c)}} e^{-\psi(\mu_\theta(\hat\gamma))}\right) ds \ll \sum_{\substack{\hat\gamma\in\Gamma_{u,C_1}\\ \psi(\mu_\theta(\hat\gamma))\le\delta T+c}} e^{-\psi(\mu_\theta(\hat\gamma))}.$$

Therefore, we have

$$\int_0^T\int_0^T \sum_{\gamma,\gamma'\in\Gamma} \tilde{\mathsf{m}}(Q\cap\gamma Q a_{-t}\cap\gamma' Q a_{-t-s})dtds \ll \left(\sum_{\substack{\gamma\in\Gamma_{u,C_1}\\ \psi(\mu_\theta(\gamma))\le\delta T+c}} e^{-\psi(\mu_\theta(\gamma))}\right)^2.$$

Since

$$\sum_{\substack{\gamma\in\Gamma_{u,C_1}\\ \delta T<\psi(\mu_\theta(\gamma))\le\delta T+c}} e^{-\psi(\mu_\theta(\gamma))} \ll \sum_{\substack{\gamma\in\Gamma_{u,C_1}\\ \delta T<\psi(\mu_\theta(\gamma))\le\delta T+c}} \nu(O_R^\theta(o,\gamma o)) \ll 1$$

for large $R = R(\nu)$ by Lemma 4.4 and Proposition 7.5, setting $r(Q) = C_1(Q)$ completes the proof. ☐

Lemma 7.8. *For any $R > 0$, there exists $0 < \ell_R < \infty$ such that any $(\xi,\eta) \in \bigcup_{\gamma\in\Gamma, \|\mu_\theta(\gamma)\|>\ell_R} O_R^\theta(o,\gamma o)\times O_R^{\mathrm{i}(\theta)}(\gamma o, o)$ satisfies $\|\mathcal{G}^\theta(\xi,\eta)\| < \ell_R$.*

Proof. Suppose not. Then there exist sequences $\gamma_i \to \infty$ in Γ and $(\xi_i,\eta_i) \in O_R^\theta(o,\gamma_i o)\times O_R^{\mathrm{i}(\theta)}(\gamma_i o, o)$ such that $\|\mathcal{G}^\theta(\xi_i,\eta_i)\| \to \infty$ as $i\to\infty$. We may assume that $\xi_i \to \xi$ and $\eta_i \to \eta$ by passing to subsequences. As $\gamma_i \to \infty$

θ-regularly, Lemma 3.5 implies that $(\xi,\eta) \in \mathcal{F}_\theta^{(2)}$. Since $\|\mathcal{G}^\theta(\xi_i,\eta_i)\| \to \|\mathcal{G}^\theta(\xi,\eta)\| < \infty$, this is a contradiction. $\square$

Lemma 7.9. *Let $u \in \mathfrak{a}_\theta^+ - \{0\}$. For any $r, R > 0$, there exists a compact subset $Q = Q(r,R) \subset \tilde{\Omega}_\theta$ such that for any $\gamma \in \Gamma_{u,r}$ with $\|\mu_\theta(\gamma)\| > \ell_R$ and*

$$(\xi,\eta) \in \left(O_R^\theta(o,\gamma o) \times O_R^{\mathrm{i}(\theta)}(\gamma o, o)\right) \cap \Lambda_\theta^{(2)},$$

there exists $v \in \mathfrak{a}_\theta$ and $t \geq 0$ such that

$$(\xi,\eta,v) \in Q \quad \text{and} \quad (\xi,\eta,v)a_{[t-1,t+1]} \subset \gamma Q.$$

Proof. Let $(\xi,\eta) \in (O_R^\theta(o,\gamma o) \times O_R^{\mathrm{i}(\theta)}(\gamma o,o)) \cap \Lambda_\theta^{(2)}$ for some $\gamma \in \Gamma_{u,r}$ with $\|\mu_\theta(\gamma)\| > \ell_R$. Then there exists $k \in K$ such that $\xi = kP_\theta$ and $d(ka_0 o, \gamma o) < R$ for some $a_0 \in A^+$. Write $a_0 = ab \in A_\theta^+ B_\theta^+$.

By Lemma 2.1, we have $\|\mu(\gamma) - \log a_0\| < D$ for some $D = D(R)$, and hence $\|\mu_\theta(\gamma) - \log a\| < D$. We also obtain from $\gamma \in \Gamma_{u,r}$ that $\|\mu_\theta(\gamma) - tu\| < r$ for some $t \geq 0$ and hence we have $\|tu - \log a\| < D + r$. Therefore, we have

$$\begin{aligned} d(ka_{tu}bo,\gamma o) &\leq d(ka_{tu}bo, ka_0 o) + d(ka_0 o,\gamma o) \\ &= d(a_{tu}o, ao) + d(ka_0 o, \gamma o) \\ &< D + r + R. \end{aligned} \tag{7.1}$$

We also note that

$$\|tu + \log b - \log a_0\| = \|tu - \log a\| < D + r.$$

Hence there exists $\tilde{a} \in A$ such that

$$\|\log \tilde{a}\| < D + r \quad \text{and} \quad a_{tu} b\tilde{a} \in A^+.$$

Let $g_0 \in G$ such that $(g_0 P_\theta, g_0 w_0 P_{\mathrm{i}(\theta)}) = (\xi,\eta)$. Since $(\xi,\eta) \in O_R^\theta(o,\gamma o) \times O_R^{\mathrm{i}(\theta)}(\gamma o, o)$ and $\|\mu_\theta(\gamma)\| > \ell_R$, we have $\|\mathcal{G}^\theta(\xi,\eta)\| < \ell_R$. By Proposition 2.8, we can replace g_0 by an element of $g_0 L_\theta$ so that we may assume that

$$d(o, g_0 o) \leq c\|\mathcal{G}^\theta(\xi,\eta)\| + c' < c\ell_R + c'.$$

Since $\xi = kP_\theta = g_0 P_\theta$, we have $g_0^{-1}k \in P_\theta$. We write the Iwasawa decomposition

$$g_0^{-1} k = m\hat{a}\hat{n} \in KAN.$$

Then we have $m = g_0^{-1}k\hat{n}^{-1}\hat{a}^{-1} \in P_\theta \hat{n}^{-1}\hat{a}^{-1} = P_\theta$. In particular, we have $m \in P_\theta \cap K = M_\theta$. We let $g = g_0 m$. Since $m \in M_\theta \subset L_\theta$, we still have $(gP_\theta, gw_0 P_{\mathrm{i}(\theta)}) = (\xi,\eta)$ and $d(o, go) = d(o, g_0 o) < c\ell_R + c'$. Moreover, we have $g^{-1}k = \hat{a}\hat{n} \in P$. Now for $s \in [t-1, t+1]$, we have

$$\begin{aligned} & d(gba_{su}o, kba_{tu}o) \leq d(gba_{su}o, gba_{tu}o) + d(gba_{tu}o, kba_{tu}o) \\ & \leq 1 + d(gba_{tu}o, gba_{tu}\tilde{a}o) + d(gba_{tu}\tilde{a}o, kba_{tu}\tilde{a}o) + d(kba_{tu}\tilde{a}o, kba_{tu}o) \\ & = 1 + 2d(o,\tilde{a}o) + d(gba_{tu}\tilde{a}o, kba_{tu}\tilde{a}o). \end{aligned}$$

Since $g^{-1}k \in P$ and $ba_{tu}\tilde{a} \in A^+$, we get $d(gba_{tu}\tilde{a}o, kba_{tu}\tilde{a}o) \le d(go, ko) = d(go, o) < c\ell_R + c'$. Together with $\|\log \tilde{a}\| < D + r$, we have

$$d(gba_{su}o, kba_{tu}o) < 1 + 2(D + r) + c\ell_R + c'.$$

Since $d(kba_{tu}o, \gamma o) < D + r + R$, we finally have

$$d(gba_{su}o, \gamma o) < 1 + 3(D + r) + R + c\ell_R + c'.$$

We set $R' = 1 + 3(D + r) + R + c\ell_R + c'$ and $Q := \{[h] \in \tilde{\Omega}_\theta : d(ho, o) \le R'\}$ which is a compact subset of $\tilde{\Omega}_\theta$.

Now the image of g under the projection $G \to \mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ is of the form (ξ, η, v) for some $v \in \mathfrak{a}_\theta$. Since $b \in S_\theta$, the product gb also projects to the same element (ξ, η, v). It follows from $d(o, go) < c\ell_R + c' \le R'$ that $(\xi, \eta, v) \in Q$. Moreover, since $d(\gamma^{-1}gba_{su}o, o) < R'$ for all $s \in [t-1, t+1]$, we have $\gamma^{-1}(\xi, \eta, v)a_{su} \in Q$ and hence $(\xi, \eta, v)a_{[t-1,t+1]} \subset \gamma Q$. This finishes the proof. $\square$

Recall the notation $\delta = \psi(u) > 0$.

Lemma 7.10. *Fix $r, R > 0$, and let $Q = Q(r, R) \subset \tilde{\Omega}_\theta$ and $C_1 = C_1(Q) > 0$ be as in Lemma 7.9 and Proposition 5.3 respectively. Let $T > 0$ and $\gamma \in \Gamma_{u,r}$ be such that*

$$\|\mu_\theta(\gamma)\| > \ell_R \quad \text{and} \quad C_1\|\psi\| + \delta < \psi(\mu_\theta(\gamma)) < \delta T - C_1\|\psi\| - \delta.$$

If $\sum_{\gamma_0 \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma_0))} = \infty$, then, for any $(\xi, \eta) \in (O_R^\theta(o, \gamma o) \times O_R^{\mathrm{i}(\theta)}(\gamma o, o)) \cap \Lambda_\theta^{(2)}$, we have

$$\int_0^T \int_{\mathfrak{a}_\theta} \mathbb{1}_{Q' \cap \gamma Q' a_{-t}}(\xi, \eta, b)\, db\, dt \ge 2 \operatorname{Vol}(A_{\theta,2})$$

where $A_{\theta,2} = \{a \in A_\theta : \|\log a\| \le 2\}$ and $Q' := QA_{\theta,2} \subset \tilde{\Omega}_\theta$.

Proof. By Lemma 7.9, there exist $v \in \mathfrak{a}_\theta$ and $t_0 \ge 0$ such that $(\xi, \eta, v) \in Q$ and $(\xi, \eta, v)a_{[t_0-1,t_0+1]} \subset \gamma Q$. In other words, $(\xi, \eta, v) \in Q \cap \gamma Q a_{-t}$ for all $t \in [t_0 - 1, t_0 + 1]$. Since $\|\mu_\theta(\gamma) - t_0 u\| < C_1$ by Proposition 5.3(1), we have $|\psi(\mu_\theta(\gamma)) - t_0\delta| < C_1\|\psi\|$. In particular, we have $[t_0 - 1, t_0 + 1] \subset [0, T]$ by the hypothesis.

We set $Q' := QA_{\theta,2}$ which is a compact subset of $\tilde{\Omega}_\theta$. We then have for each $t \in [t_0 - 1, t_0 + 1]$ that

$$\int_{A_\theta} \mathbb{1}_{Q' \cap \gamma Q' a_{-t}}((\xi, \eta, v)b)\, db \ge \int_{A_{\theta,2}} \mathbb{1}_{\gamma Q'}((\xi, \eta, v)ba_t)\, db \ge \operatorname{Vol}(A_{\theta,2})$$

where the last inequality follows from $(\xi,\eta,v)a_t \in \gamma Q$. Therefore, we have

$$\begin{aligned}\int_0^T \int_{\mathfrak{a}_\theta} \mathbb{1}_{Q'\cap\gamma Q' a_{-t}}(\xi,\eta,b)dbdt &= \int_0^T \int_{A_\theta} \mathbb{1}_{Q'\cap\gamma Q' a_{-t}}((\xi,\eta,v)b)dbdt \\ &\geq \int_{t_0-1}^{t_0+1} \int_{A_\theta} \mathbb{1}_{Q'\cap\gamma Q' a_{-t}}((\xi,\eta,v)b)dbdt \\ &\geq 2\operatorname{Vol}(A_{\theta,2})\end{aligned}$$

as desired. ☐

Proof of Proposition 7.4(2). Fix $R > \max(R(\nu), R(\nu_{\mathrm{i}}))$ where $R(\nu), R(\nu_{\mathrm{i}})$ are defined as in Lemma 4.4. Let $Q' = Q(r,R)A_{\theta,2}$ where $Q(r,R)$ is given in Lemma 7.9, so that Q' satisfies the conclusion of Lemma 7.10. For any $\gamma \in \Gamma$ and $t > 0$, we have

$$\begin{aligned}&\tilde{\mathsf{m}}(Q'\cap\gamma Q' a_{-t}) \\ &= \int_{\mathcal{F}_\theta^{(2)}} \left(\int_{\mathfrak{a}_\theta} \mathbb{1}_{Q'\cap\gamma Q' a_{-t}}(\xi,\eta,b)db\right) e^{\psi(\mathcal{G}^\theta(\xi,\eta))} d\nu(\xi)d\nu_{\mathrm{i}}(\eta) \\ &\geq \int_{O_R^\theta(o,\gamma o)\times O_R^{\mathrm{i}(\theta)}(\gamma o,o)} \left(\int_{\mathfrak{a}_\theta} \mathbb{1}_{Q'\cap\gamma Q' a_{-t}}(\xi,\eta,b)db\right) e^{\psi(\mathcal{G}^\theta(\xi,\eta))} d\nu(\xi)d\nu_{\mathrm{i}}(\eta).\end{aligned}$$

By Lemma 7.10, if $\gamma \in \Gamma_{u,r}$, $\|\mu_\theta(\gamma)\| > \ell_R$ and $C_1\|\psi\| + \delta < \psi(\mu_\theta(\gamma)) < \delta T - C_1\|\psi\| - \delta$ where $C_1 = C_1(Q)$, then

$$\begin{aligned}&\int_0^T \tilde{\mathsf{m}}(Q'\cap\gamma Q' a_{-t})dt \\ &\geq 2\operatorname{Vol}(A_{\theta,2}) \int_{O_R^\theta(o,\gamma o)\times O_R^{\mathrm{i}(\theta)}(\gamma o,o)} e^{\psi(\mathcal{G}^\theta(\xi,\eta))} d\nu(\xi)d\nu_{\mathrm{i}}(\eta) \\ &\geq 2\operatorname{Vol}(A_{\theta,2}) e^{-\|\psi\|\ell_R} \nu(O_R^\theta(o,\gamma o))\nu_{\mathrm{i}}(O_R^{\mathrm{i}(\theta)}(\gamma o,o))\end{aligned}$$

where the last inequality follows from $\|\mathcal{G}^\theta(\xi,\eta)\| < \ell_R$. By Lemma 4.4, we conclude

$$\int_0^T \tilde{\mathsf{m}}(Q'\cap\gamma Q' a_{-t})dt \gg e^{-\psi(\mu_\theta(\gamma))}.$$

For each $T \geq 1$, we define

$$\Gamma_T = \{\gamma\in\Gamma : \|\mu_\theta(\gamma)\| > \ell_R, C_1\|\psi\| + \delta < \psi(\mu_\theta(\gamma)) < \delta T - (C_1\|\psi\| + \delta)\}.$$

Since both $\{\gamma \in \Gamma : \|\mu_\theta(\gamma)\| \leq \ell_R\}$ and $\{\gamma \in \Gamma : \psi(\mu_\theta(\gamma)) \leq C_1\|\psi\| + \delta\}$ are finite sets, we have

$$\begin{aligned}\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q' \cap \gamma Q' a_{-t})dt &\geq \int_0^T \sum_{\gamma\in\Gamma_{u,r}\cap\Gamma_T} \tilde{\mathsf{m}}(Q' \cap \gamma Q' a_{-t})dt \\ &\gg \sum_{\gamma\in\Gamma_{u,r}\cap\Gamma_T} e^{-\psi(\mu_\theta(\gamma))} \\ &\gg \sum_{\substack{\gamma\in\Gamma_{u,r} \\ \psi(\mu_\theta(\gamma))<\delta T-(C_1\|\psi\|+\delta)}} e^{-\psi(\mu_\theta(\gamma))}.\end{aligned}$$

By Lemma 4.4 and Proposition 7.5,

$$\sum_{\substack{\gamma\in\Gamma_{u,r} \\ \delta T-(C_1\|\psi\|+\delta)\leq\psi(\mu_\theta(\gamma))\leq\delta T}} e^{-\psi(\mu_\theta(\gamma))} \ll \sum_{\substack{\gamma\in\Gamma_{u,r} \\ \delta T-(C_1\|\psi\|+\delta)\leq\psi(\mu_\theta(\gamma))\leq\delta T}} \nu(O_R^\theta(o,\gamma o)) \ll 1.$$

Therefore, we obtain

$$\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q' \cap \gamma Q' a_{-t})dt \gg \sum_{\substack{\gamma\in\Gamma_{u,r} \\ \psi(\mu_\theta(\gamma))\leq\delta T}} e^{-\psi(\mu_\theta(\gamma))}.$$

$\square$

We will apply the following version of Borel-Cantelli lemma.

Lemma 7.11. [1, Lem. 2] *Let (Ω, M) be a finite Borel measure space and $\{P_t : t \geq 0\}$ be a collection of subsets of Ω such that the map $(t, \omega) \mapsto \mathbb{1}_{P_t}(\omega)$ is measurable on $\mathbb{R}_+ \times \Omega$. Suppose that*

(1) *$\int_0^\infty \mathsf{M}(P_t)dt = \infty$, and*
(2) *for all large enough T,*

$$\int_0^T \int_0^T \mathsf{M}(P_t \cap P_s)dtds \ll \left(\int_0^T \mathsf{M}(P_t)dt\right)^2$$

where the implied constant is independent of T.

Then we have

$$\mathsf{M}\left(\left\{\omega \in \Omega : \int_0^\infty \mathbb{1}_{P_t}(\omega)dt = \infty\right\}\right) > 0.$$

Proof of Theorem 7.2. Let $Q \subset \tilde{\Omega}_\theta$ be a compact subset with $\tilde{\mathsf{m}}(Q) > 0$. Let $r = r(Q) > 1$ be large enough so that $\sum_{\gamma\in\Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ and that Proposition 7.4(1) holds. Let $Q' = Q'(r)$ be a compact subset of $\tilde{\Omega}_\theta$ given by Proposition 7.4(2). Replacing Q' with a larger compact subset if necessary, we may assume that $\tilde{\mathsf{m}}(Q') > 0$.

Since m is u-balanced, we have for $T > 1$ that

$$\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q \cap \gamma Q a_{-t})dt \asymp \int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q' \cap \gamma Q' a_{-t})dt \tag{7.2}$$

with the implied constant independent of T. Since we already have

$$\int_0^T \int_0^T \sum_{\gamma,\gamma'\in\Gamma} \tilde{\mathsf{m}}(Q\cap\gamma Q a_{-t}\cap \gamma' Q a_{-t-s}) dtds \ll \left(\sum_{\substack{\gamma\in\Gamma_{u,r} \\ \psi(\mu_\theta(\gamma))\le \delta T}} e^{-\psi(\mu_\theta(\gamma))} \right)^2$$

and

$$\sum_{\substack{\gamma\in\Gamma_{u,r} \\ \psi(\mu_\theta(\gamma))\le \delta T}} e^{-\psi(\mu_\theta(\gamma))} \ll \int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q'\cap \gamma Q' a_{-t}) dt \tag{7.3}$$

by Proposition 7.4, it follows from (7.2) that

$$\int_0^T \int_0^T \sum_{\gamma,\gamma'\in\Gamma} \tilde{\mathsf{m}}(Q\cap\gamma Q a_{-t}\cap \gamma' Q a_{-t-s}) dtds \ll \left(\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q\cap\gamma Q a_{-t}) dt \right)^2 . \tag{7.4}$$

By abusing notation, for a subset $U\subset \tilde{\Omega}_\theta$, we denote by $[U]$ the image of U under the projection $\tilde{\Omega}_\theta \to \Omega_\theta$, i.e., $[U]=\Gamma\backslash \Gamma U$. We set $\mathsf{M}=\mathsf{m}|_{[Q]}$ which is a finite Borel measure. We let $P_t = [Q\cap \Gamma Q a_{-t}]$ for $t\ge 0$. Since $\#\{\gamma\in\Gamma : Qa_{-t}\cap \gamma Q a_{-t}\neq\emptyset\}$ is uniformly bounded independent of t, we have $\mathsf{M}(P_t)\asymp \sum_{\gamma\in\Gamma}\tilde{\mathsf{m}}(Q\cap\gamma Q a_{-t})$ with the implied constant independent of t. Noting that $\sum_{\gamma\in\Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))}=\infty$, it follows from (7.2) and (7.3) that

$$\int_0^\infty \mathsf{M}(P_t) dt = \infty$$

and hence the condition (1) in Lemma 7.11 is satisfied.

The following is a rephrase of (7.4):

$$\int_0^T \int_0^T \mathsf{M}(P_t\cap P_{t+s}) dsdt \ll \left(\int_0^T \mathsf{M}(P_t)dt\right)^2 .$$

It implies

$$\begin{aligned}\int_0^T \int_0^T \mathsf{M}(P_t\cap P_s) dsdt &= 2\int_0^T\int_t^T \mathsf{M}(P_t\cap P_s)dsdt \\ &\le 2\int_0^T\int_0^T \mathsf{M}(P_t\cap P_{t+s})dsdt \\ &\ll \left(\int_0^T \mathsf{M}(P_t)dt\right)^2,\end{aligned}$$

showing that the condition (2) in Lemma 7.11 is satisfied.

Hence, by Lemma 7.11, we have

$$\mathsf{M}\left(\left\{[(\xi,\eta,v)]\in[Q] : \int_0^\infty \mathbb{1}_{[Q]}([(\xi,\eta,v)]a_t)dt=\infty\right\}\right)>0.$$

In particular, there exists a subset $Q_0 \subset Q$ such that $\tilde{\mathsf{m}}(Q_0) > 0$ and for all $(\xi, \eta, v) \in Q_0$, there exist sequences $\gamma_i \in \Gamma$ and $t_i \to \infty$ such that $\gamma_i^{-1}(\xi, \eta, v)a_{t_i} \in Q$ for all $i \geq 1$. In particular,

$$(\xi, \eta, v) \in Q \cap \gamma_i Q a_{-t_i} \quad \text{for all } i \geq 1,$$

which implies $\xi \in \Lambda_\theta^u$ by Lemma 5.5. Since this holds for all $(\xi, \eta, v) \in Q_0$, we have that

$$\xi \in \Lambda_\theta^u \quad \text{for all } (\xi, \eta, v) \in Q_0.$$

Since $\tilde{\mathsf{m}}(Q_0) > 0$ and $\tilde{\mathsf{m}}$ is equivalent to the product measure $\nu \otimes \nu_{\mathrm{i}} \otimes db$, it follows that $\nu(\Lambda_\theta^u) > 0$ as desired. Since m is A_u-invariant, the u-balanced condition remains same after changing the sign of T. Then the same argument with the negative T gives $\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)}) > 0$. ☐

Lemma 7.12. *We have either*

$$\nu(\Lambda_\theta^u) = 0 \quad \text{or} \quad \nu(\Lambda_\theta^u) = 1.$$

Proof. Suppose that $\nu(\Lambda_\theta^u) > 0$. Then by Theorem 5.6, we must have $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ for some $r > 0$. This implies that ν is the unique (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$ ([10], [22, Thm. 1.5]). On the other hand, if $0 < \nu(\Lambda_\theta^u) < 1$, then $\tilde{\nu} := \frac{1}{\nu(\mathcal{F}_\theta - \Lambda_\theta^u)} \nu|_{\mathcal{F}_\theta - \Lambda_\theta^u}$ defines another (Γ, ψ)-conformal measure, which would contradict the uniqueness of the (Γ, ψ)-conformal measure. Therefore, $\nu(\Lambda_\theta^u)$ must be either 0 or 1. ☐

Corollary 7.13. *If* m *is u-balanced, the following are equivalent:*

(1) $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ *for some* $r > 0$*;*
(2) $\nu(\Lambda_\theta^u) = 1 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})$.

Similarly, if m *is u-balanced, the following are also equivalent:*

(1) $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} < \infty$ *for all* $r > 0$*;*
(2) $\nu(\Lambda_\theta^u) = 0 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})$.

Proof. By Lemma 7.12, we have $\nu(\Lambda_\theta^u) = 0$ or $\nu(\Lambda_\theta^u) = 1$. Similarly, noting that $\psi \circ \mathrm{i} \in \mathfrak{a}_{\mathrm{i}(\theta)}^*$ is $(\Gamma, \mathrm{i}(\theta))$-proper as well, we also have either $\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)}) = 0$ or $\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)}) = 1$. Therefore Theorem 7.2 implies that if $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ for some $r > 0$, then $\nu(\Lambda_\theta^u) = 1 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})$. On the other hand Theorem 5.6 implies that if $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} < \infty$ for all $r > 0$, then $\nu(\Lambda_\theta^u) = 0 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})$. This proves the corollary. ☐

We finish the section with the following corollary of Proposition 7.4, which will be used later. The following estimate reduces the divergence of the series $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))}$ to the local mixing rate for the a_t-flow:

Corollary 7.14. *For all sufficiently large $r > 0$, there exist compact subsets Q_1, Q_2 of Ω_θ with non-empty interior such that for all $T \geq 1$,*

$$\left(\int_0^T \mathsf{m}(Q_1 \cap Q_1 a_{-t}) dt \right)^{1/2} \ll \sum_{\substack{\gamma \in \Gamma_{u,r} \\ \psi(\mu_\theta(\gamma)) \leq \delta T}} e^{-\psi(\mu_\theta(\gamma))} \ll \int_0^T \mathsf{m}(Q_2 \cap Q_2 a_{-t}) dt.$$

Proof. Let $Q \subset \tilde{\Omega}_\theta$ be a compact subset with non-empty interior. By Proposition 7.4(1), there exists $r_0 = r_0(Q) > 0$ such that for all $T \geq 1$ and for all $r \geq r_0$,

(7.5)
$$\int_0^T \int_0^T \sum_{\gamma, \gamma' \in \Gamma} \tilde{\mathsf{m}}(Q \cap \gamma Q a_{-t} \cap \gamma' Q a_{-t-s}) dt ds \ll \left(\sum_{\substack{\gamma \in \Gamma_{u,r} \\ \psi(\mu_\theta(\gamma)) \leq \delta T}} e^{-\psi(\mu_\theta(\gamma))} \right)^2.$$

Fix a small $\varepsilon > 0$ so that $Q^- := \bigcap_{0 \leq s \leq \varepsilon} Q a_{-s}$ has non-empty interior. Since we have

$$\varepsilon \int_0^T \sum_{\gamma \in \Gamma} \tilde{\mathsf{m}}(Q^- \cap \gamma Q^- a_{-t}) dt \leq \int_0^T \int_0^\varepsilon \sum_{\gamma \in \Gamma} \tilde{\mathsf{m}}(Q \cap \gamma (Q \cap Q a_{-s}) a_{-t}) ds dt,$$

it follows from (7.5) that for all $r \geq r_0$,

$$\int_0^T \sum_{\gamma \in \Gamma} \tilde{\mathsf{m}}(Q^- \cap \gamma Q^- a_{-t}) dt \ll \left(\sum_{\substack{\gamma \in \Gamma_{u,r} \\ \psi(\mu_\theta(\gamma)) \leq \delta T}} e^{-\psi(\mu_\theta(\gamma))} \right)^2.$$

Now let $Q' = Q'(r) \subset \tilde{\Omega}_\theta$ be a compact subset given in Proposition 7.4(2) such that for any $T > 1$,

$$\int_0^T \sum_{\gamma \in \Gamma} \tilde{\mathsf{m}}(Q' \cap \gamma Q' a_{-t}) dt \gg \sum_{\substack{\gamma \in \Gamma_{u,r} \\ \psi(\mu_\theta(\gamma)) \leq \delta T}} e^{-\psi(\mu_\theta(\gamma))}. \tag{7.6}$$

Replacing Q' with a larger compact subset, we may assume that $\operatorname{int} Q' \neq \emptyset$. Hence it suffices to set $Q_1 = \Gamma \backslash \Gamma Q^-$ and $Q_2 = \Gamma \backslash \Gamma Q'$ to finish the proof. □

Remark 7.15. For $\theta = \Pi$, Corollary 7.14 was established in [9] for any Zariski dense discrete subgroup of G (see [9, Proof of Thm. 6.3]). If Γ is a lattice of G, then, together with the Howe-Moore mixing property of the (finite) Haar measure [18], it implies that for any non-zero $u \in \mathfrak{a}^+$, we have $\sum_{\gamma \in \Gamma_{u,r}} e^{-2\rho(\mu(\gamma))} = \infty$ for all $r > 1$ large enough where 2ρ denotes the sum of all positive roots counted with multiplicity.

8. Transitivity subgroup and ergodicity of directional flows

In this section, we complete the proof of Theorem 1.1, by establishing the equivalence between co-nullity of directional conical sets and conservativity/ergodicity of directional flows. We use the notion of transitivity subgroup to carry out the Hopf argument in our setting.

Let $\Gamma < G$ be a Zariski dense θ-transverse subgroup. We fix a non-zero vector $u \in \mathfrak{a}_\theta^+$ and a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$. We also fix a pair ν, ν_{i} of (Γ, ψ) and $(\Gamma, \psi \circ \mathrm{i})$-conformal measures on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively. Denote by $\mathsf{m} = \mathsf{m}(\nu, \nu_{\mathrm{i}})$ the associated BMS measure on Ω_θ. In this section, we discuss the ergodicity and conservativity of the directional flow

$$A_u = \{a_t := \exp(tu) : t \in \mathbb{R}\}$$

on Ω_θ with respect to m. We emphasize that the notion of a transitivity subgroup plays a key role in showing the A_u-ergodicity.

Conservativity of directional flows. Recall the following definitions:

(1) A Borel subset $B \subset \Omega_\theta$ is called a wandering set for m if for m-a.e. $x \in B$, we have $\int_{-\infty}^{\infty} \mathbb{1}_B(xa_t)dt < \infty$.
(2) We say that $(\Omega_\theta, A_u, \mathsf{m})$ is completely conservative if there is no wandering set $B \subset \Omega_\theta$ with $\mathsf{m}(B) > 0$.
(3) We say that $(\Omega_\theta, A_u, \mathsf{m})$ is completely dissipative if Ω_θ is a countable union of wandering sets modulo m.

The following is proved for $\theta = \Pi$ in [9, Prop. 4.2] and a similar proof works for general θ:

Proposition 8.1. *The flow $(\Omega_\theta, A_u, \mathsf{m})$ is completely conservative (resp. completely dissipative) if and only if $\max\left(\nu(\Lambda_\theta^u), \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})\right) > 0$ (resp. $\nu(\Lambda_\theta^u) = 0 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})$).*

Proof. Suppose that there exists a non-wandering subset B with $\mathsf{m}(B) > 0$. Setting $B^\pm := \{x \in B : \limsup_{t \to \pm\infty} xa_t \cap B \neq \emptyset\}$, we have $\mathsf{m}(B^+ \cup B^-) > 0$. Since m is locally equivalent to $\nu \otimes \nu_{\mathrm{i}} \otimes db$, if we have $\mathsf{m}(B^+) > 0$, then $\nu(\Lambda_\theta^u) > 0$ by Lemma 5.5. Otherwise, if $\mathsf{m}(B^-) > 0$, then $\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)}) > 0$. It shows the following two implications:

$$(\Omega_\theta, A_u, \mathsf{m}) \text{ is completely conservative} \Rightarrow \max\left(\nu(\Lambda_\theta^u), \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})\right) > 0;$$

$$(\Omega_\theta, A_u, \mathsf{m}) \text{ is completely dissipative} \Leftarrow \nu(\Lambda_\theta^u) = 0 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)})$$

where the second implication is due to the σ-compactness of Ω_θ.

Now suppose that $\nu(\Lambda_\theta^u) > 0$ (resp. $\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)} > 0$). By Theorem 5.6, $\sum_{\gamma \in \Gamma_{u,r}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ (resp. $\sum_{\gamma \in \Gamma_{u,r}^{-1}} e^{-(\psi \circ \mathrm{i})(\mu_{\mathrm{i}(\theta)}(\gamma))} = \infty$) for some $r > 0$. Note that $\gamma \in \Gamma_{u,r}^{-1}$ if and only if $\|\mu_{\mathrm{i}(\theta)}(\gamma) - t\,\mathrm{i}(u)\| < r$ for some $t \geq 0$. Hence it follows from (7.12) that $\nu(\Lambda_\theta^u) = 1$ (resp. $\nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)}) = 1$). It implies that for m-a.e. $\Gamma[g] \in \Omega_\theta$, we have $g^+ \in \Lambda_\theta^u$ (resp. $g^- \in \Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(u)}$) and hence $\Gamma[g]a_{t_i u}$ is a convergent sequence for some sequence $t_i \to \infty$ (resp. $t_i \to -\infty$). Hence, for m-a.e. $x \in \Omega_\theta$, there exists a compact subset B such that $\int_{-\infty}^{\infty} \mathbb{1}_B(xa_t)dt = \infty$. This implies the conservativity of $(\Omega_\theta, A_u, \mathsf{m})$ by [25, Lem. 6.1]. $\square$

Density of θ-transitivity subgroups.

Definition 8.2 (θ-transitivity subgroup)**.** For $g \in G$ with $(g^+, g^-) \in \Lambda_\theta^{(2)}$, define $\mathcal{H}_\Gamma^\theta(g)$ to be the set of all elements $a \in A_\theta$ such that there exist $\gamma \in \Gamma$, $s \in S_\theta$ and a sequence $n_1, \cdots, n_k \in N_\theta \cup \check{N}_\theta$ such that

(1) $((gn_1 \cdots n_r)^+, (gn_1 \cdots n_r)^-) \in \Lambda_\theta^{(2)}$ for all $1 \le r \le k$; and
(2) $\gamma g n_1 \cdots n_k = gas$.

It is not hard to see that $\mathcal{H}_\Gamma^\theta(g)$ is a subgroup (cf. [38, Lem. 3.1]).

We deduce the density of transitive subgroups from Theorem 2.6:

Proposition 8.3. *For any $g \in G$ with $(g^+, g^-) \in \Lambda_\theta^{(2)}$, the subgroup $\mathcal{H}_\Gamma^\theta(g)$ is dense in A_θ.*

Proof. Note that we have a Zariski dense open subset $g\check{N}_\theta P_\theta/P \subset \mathcal{F}$; this is well-defined since $P \subset P_\theta$. Hence there exists a Zariski dense Schottky subgroup $\Gamma_0 < \Gamma$ so that for any loxodromic element $\gamma \in \Gamma_0$, its attracting fixed point y_γ belongs to $g\check{N}_\theta P_\theta$ (cf. [14, Lem. 7.3], [3]). Note that any non-trivial element of Γ_0 is loxodromic. By Theorem 2.6, it suffices to prove:

$$\{p_\theta(\lambda(\gamma)) : \gamma \in \Gamma_0\} \subset \log \mathcal{H}_\Gamma^\theta(g). \tag{8.1}$$

Fixing any non-trivial element $\gamma \in \Gamma_0$, write $\gamma = ha_\gamma m h^{-1} \in hA^+Mh^{-1}$ for some $h \in G$. Then $\lambda(\gamma) = \log a_\gamma$ and $y_\gamma = hP \in \Lambda$; hence $y_\gamma^\theta := hP_\theta \in g\check{N}_\theta P_\theta$. Using $P_\theta = N_\theta A_\theta S_\theta$, we can write $h \in g\check{n}nA_\theta S_\theta$ for some $\check{n} \in \check{N}_\theta$ and $n \in N_\theta$. By replacing h with $g\check{n}n$, we may assume that

$$h = g\check{n}n \in g\check{N}_\theta N_\theta \quad \text{and} \quad \gamma = hash^{-1}$$

for some $s \in S_\theta$ where a is the A_θ-component of a_γ in the decomposition $a_\gamma \in A_\theta^+ B_\theta^+$ so that $p_\theta(\log a_\gamma) = \log a$. It remains to show that $a \in \mathcal{H}_\Gamma^\theta(g)$. We first note from $\gamma = hash^{-1}$ and $h = g\check{n}n$ that

$$\gamma = (gas)\left((as)^{-1}\check{n}(as)\right)\left((as)^{-1}n(as)\right)n^{-1}\check{n}^{-1}g^{-1}$$

and hence

$$\gamma g\check{n}n\left((as)^{-1}n^{-1}(as)\right)\left((as)^{-1}\check{n}^{-1}(as)\right) = gas. \tag{8.2}$$

Writing $n_1 = \check{n}$, $n_2 = n$, $n_3 = (as)^{-1}n^{-1}(as)$ and $n_4 = (as)^{-1}\check{n}^{-1}(as)$, we have $n_1, n_4 \in \check{N}_\theta$ and $n_2, n_3 \in N_\theta$. By (8.2), the elements n_i, $1 \le i \le 4$, satisfy the second condition for $a \in \mathcal{H}_\Gamma^\theta(g)$. We now check the first condition:

- $gn_1P_\theta = g\check{n}P_\theta = hP_\theta = y_\gamma^\theta \in \Lambda_\theta$ and $gn_1w_0P_{\mathrm{i}(\theta)} = gw_0P_{\mathrm{i}(\theta)} \in \Lambda_{\mathrm{i}(\theta)}$;
- $gn_1n_2P_\theta = hP_\theta \in \Lambda_\theta$ and $gn_1n_2w_0P_{\mathrm{i}(\theta)} = hw_0P_{\mathrm{i}(\theta)} = y_{\gamma^{-1}}^{\mathrm{i}(\theta)} \in \Lambda_{\mathrm{i}(\theta)}$;

- $gn_1n_2n_3P_\theta = gn_1n_2P_\theta \in \Lambda_\theta$ and $gn_1n_2n_3w_0P_{\mathrm{i}(\theta)} = \gamma^{-1}gasn_4^{-1}w_0P_{\mathrm{i}(\theta)} = \gamma^{-1}gasw_0P_{\mathrm{i}(\theta)} = \gamma^{-1}gw_0P_{\mathrm{i}(\theta)} \in \Lambda_{\mathrm{i}(\theta)}$ by (8.2);
- $gn_1n_2n_3n_4P_\theta = \gamma^{-1}gasP_\theta = \gamma^{-1}gP_\theta \in \Lambda_\theta$ and $gn_1n_2n_3n_4w_0P_{\mathrm{i}(\theta)} = gn_1n_2n_3w_0P_{\mathrm{i}(\theta)} \in \Lambda_{\mathrm{i}(\theta)}$.

This proves that $a \in \mathcal{H}_\Gamma^\theta(g)$ and completes the proof. □

Stable and unstable foliations for directional flows. Recall the notation that for $g \in G$, we set

$$[g] = (g^+, g^-, \beta_{g^+}^\theta(e, g)) \in \mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta.$$

Lemma 8.4. *Let $g \in G$, $n \in N_\theta$ and $\check{n} \in \check{N}_\theta$. Then*

$$[gn] = (g^+, (gn)^-, \beta_{g^+}^\theta(e, g));$$
$$[g\check{n}] = ((g\check{n})^+, g^-, \beta_{g^+}^\theta(e, g) + \mathcal{G}^\theta((g\check{n})^+, g^-) - \mathcal{G}^\theta(g^+, g^-)).$$

Proof. Since $(gn)^+ = gnP_\theta = gP_\theta$, we have

$$\beta_{(gn)^+}^\theta(e, gn) - \beta_{g^+}^\theta(e, g) = \beta_{e^+}^\theta(e, n) = 0$$

and therefore $[gn] = (g^+, (gn)^-, \beta_{g^+}^\theta(e, g))$. To see the second identity, we first note that $g\check{n}w_0P_{\mathrm{i}(\theta)} = gw_0P_{\mathrm{i}(\theta)}$, that is, $(g\check{n})^- = g^-$. Since $\beta_{e^-}^{\mathrm{i}(\theta)}(e, \check{n}) = 0$, we have

$$\begin{aligned}\mathcal{G}^\theta((g\check{n})^+, g^-) &= \beta_{(g\check{n})^+}^\theta(e, g\check{n}) + \mathrm{i}(\beta_{g^-}^{\mathrm{i}(\theta)}(e, g)) + \mathrm{i}(\beta_{e^-}^{\mathrm{i}(\theta)}(e, \check{n})) \\ &= \beta_{(g\check{n})^+}^\theta(e, g\check{n}) + \mathrm{i}(\beta_{g^-}^{\mathrm{i}(\theta)}(e, g)).\end{aligned}$$

Since $\mathcal{G}^\theta(g^+, g^-) = \beta_{g^+}^\theta(e, g) + \mathrm{i}(\beta_{g^-}^{\mathrm{i}(\theta)}(e, g))$, we get

$$\beta_{(g\check{n})^+}^\theta(e, g\check{n}) = \beta_{g^+}^\theta(e, g) + \mathcal{G}^\theta((g\check{n})^+, g^-) - \mathcal{G}^\theta(g^+, g^-)$$

proving the second identity. □

We say a metric d on Ω_θ *admissible* if it extends to a metric of the one-point compactification of Ω_θ (if Ω_θ is compact, any metric is admissible). Since Ω_θ is a second countable locally compact Hausdorff space (Theorem 5.1), there exists an admissible metric.

For $x \in \Omega_\theta$, we define $W^{ss}(x)$ (resp. $W^{su}(x)$) to be the set of all $y \in \Omega_\theta$ such that $\mathsf{d}(xa_t, ya_t) \to 0$ as $t \to +\infty$ (resp. $t \to -\infty$). They form strongly stable and unstable foliations in Ω_θ with respect to the flow $\{a_t\}$ respectively.

In turns out that with respect to any admissible metric d on Ω_θ, the N_θ and $\check{N}_\theta$-orbits are contained in the stable and unstable foliations of the directional flow $\{a_t\}$ on Ω_θ respectively. The following proposition is important in applying Hopf-type arguments; the observation that one can use an admissible metric in this context is due to Blayac-Canary-Zhu-Zimmer [5].

Proposition 8.5. *Let $g \in G$ be such that $[g] \in \tilde{\Omega}_\theta$. For any compact subsets $\mathcal{U} \subset N_\theta$ and $\check{\mathcal{U}} \subset \check{N}_\theta$, we have, as $t \to +\infty$*

$$\operatorname{diam}\left(\{\Gamma[gn] \in \Omega_\theta : n \in \mathcal{U}\} \cdot a_t\right) \to 0;$$
$$\operatorname{diam}\left(\{\Gamma[g\check{n}] \in \Omega_\theta : \check{n} \in \check{\mathcal{U}}\} \cdot a_{-t}\right) \to 0$$

where the diameter is computed with respect to an admissible metric d on Ω_θ. In particular,

1. *$\{\Gamma[gn] \in \Omega_\theta : n \in N_\theta\} \subset W^{ss}(\Gamma[g])$;*
2. *$\{\Gamma[g\check{n}] \in \Omega_\theta : \check{n} \in \check{N}_\theta\} \subset W^{su}(\Gamma[g])$.*

Proof. Let ♠ be the point at infinity in the one-point compactification of Ω_θ. For each $\varepsilon > 0$, set $\mathsf{Q}_\varepsilon = \Omega_\theta$ if Ω_θ is compact and $\mathsf{Q}_\varepsilon = \{x \in \Omega_\theta : \mathsf{d}(x, ♠) \geq \varepsilon/2\}$ otherwise, and choose a compact lift $\tilde{\mathsf{Q}}_\varepsilon \subset \tilde{\Omega}_\theta$ of Q_ε. Let $[g] = (\xi, \eta, v) \in \tilde{\Omega}_\theta$. To show the first claim, suppose not. Then there exist $\varepsilon > 0$, a sequence $t_i \to \infty$ and convergent sequences $n_i, n_i' \in N_\theta$ such that $[gn_i], [gn_i'] \in \tilde{\Omega}_\theta$ and $\mathsf{d}(\Gamma[gn_i]a_{t_i}, \Gamma[gn_i']a_{t_i}) > \varepsilon$ for all $i \geq 1$. By passing to a subsequence and switching n_i and n_i' if necessary, we may assume that for all $i \geq 1$, $\gamma_i[gn_i]a_{t_i} \in \tilde{\mathsf{Q}}_\varepsilon$ for some $\gamma_i \in \Gamma$. After passing to a subsequence, we have the convergence

(8.3)
$$\gamma_i[gn_i]a_{t_i} = (\gamma_i\xi, \gamma_i(gn_i)^-, v + \beta_\xi^\theta(\gamma_i^{-1}, e) + t_i u) \to (\xi_0, \eta_0, v_0) \text{ as } i \to \infty,$$

for some $(\xi_0, \eta_0, v_0) \in \tilde{\mathsf{Q}}_\varepsilon$. In particular, for any linear form $\phi \in \mathfrak{a}_\theta^*$ positive on $\mathfrak{a}_\theta^+$, we must have $\phi(\beta_\xi^\theta(\gamma_i^{-1}, e)) \to -\infty$ as $i \to \infty$ and the sequence γ_i is unbounded.

Since the sequence $n_i \in N_\theta$ converges, the sequence $(\xi, (gn_i)^-) \in \Lambda_\theta^{(2)}$ is convergent as well. Moreover, (8.3) implies that the sequence $\gamma_i(\xi, (gn_i)^-) \in \Lambda_\theta^{(2)}$ is precompact. By the argument as in the proof of [22, Lem. 9.10, Prop. 9.11], for any compact subset $C \subset \Lambda_{\mathrm{i}(\theta)}$ with $\{\xi\} \times C \subset \Lambda_\theta^{(2)}$, we have $\gamma_i C \to \eta_0$ as $i \to \infty$. Since $n_i' \in N_\theta$ is a convergent sequence and $(\xi, \{(gn_i')^-\}) \subset \Lambda_\theta^{(2)}$, we have $\gamma_i(gn_i')^- \to \eta_0$. Since $[gn_i'] = (\xi, (gn_i')^-, v)$ by Lemma 8.4, we deduce from (8.3) that

$$\gamma_i[gn_i']a_{t_i} = (\gamma_i\xi, \gamma_i(gn_i')^-, v + \beta_\xi^\theta(\gamma_i^{-1}, e) + t_i u) \to (\xi_0, \eta_0, v_0) \text{ as } i \to \infty.$$

Therefore, two sequences $\gamma_i[gn_i]a_{t_i}$ and $\gamma_i[gn_i']a_{t_i}$ converge to the same limit, which is a contradiction to the assumption $\mathsf{d}(\Gamma[gn_i]a_{t_i}, \Gamma[gn_i']a_{t_i}) > \varepsilon$ for all $i \geq 1$. Hence the first claim is proved.

For the second claim, suppose to the contrary that for some $\varepsilon > 0$, there exist a sequence $t_i \to \infty$ and convergent sequences $\check{n}_i, \check{n}_i' \in \check{N}_\theta$ such that $[g\check{n}_i], [g\check{n}_i'] \in \tilde{\Omega}_\theta$ and $\mathsf{d}(\Gamma[g\check{n}_i]a_{-t_i}, \Gamma[g\check{n}_i']a_{-t_i}) > \varepsilon$ for all $i \geq 1$. As above, we may then assume that for all $i \geq 1$, $\gamma_i[g\check{n}_i]a_{-t_i} \in \tilde{\mathsf{Q}}_\varepsilon$ for some sequence $\gamma_i \in \Gamma$. By passing to a subsequence, we have the convergence

$$\gamma_i[g\check{n}_i]a_{-t_i} \to (\xi_1, \eta_1, v_1) \text{ as } i \to \infty$$

for some $(\xi_1, \eta_1, v_1) \in \tilde{\mathsf{Q}}_\varepsilon$. By Lemma 8.4, we have for each $i \geq 1$ that

$$\begin{aligned}\gamma_i[g\check{n}_i] &= \gamma_i \left((g\check{n}_i)^+, \eta, v + \mathcal{G}^\theta((g\check{n}_i)^+, \eta) - \mathcal{G}^\theta(\xi, \eta)\right) \\ &= \left(\gamma_i(g\check{n}_i)^+, \gamma_i\eta, v + \mathcal{G}^\theta((g\check{n}_i)^+, \eta) - \mathcal{G}^\theta(\xi, \eta) + \beta^\theta_{(g\check{n}_i)^+}(\gamma_i^{-1}, e)\right),\end{aligned}$$

and therefore we have that as $i \to \infty$,

$$\begin{aligned}&\gamma_i(g\check{n}_i)^+ \to \xi_1; \\ &\gamma_i\eta \to \eta_1; \\ &v + \mathcal{G}^\theta((g\check{n}_i)^+, \eta) - \mathcal{G}^\theta(\xi, \eta) + \beta^\theta_{(g\check{n}_i)^+}(\gamma_i^{-1}, e) - t_i u \to v_1.\end{aligned} \tag{8.4}$$

Since the sequence $\check{n}_i \in \check{N}_\theta$ converges, the sequence $((g\check{n}_i)^+, \eta) \in \Lambda_\theta^{(2)}$ is convergent as well. Hence $\mathcal{G}^\theta((g\check{n}_i)^+, \eta)$ is a bounded sequence in $\mathfrak{a}_\theta$. It then follows from (8.4) that for any linear form $\phi \in \mathfrak{a}_\theta^*$ positive on $\mathfrak{a}_\theta^+$, we have

$$\phi(\beta^\theta_{(g\check{n}_i)^+}(\gamma_i^{-1}, e)) \to \infty \quad \text{as } i \to \infty$$

and the sequence γ_i is unbounded.

Again, by the same argument as in the proof of [22, Lem. 9.10, Prop. 9.11], we obtain that for any compact subset $C \subset \Lambda_\theta$ such that $C \times \{\eta\} \subset \Lambda_\theta^{(2)}$, we have $\gamma_i C \to \xi_1$ as $i \to \infty$. Since the sequence $((g\check{n}'_i)^+, \eta) \in \Lambda_\theta^{(2)}$ is convergent as mentioned above, we also have $\gamma_i(g\check{n}'_i)^+ \to \xi_1$ as $i \to \infty$. It then follows from Lemma 8.4 that

$$\gamma_i[g\check{n}_i] = \left(\gamma_i(g\check{n}_i)^+, \gamma_i\eta, v + \beta^\theta_\xi(\gamma_i^{-1}, e) + \mathcal{G}^\theta(\gamma_i(g\check{n}_i)^+, \gamma_i\eta) - \mathcal{G}^\theta(\gamma_i\xi, \gamma_i\eta)\right);$$

$$\gamma_i[g\check{n}'_i] = \left(\gamma_i(g\check{n}'_i)^+, \gamma_i\eta, v + \beta^\theta_\xi(\gamma_i^{-1}, e) + \mathcal{G}^\theta(\gamma_i(g\check{n}'_i)^+, \gamma_i\eta) - \mathcal{G}^\theta(\gamma_i\xi, \gamma_i\eta)\right).$$

Since both sequences $(\gamma_i(g\check{n}_i)^+, \gamma_i\eta)$ and $(\gamma_i(g\check{n}'_i)^+, \gamma_i\eta)$ converge to (ξ_1, η_1) and $\gamma_i[g\check{n}_i]a_{-t_i} \to (\xi_1, \eta_1, v_1)$ as $i \to \infty$, it follows that

$$\gamma_i[g\check{n}'_i]a_{-t_i} \to (\xi_1, \eta_1, v_1) \quad \text{as } i \to \infty.$$

Again, two sequences $\gamma_i[g\check{n}_i]a_{-t_i}$ and $\gamma_i[g\check{n}'_i]a_{-t_i}$ converge to the same limit, contradicting the assumption that $\mathsf{d}(\Gamma[g\check{n}_i]a_{-t_i}, \Gamma[g\check{n}'_i]a_{-t_i}) > \varepsilon$ for all $i \geq 1$. This proves (2). □

For a (Γ, θ)-proper form $\phi \in \mathfrak{a}_\theta^*$, the action of $A_u = \{a_t : t \in \mathbb{R}\}$ on Ω_θ induces a right A_u-action on Ω_ϕ via the projection $\Omega_\theta \to \Omega_\phi$ where Ω_ϕ is defined in (6.4). Note that when $u \in \operatorname{int} \mathcal{L}_\theta$, the condition $\phi(u) > 0$ is satisfied for any (Γ, θ)-proper $\phi \in \mathfrak{a}_\theta^*$ [22, Lem. 4.3].

Proposition 8.6. *Let $\phi \in \mathfrak{a}_\theta^*$ be a (Γ, θ)-proper form such that $\phi(u) > 0$ and $g \in G$ be such that $[g]_\phi \in \tilde{\Omega}_\phi$. For any compact subsets $\mathcal{U} \subset N_\theta$ and $\check{\mathcal{U}} \subset \check{N}_\theta$, we have, as $t \to +\infty$,*

$$\operatorname{diam}\left(\{\Gamma[gn]_\phi \in \Omega_\phi : n \in \mathcal{U}\} \cdot a_t\right) \to 0;$$

$$\operatorname{diam}\left(\{\Gamma[g\check{n}]_\phi \in \Omega_\phi : \check{n} \in \check{\mathcal{U}}\} \cdot a_{-t}\right) \to 0$$

where the diameter is computed with respect to an admissible[5] *metric* d *on* Ω_ϕ. *In particular, we have*

$$\{\Gamma[gn]_\phi \in \Omega_\phi : n \in N_\theta\} \subset W^{ss}(\Gamma[g]_\phi);$$
$$\{\Gamma[g\check{n}]_\phi \in \Omega_\phi : \check{n} \in \check{N}_\theta\} \subset W^{su}(\Gamma[g]_\phi),$$

where $W^{ss}(x)$ *(resp.* $W^{su}(x)$*) is the set of all* $y \in \Omega_\phi$ *such that* $\mathsf{d}(xa_t, ya_t) \to 0$ *as* $t \to +\infty$ *(resp.* $t \to -\infty$*) for* $x \in \Omega_\phi$.

Proof. The condition $\phi(u) > 0$ ensures that the convergence of the sequences $\phi(\beta^\theta_\xi(\gamma_i^{-1}, e)) + t_i\phi(u)$ in (8.3) and $\phi(\beta^\theta_{(g\check{n}_i)^+}(\gamma_i^{-1}, e)) - t_i\phi(u)$ in (8.4) implies that $\phi(\beta^\theta_\xi(\gamma_i^{-1}, e)) \to -\infty$ and $\phi(\beta^\theta_{(g\check{n}_i)^+}(\gamma_i^{-1}, e)) \to +\infty$ respectively. Given this, we can proceed exactly as in the proof of Proposition 8.5, replacing Ω_θ by Ω_ϕ. □

Conservativity of general actions. Let H be a connected subgroup of A. Denote by dh the Haar measure on H. Consider the dynamical system (H, Ω, λ) where Ω is a separable, locally compact and σ-compact topological space on which H acts continuously and preserving a Radon measure λ on Ω. A Borel subset $B \subset \Omega$ is called wandering if $\int_H \mathbb{1}_B(h.w)dh < \infty$ for λ-almost all $w \in B$. If there is no wandering subset of positive measure, the system is called completely conservative. If Ω is a countable union of wandering subsets, then the system is called completely dissipative. An ergodic system (H, Ω, λ) is either completely conservative or completely dissipative by the Hopf decomposition theorem.

Lemma 8.7. *If* $(\Omega_\theta, A_\theta, \mathsf{m})$ *is completely conservative, then it is* A_θ*-ergodic.*

Proof. Choose any $\phi \in \mathfrak{a}_\theta^*$ which is positive on $\mathfrak{a}_\theta^+$; in particular, ϕ is (Γ, θ)-proper. Consider $\tilde{\Omega}_\phi$, Ω_ϕ and $\mathsf{m}^\phi = \mathsf{m}^\phi_{\nu,\nu_{\mathrm{i}}}$ as defined in (6.4) and (6.6). The conservativity of the A_θ-action on $(\Omega_\theta, \mathsf{m})$ then implies the conservativity of the $\mathbb{R}$-action on $(\Omega_\phi, \mathsf{m}^\phi)$, and the A_θ-ergodicity on $(\Omega_\theta, \mathsf{m})$ follows if we show the ergodicity of $(\Omega_\phi, \mathbb{R}, \mathsf{m}^\phi)$.

Let f be a bounded m^ϕ-measurable $\mathbb{R}$-invariant function on Ω_ϕ. We need to show that f is constant m^ϕ-a.e. Choose any admissible metric on Ω_ϕ which exists by Theorem 6.1 and apply Proposition 8.6. By a theorem of Coudéne [13, Sec. 2], it follows that there exists an m^ϕ-conull subset $W_0 \subset \Omega_\phi$ such that if $\Gamma[g]_\phi, \Gamma[gn]_\phi \in W_0$ for $g \in G$ and $n \in N_\theta \cup \check{N}_\theta$, then

$$f(\Gamma[g]_\phi) = f(\Gamma[gn]_\phi).$$

Let $\tilde{f} : \tilde{\Omega}_\phi \to \mathbb{R}$ and $\tilde{W}_0 \subset \tilde{\Omega}_\phi$ be Γ-invariant lifts of f and W_0 respectively. Since f is $\mathbb{R}$-invariant, we may assume that $\tilde{W}_0$ is $\mathbb{R}$-invariant as well. For any $[g]_\phi, [h]_\phi \in \tilde{\Omega}_\phi$ with $g^+ = h^+$, we can find $n \in N_\theta$ and $a \in A_\theta$ such that $[gna]_\phi = [h]_\phi$ by (2.6). Similarly, if $g^- = h^-$, we can find $n \in \check{N}_\theta$ and $a \in A_\theta$ such that $[gna]_\phi = [h]_\phi$. Hence, by the $\mathbb{R}$-invariance of f and hence

[5]I.e., it extends to a metric on the one-point compactification of Ω_ϕ

of $\tilde{f}$, for any $(\xi,\eta,s),(\xi',\eta',s')\in\tilde{W}_0$ such that $\xi=\xi'$ or $\eta=\eta'$, we have $\tilde{f}(\xi,\eta,s)=\tilde{f}(\xi',\eta',s')$.

Let

$$W^+:=\{\xi\in\Lambda_\theta:(\xi,\eta',s)\in\tilde{W}_0 \text{ for all } s\in\mathbb{R} \text{ and } \nu_{\mathrm{i}}\text{-a.e. } \eta'\};$$
$$W^-:=\{\eta\in\Lambda_{\mathrm{i}(\theta)}:(\xi',\eta,s)\in\tilde{W}_0 \text{ for all } s\in\mathbb{R} \text{ and } \nu\text{-a.e. } \xi'\}.$$

Then $\nu(W^+)=\nu_{\mathrm{i}}(W^-)=1$ by Fubini's theorem. Hence the set $W':=(W^+\times W^-)\cap\Lambda_\theta^{(2)}$ has full $\nu\otimes\nu_{\mathrm{i}}$-measure. We choose a $\nu\otimes\nu_{\mathrm{i}}$-conull subset $W\subset W'$ such that $W\times\mathbb{R}\subset\tilde{W}_0$. Let $(\xi,\eta),(\xi',\eta')\in W$. Then there exists $\eta_1\in\Lambda_{\mathrm{i}(\theta)}$ so that $(\xi,\eta_1),(\xi',\eta_1)\in W$. Hence for any $s\in\mathbb{R}$, we get

$$\tilde{f}(\xi,\eta,s)=\tilde{f}(\xi,\eta_1,s)=\tilde{f}(\xi',\eta_1,s)=\tilde{f}(\xi',\eta',s).$$

Therefore, $\tilde{f}$ is constant on $W\times\mathbb{R}$, and hence f is constant m^ϕ-a.e., completing the proof. ☐

Ergodicity of directional flows. We now prove the following analog of the Hopf dichotomy:

Proposition 8.8. *The directional flow $(\Omega_\theta, A_u, \mathsf{m})$ is completely conservative if and only if $(\Omega_\theta, A_u, \mathsf{m})$ is ergodic.*

Proof. Suppose that $(\Omega_\theta, A_u, \mathsf{m})$ is completely conservative. Since this implies that $(\Omega_\theta, A_\theta, \mathsf{m})$ is completely conservative, we have $(\Omega_\theta, A_\theta, \mathsf{m})$ is ergodic by Lemma 8.7. Let $f:\Omega_\theta\to\mathbb{R}$ be a bounded measurable function which is A_u-invariant. By the A_θ-ergodicity, it suffices to prove that f is A_θ-invariant.

Choose any admissible metric on Ω_θ which exists by Theorem 5.1. Similarly to the proof of Lemma 8.7, Proposition 8.5 and [13] imply that there exists an m-conull subset $W_0\subset\Omega_\theta$ such that if $\Gamma[g],\Gamma[gn]\in W_0$ for $g\in G$ and $n\in N_\theta\cup\check{N}_\theta$, then

$$f(\Gamma[g])=f(\Gamma[gn]).$$

Consider the Γ-invariant lifts $\tilde{f}:\tilde{\Omega}_\theta\to\mathbb{R}$ and the $\tilde{\mathsf{m}}$-conull subset $\tilde{W}_0\subset\tilde{\Omega}_\theta$ of f and W_0 respectively. Let

$$W_1:=\{(\xi,\eta)\in\Lambda_\theta^{(2)}:(\xi,\eta,b)\in\tilde{W}_0 \text{ for } db\text{-a.e. } b\in\mathfrak{a}_\theta\};$$

$$W:=\{(\xi,\eta)\in W_1:(\xi,\eta'),(\xi',\eta)\in W_1 \text{ for } \nu\text{-a.e. } \xi'\in\Lambda_\theta, \nu_{\mathrm{i}}\text{-a.e. } \eta'\in\Lambda_{\mathrm{i}(\theta)}\}.$$

By Fubini's theorem, W has the full $\nu\otimes\nu_{\mathrm{i}}$-measure and we may assume that W is Γ-invariant as well. For all small $\varepsilon>0$, we define $\tilde{f}_\varepsilon:\tilde{\Omega}\to\mathbb{R}$ by

$$\tilde{f}_\varepsilon([g])=\frac{1}{\mathrm{Vol}(A_{\theta,\varepsilon})}\int_{A_{\theta,\varepsilon}}\tilde{f}([g]b)db$$

where $A_{\theta,\varepsilon}=\{a\in A_\theta:\|\log a\|\le\varepsilon\}$. Then for $g\in G$ and $n\in N_\theta\cup\check{N}_\theta$ such that $(g^+,g^-),((gn)^+,(gn)^-)\in W$, we have $\tilde{f}_\varepsilon([g])=\tilde{f}_\varepsilon([gn])$ and $\tilde{f}_\varepsilon$ is continuous on $[g]A_\theta$.

Since $\tilde{f} = \lim_{\varepsilon\to 0} \tilde{f}_\varepsilon$ $\tilde{\mathsf{m}}$-a.e., it suffices to show that $\tilde{f}_\varepsilon$ is A_θ-invariant. Fix $g \in G$ such that $(g^+, g^-) \in W$. By Proposition 8.3 and the continuity of $\tilde{f}_\varepsilon$ on each A_θ-orbit, it is again sufficient to show that $\tilde{f}_\varepsilon$ is invariant under $\mathcal{H}_\Gamma^\theta(g)$. Let $a \in \mathcal{H}_\Gamma^\theta(g)$. Then there exist $\gamma \in \Gamma$ and a sequence $n_1, \cdots, n_k \in N_\theta \cup \check{N}_\theta$ such that

(1) $(gn_1 \cdots n_r)^+ \in \Lambda_\theta$ and $(gn_1 \cdots n_r)^- \in \Lambda_{\mathrm{i}(\theta)}$ for all $1 \le r \le k$; and
(2) $gn_1 \cdots n_k = \gamma g a s$ for some $s \in S_\theta$.

For each $i = 1, \cdots, k$, we denote by $N_i = N_\theta$ if $n_i \in N_\theta$ and $N_i = \check{N}_\theta$ if $n_i \in \check{N}_\theta$. We may assume that $N_i \ne N_{i+1}$ for all $1 \le i \le k-1$. Noting that W is Γ-invariant, we consider a sequence of k-tuples $(n_{1,j}, \cdots, n_{k,j}) \in N_1 \times \cdots \times N_k$ as follows:

Case 1: $N_k = \check{N}_\theta$**.** In this case, we have

$$(\gamma g)^+ = (gn_1 \cdots n_k)^+ \quad \text{and} \quad (\gamma g)^- = (gn_1 \cdots n_{k-1})^-.$$

Take a sequence of k-tuples $(n_{1,j}, \cdots, n_{k,j}) \in N_1 \times \cdots \times N_k$ converging to $(n_1, \cdots, n_k)$ as $j \to \infty$ so that for each j, we have

(1) $((gn_{1,j} \cdots n_{r,j})^+, (gn_{1,j} \cdots n_{r,j})^-) \in W$ for all $1 \le r \le k$;
(2) $(\gamma g)^- = (gn_{1,j} \cdots n_{k-1,j})^-$; and
(3) $(\gamma g)^+ = (gn_{1,j} \cdots n_{k,j})^+$.

This is possible since $(g^+, g^-), ((\gamma g)^+, (\gamma g)^-) \in W$ and W has the full $\nu \otimes \nu_{\mathrm{i}}$-measure. Since $n_{k,j} \in \check{N}_\theta$, we indeed have $(\gamma g)^- = (gn_{1,j} \cdots n_{k,j})^-$ as well, and therefore $gn_{1,j} \cdots n_{k,j} = \gamma g a_j s_j$ for some $a_j \in A_\theta$ and $s_j \in S_\theta$. In particular, we have

$$[gn_{1,j} \cdots n_{k,j}] = [\gamma g a_j] \in \tilde{\Omega}_\theta \quad \text{for all } j \ge 1.$$

Case 2: $N_k = N_\theta$**.** In this case, we have

$$(\gamma g)^+ = (gn_1 \cdots n_{k-1})^+ \quad \text{and} \quad (\gamma g)^- = (gn_1 \cdots n_k)^-.$$

We then take a sequence of k-tuples $(n_{1,j}, \cdots, n_{k,j}) \in N_1 \times \cdots \times N_k$ converging to $(n_1, \cdots, n_k)$ as $j \to \infty$ so that for each j, we have

(1) $((gn_{1,j} \cdots n_{r,j})^+, (gn_{1,j} \cdots n_{r,j})^-) \in W$ for all $1 \le r \le k$;
(2) $(\gamma g)^+ = (gn_{1,j} \cdots n_{k-1,j})^+$; and
(3) $(\gamma g)^- = (gn_{1,j} \cdots n_{k,j})^-$.

Since $n_{k,j} \in N_\theta$, we have $(\gamma g)^+ = (gn_{1,j} \cdots n_{k,j})^+$ as well, and therefore $gn_{1,j} \cdots n_{k,j} = \gamma g a_j s_j$ for some $a_j \in A_\theta$ and $s_j \in S_\theta$. In particular, we have

$$[gn_{1,j} \cdots n_{k,j}] = [\gamma g a_j] \in \tilde{\Omega}_\theta \quad \text{for all } j \ge 1.$$

In either case, we have that for each $j \ge 1$,

$$\tilde{f}_\varepsilon([\gamma g a_j]) = \tilde{f}_\varepsilon([gn_{1,j} \cdots n_{k,j}]) = \tilde{f}_\varepsilon([gn_{1,j} \cdots n_{k-1,j}]) = \cdots = \tilde{f}_\varepsilon([g]).$$

Since $\tilde{f}_\varepsilon$ is Γ-invariant, it implies

$$\tilde{f}_\varepsilon([ga_j]) = \tilde{f}_\varepsilon([g]) \quad \text{for all } j \ge 1.$$

Since a_j converges to a as $j \to \infty$, we get $\tilde{f}_\varepsilon([ga]) = \tilde{f}_\varepsilon([g])$ by the continuity of $\tilde{f}_\varepsilon$ on gA_θ. This shows that $\tilde{f}_\varepsilon$ is invariant under $\mathcal{H}_\Gamma^\theta(g)$, finishing the proof of ergodicity.

Now suppose that the flow $(\Omega_\theta, A_u, \mathsf{m})$ is ergodic. Then by the Hopf decomposition theorem, it is either completely conservative or completely dissipative. Suppose to the contrary that $(\Omega_\theta, A_u, \mathsf{m})$ is completely dissipative. Then it is isomorphic to a translation on $\mathbb{R}$ with respect to the Lebesgue measure. This yields a contradiction as in proof of [22, Thm. 10.2], as we recall for readers' convenience. Since $(\Omega_\theta, A_u, \mathsf{m})$ is isomorphic to a translation on $\mathbb{R}$, $(\nu \times \nu_{\mathrm{i}})|_{\Lambda_\theta^{(2)}}$ is supported on the single Γ-orbit $\Gamma(\xi_0, \eta_0)$ by the ergodicity of $(\Gamma, \Lambda_\theta^{(2)}, \nu \times \nu_{\mathrm{i}})$. Since ν and ν_{i} also have atoms on ξ_0 and η_0 respectively, we have

$$(\Gamma\xi_0 \times \Gamma\eta_0) \cap \Lambda_\theta^{(2)} \subset \Gamma(\xi_0, \eta_0).$$

We deduce from the θ-antipodality of Γ that $\Gamma\xi_0 \subset \Gamma_{\eta_0}\xi_0 \cup \{\eta_0'\}$ where $\Gamma_{\eta_0} = \mathrm{Stab}_\Gamma(\eta_0)$ and η_0' is the image of η_0 under the Γ-equivariant homeomorphism $\Lambda_{\mathrm{i}(\theta)} \to \Lambda_\theta$ obtained in [22, Lem. 9.5]. Since $\Gamma_{\eta_0} = \Gamma_{\eta_0'}$, we have

$$\Gamma\xi_0 \subset \Gamma_{\eta_0'}\xi_0 \cup \{\eta_0'\}. \tag{8.5}$$

Recall that the Γ-action on Λ_θ is a non-elementary convergence group action [19, Thm. 4.16] and hence there must be infintely many accumulation points of $\Gamma\xi_0$. On the other hand, as $\Gamma_{\eta_0'}$ is an elementary subgroup, the orbit $\Gamma_{\eta_0'}\xi_0$ accumulates at most at two points of Λ_θ ([37], [8]). This yields a contradiction, and therefore $(\Omega_\theta, A_u, \mathsf{m})$ is completely conservative. $\square$

Proof of Theorem 1.1. The equivalences between (1)-(3) follow from Proposition 8.1 and Proposition 8.8. Suppose that m is u-balanced. Corollary 7.13 implies that (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). That the first case occurs only when $\psi(u) = \psi_\Gamma^\theta(u) > 0$ is a consequence of Lemma 5.7.

9. Ergodic dichotomy for subspace flows

In this section, we extend our ergodic dichotomy to the action of any connected subgroup of A_θ of arbitrary dimension. In fact, we deduce this from the ergodic dichotomy for directional flows.

Let Γ be a Zariski dense θ-transverse subgroup of G. Let $W < \mathfrak{a}_\theta$ be a non-zero linear subspace and set $A_W = \exp W$. We consider the subspace flow A_W on Ω_θ and explain how the proof of Theorem 1.1 extends to this setting so that we obtain Theorem 1.3, adapting the argument of Pozzetti-Sambarino [28] on relating the subspace flows with directional flows.

For $R > 0$, we set

$$\Gamma_{W,R} = \{\gamma \in \Gamma : \|\mu_\theta(\gamma) - W\| < R\}.$$

If $W = \mathfrak{a}_\theta$, then $\Gamma_{W,R} = \Gamma$ for all $R > 0$.

Definition 9.1 (W-conical points)**.** We say that $\xi \in \mathcal{F}_\theta$ is a W-conical point of Γ if there exist $R > 0$ and a sequence $\gamma_i \in \Gamma_{W,R}$ such that $\xi \in O_R^\theta(o, \gamma_i o)$ for all $i \geq 1$. We denote by Λ_θ^W the set of all W-conical points of Γ.

Fix a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$. Let ν, ν_{i} be a pair of (Γ, ψ) and $(\Gamma, \psi \circ \mathrm{i})$-conformal measures on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively, and let $\mathsf{m} = \mathsf{m}_{\nu, \nu_{\mathrm{i}}}$ denote the associated BMS measure on Ω_θ.

If $W \cap \mathcal{L}_\theta = \{0\}$ or $W \subset \ker \psi$, then the (Γ, θ)-proper hypothesis on ψ implies that $\Gamma_{W,R}$ is finite for all $R > 0$, and hence $\Lambda_\theta^W = \Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(W)} = \emptyset$ and $(\Omega_\theta, A_W, \mathsf{m})$ is completely dissipative and non-ergodic.

The rest of this section is now devoted to proving Theorem 1.3, assuming that

- $W \cap \mathcal{L}_\theta \neq \{0\}$;
- $W \not\subset \ker \psi$.

Recalling that $\psi \geq 0$ on $\mathcal{L}_\theta$ by [22, Lem. 4.3], the intersection $W \cap \ker \psi$ has codimension one in W and intersects $\operatorname{int} \mathcal{L}_\theta$ only at 0.

Set
$$W^\diamond = \mathfrak{a}_\theta / (W \cap \ker \psi) \quad \text{ and } \quad \tilde{\Omega}_{W^\diamond} := \Lambda_\theta^{(2)} \times W^\diamond.$$
Recalling the spaces $\tilde{\Omega}_\psi$ and Ω_ψ defined in (6.4), the projection $\tilde{\Omega}_\theta \to \tilde{\Omega}_\psi$ factors through $\tilde{\Omega}_{W^\diamond}$ in a Γ-equivariant way. Since the Γ-action on $\tilde{\Omega}_\psi$ is properly discontinuous (Theorem 6.1), the induced Γ-action on $\tilde{\Omega}_{W^\diamond}$ is also properly discontinuous. Moreover, the trivial vector bundle $\Omega_\theta \to \Omega_\psi$ in (6.5) factors through
$$\Omega_{W^\diamond} := \Gamma \backslash \tilde{\Omega}_{W^\diamond}. \tag{9.1}$$
Hence we have a $W \cap \ker \psi$-equivariant homeomorphism:
$$\Omega_\theta \simeq \Omega_{W^\diamond} \times (W \cap \ker \psi).$$

Denote by m' the A_θ-invariant Radon measure on $\Omega_{W^\diamond}$ such that $\mathsf{m} = \mathsf{m}' \otimes \mathrm{Leb}_{W \cap \ker \psi}$.

The main point of the proof of Theorem 1.3 is to relate the action of A_W on Ω_θ with that of a directional flow on $\Omega_{W^\diamond}$. Once we do that, we can proceed similarly to the proof of Theorem 1.1.

Since $W \not\subset \ker \psi$, there exists $u \in W$ with $\psi(u) \neq 0$. By replacing u by $-u$ if necessary, we fix $u \in W$ such that
$$\psi(u) > 0.$$
Set $A_u = A_{\mathbb{R}u} = \{a_{tu} = \exp(tu) : t \in \mathbb{R}\}$ and consider the A_u-action on $(\Omega_{W^\diamond}, \mathsf{m}')$. Since $W = \mathbb{R}u + (W \cap \ker \psi)$, we have:

Lemma 9.2. *The A_W-action on $(\Omega_\theta, \mathsf{m})$ is ergodic (resp. completely conservative, non-ergodic, completely dissipative) if and only if the A_u-action on $(\Omega_{W^\diamond}, \mathsf{m}')$ ergodic (resp. completely conservative, non-ergodic, completely dissipative).*

Among the ingredients for the proof of Theorem 1.1, Lemma 5.2 and Proposition 5.3 were repeatedly used and played basic roles in the proof. The following analogue of Lemma 5.2 can be proved by a similar argument as in the proof of Lemma 5.2:

Lemma 9.3. *Suppose that $d_i \in a_{t_i u} \exp(W \cap \ker \psi) B_\theta^+$, $t_i > 0$ and $\gamma_i \in \Gamma$ are sequences such that $\gamma_i h_i m_i d_i$ is bounded for some bounded sequence $h_i \in G$ with $h_i P \in \Lambda$ and $m_i \in M_\theta$. Then there exists $w \in \mathcal{W} \cap M_\theta$ such that, after passing to a subsequence, we have that for all $i \geq 1$,*

$$d_i \in wA^+w^{-1}.$$

Proof. As in the proof of Lemma 5.2, there exists a Weyl element $w \in \mathcal{W}$ such that $d_i \in wA^+w^{-1}$ for all $i \geq 1$ after passing to a subsequence, and moreover $w \in M_\theta$ or $w \in M_\theta w_0$. We claim that the latter case $w \in M_\theta w_0$ cannot happen. Suppose that $w \in M_\theta w_0$ and write $d_i = a_{t_i u} a_i b_i$ for $a_i \in \exp(W \cap \ker \psi)$ and $b_i \in B_\theta^+$. Since $w \in M_\theta w_0$, we get $\mu_{\mathrm{i}(\theta)}(d_i) = \log(w_0^{-1} a_{t_i u} a_i w_0)$ for all $i \geq 1$. In particular, $t_i u + \log a_i \in -\mathfrak{a}_\theta^+$.

Since the sequence $\gamma_i h_i m_i d_i$ is bounded by the hypothesis, the sequence $\mu_{\mathrm{i}(\theta)}(\gamma_i^{-1}) - \mu_{\mathrm{i}(\theta)}(d_i)$ is bounded as well by Lemma 2.1. Since $\mu_{\mathrm{i}(\theta)}(\gamma_i^{-1}) = -\operatorname{Ad}_{w_0}(\mu_\theta(\gamma_i))$ and $\mu_{\mathrm{i}(\theta)}(d_i) = \operatorname{Ad}_{w_0}(t_i u + \log a_i)$, it follows that $\mu_\theta(\gamma_i) = -(t_i u + \log a_i) + q_i$ for some bounded sequence $q_i \in \mathfrak{a}_\theta$. Applying ψ, we get $\psi(\mu_\theta(\gamma_i)) = -t_i \psi(u) + \psi(q_i)$ since $\log a_i \in \ker \psi$. Since $\psi(u) > 0$, $\psi(\mu_\theta(\gamma_i))$ is uniformly bounded. The (Γ, θ)-properness of ψ implies that γ_i is a finite sequence, yielding a contradiction. Therefore, the case $w \in M_\theta w_0$ cannot occur; so $w \in M_\theta$. $\square$

Let $p : \mathfrak{a}_\theta \to W^\diamond$ denote the natural projection map. Choosing a norm $\|\cdot\|$ on $W^\diamond$, the map p is Lipschitz. Then for a constant $c > 1$ depending on the Lipschitz constant of p as well as norms on $\mathfrak{a}_\theta$ and $W^\diamond$, we have for all $R > 0$,

$$\{\gamma \in \Gamma : \|p(\mu_\theta(\gamma)) - \mathbb{R}u\| < R/c\} \subset \Gamma_{W,R} \subset \{\gamma \in \Gamma : \|p(\mu_\theta(\gamma)) - \mathbb{R}u\| < cR\}.$$

Note also that $\psi(p(\mu_\theta(\gamma))) = \psi(\mu_\theta(\gamma))$ for all $\gamma \in \Gamma$.

Using this relation and Lemma 9.3, similar arguments as in Sections 5 and 7 apply to the A_u-flow on $\Omega_{W^\diamond}$, replacing $\Gamma_{u,r}$ with $\Gamma_{W,R}$. In particular, applying Lemma 9.3 in place of Lemma 5.2, the following analogs of Proposition 5.3 and Lemma 5.5(2) respectively can be proved similarly.

Proposition 9.4. *Let $Q \subset \tilde{\Omega}_{W^\diamond}$ be a compact subset. There are positive constants $C_1 = C_1(Q), C_2 = C_2(Q)$ and $R = R(Q)$ such that if $[h] \in Q \cap \gamma Q a_{-tu}$ for some $h \in G$, $\gamma \in \Gamma$ and $t > 0$, then the following hold:*

1. $\|p(\mu_\theta(\gamma)) - tu\| < C_1$;
2. $(h^+, h^-) \in O_R^\theta(o, \gamma o) \times O_R^{\mathrm{i}(\theta)}(\gamma o, o)$;
3. $\|\mathcal{G}^\theta(h^+, h^-)\| < C_2$.

Lemma 9.5. *The following are equivalent for any $\xi \in \Lambda_\theta$:*

1. $\xi \in \Lambda_\theta^W$;

(2) *$\xi = gP_\theta \in \mathcal{F}_\theta$ for some $g \in G$ such that $[g] \in \Omega_\theta$ and* $\limsup [g](A_W \cap A^+) \neq \emptyset$*;*
(3) *the sequence $[(\xi, \eta, v)]a_{t_i u}$ is precompact in $\Omega_{W^\diamond}$ for some $\eta \in \Lambda_{\mathrm{i}(\theta)}$, $v \in W^\diamond$ and $t_i \to \infty$.*

In particular, a W-conical point of Γ is a u-conical point for the action of A_u on $\Omega_{W^\diamond}$ and vice versa.

Since the recurrence of the A_u-flow on $\Omega_{W^\diamond}$ is related to the W-conical set as stated in Lemma 9.5, the arguments in Section 8 for the directional flow $(\Omega_{W^\diamond}, A_u, \mathsf{m}')$ yield the following equivalences:
(9.2)
$$\begin{aligned}
\max\left(\nu(\Lambda_\theta^W), \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(W)})\right) > 0 &\Leftrightarrow (\Omega_{W^\diamond}, A_u, \mathsf{m}') \text{ is completely conservative}\\
&\Leftrightarrow (\Omega_{W^\diamond}, A_u, \mathsf{m}') \text{ is ergodic;}\\
\max\left(\nu(\Lambda_\theta^W), \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(W)})\right) = 0 &\Leftrightarrow (\Omega_{W^\diamond}, A_u, \mathsf{m}') \text{ is completely dissipative}\\
&\Leftrightarrow (\Omega_{W^\diamond}, A_u, \mathsf{m}') \text{ is non-ergodic.}
\end{aligned}$$

This proves the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) of Theorem 1.3.

Definition 9.6. We say that m is W-balanced if there exists $u \in W$ with $\psi(u) > 0$ such that $(\Omega_{W^\diamond}, \mathsf{m}')$ is u-balanced.

To complete the proof of Theorem 1.3, it remains to prove the following:

Theorem 9.7. *Suppose that m is W-balanced. The following are equivalent:*
(1) $\sum_{\gamma \in \Gamma_{W,R}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ *for some $R > 0$;*
(2) $\nu(\Lambda_\theta^W) = 1 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(W)})$.

Similarly, the following are also equivalent:
(1) $\sum_{\gamma \in \Gamma_{W,R}} e^{-\psi(\mu_\theta(\gamma))} < \infty$ *for all $R > 0$;*
(2) $\nu(\Lambda_\theta^W) = 0 = \nu_{\mathrm{i}}(\Lambda_{\mathrm{i}(\theta)}^{\mathrm{i}(W)})$.

In the rest of this section, we assume that m is W-balanced, and choose $u \in W$ with $\psi(u) > 0$ so that m' is u-balanced. Following the proof of Proposition 7.4 while applying Proposition 9.4 in the place of Proposition 5.3, we get:

Proposition 9.8. *Suppose that $\sum_{\gamma \in \Gamma_{W,R}} e^{-\psi(\mu_\theta(\gamma))} = \infty$ for some $R > 0$. Set $\delta = \psi(u) > 0$.*

(1) *For any compact subset $Q \subset \tilde{\Omega}_{W^\diamond}$, there exists $R = R(Q) > 0$ such that for any $T > 1$, we have*

$$\int_0^T \int_0^T \sum_{\gamma, \gamma' \in \Gamma} \tilde{\mathsf{m}}'(Q \cap \gamma Q a_{-tu} \cap \gamma' Q a_{-(t+s)u}) dt ds \ll \left(\sum_{\substack{\gamma \in \Gamma_{W,R} \\ \psi(\mu_\theta(\gamma)) \le \delta T}} e^{-\psi(\mu_\theta(\gamma))} \right)^2.$$

(2) *For any $R>0$, there exists a compact subset $Q'=Q'(R)\subset\tilde\Omega_{W^\diamond}$ such that*
$$\int_0^T\sum_{\gamma\in\Gamma}\tilde{\mathsf{m}}'(Q'\cap\gamma Q'a_{-tu})dt\gg\sum_{\substack{\gamma\in\Gamma_{W,R}\\ \psi(\mu_\theta(\gamma))\le\delta T}}e^{-\psi(\mu_\theta(\gamma))}.$$

The proof of Theorem 5.6 works verbatim for Λ_θ^W so that the convergence $\sum_{\gamma\in\Gamma_{W,R}}e^{-\psi(\mu_\theta(\gamma))}<\infty$ for all $R>0$ implies that $\nu(\Lambda_\theta^W)=0$. Using Proposition 9.8 together with the W-balanced condition, Theorem 9.7 can now be proved by the same argument as in the proof of Corollary 7.13.

Remark 9.9. The W-balanced condition on m was needed because Q and Q' in Proposition 9.8 may not be the same in principle. However when $W=\mathfrak{a}_\theta$, we have $\Gamma_{W,R}=\Gamma$ for any $R>0$ and Q and Q' in Proposition 9.8 can be taken to be the same set, and hence the W-balanced condition is not needed in the proof of Theorem 1.3.

Similarly to Corollary 7.14, we have the following estimates which reduce the divergence of the series $\sum_{\gamma\in\Gamma_{W,R}}e^{-\psi(\mu_\theta(\gamma))}$ to the local mixing rate for the a_t-flow:

Corollary 9.10. *For all sufficiently large $R>0$, there exist compact subsets Q_1,Q_2 of $\Omega_{W^\diamond}$ with non-empty interior such that for all $T\ge 1$,*
$$\left(\int_0^T\mathsf{m}'(Q_1\cap Q_1a_{-t})dt\right)^{1/2}\ll\sum_{\substack{\gamma\in\Gamma_{W,R}\\ \psi(\mu_\theta(\gamma))\le\delta T}}e^{-\psi(\mu_\theta(\gamma))}\ll\int_0^T\mathsf{m}'(Q_2\cap Q_2a_{-t})dt.$$

10. Dichotomy theorems for Anosov subgroups

In this last section, we focus on Anosov subgroups and establish the codimension dichotomy for ergodicity of the subspace flow given by $\exp W$, for a linear subspace $W<\mathfrak{a}_\theta$. Using the local mixing theorem for directional flows for Anosov subgroups, we show that the Poincaré series associated to W diverges if and only if the codimension of the subspace W in $\mathfrak{a}_\theta$ is at most 2.

Let $\Gamma<G$ be a Zariski dense θ-Anosov subgroup defined as in the introduction. Recall that $\mathcal{L}_\theta\subset\mathfrak{a}_\theta^+$ denotes the θ-limit cone of Γ. Denote by $\mathcal{T}_\Gamma^\theta\subset\mathfrak{a}_\theta^*$ the set of all linear forms tangent to the growth indicator ψ_Γ^θ and by $\mathcal{M}_\Gamma^\theta$ the set of all Γ-conformal measures on Λ_θ. There are one-to-one correspondences between the following sets ([22, Coro. 1.12], [34, Thm. A]):
$$\mathbb{P}(\operatorname{int}\mathcal{L}_\theta)\longleftrightarrow\mathcal{T}_\Gamma^\theta\longleftrightarrow\mathcal{M}_\Gamma^\theta.$$
Namely, for each unit vector $v\in\operatorname{int}\mathcal{L}_\theta$, there exists a unique $\psi_v\in\mathfrak{a}_\theta^*$ which is tangent to ψ_Γ^θ at v and a unique (Γ,ψ_v)-conformal measure ν_v supported on Λ_θ. The linear form $\psi_v\circ\mathrm{i}\in\mathfrak{a}_{\mathrm{i}(\theta)}^*$ is tangent to $\psi_\Gamma^{\mathrm{i}(\theta)}$ at $\mathrm{i}(v)$ and the

measure $\nu_{\mathrm{i}(v)}$ is a $(\Gamma, \psi_v \circ \mathrm{i})$-conformal measure on $\Lambda_{\mathrm{i}(\theta)}$. Denote by m_v the BMS measure on Ω_θ associated with the pair $(\nu_v, \nu_{\mathrm{i}(v)})$.

What distinguishes θ-Anosov subgroups from general θ-transverse subgroups is that Ω_{ψ_v} is a *compact* metric space ([33] and [11, Appendix]) and hence Ω_θ is a vector bundle over a *compact* space Ω_{ψ_v} with fiber $\ker \psi_v \simeq \mathbb{R}^{\#\theta-1}$. We use the the following local mixing for directional flows due to Sambarino.

Theorem 10.1 ([34, Thm. 2.5.2], see also [12] for $\theta = \Pi$)**.** *Let $\Gamma < G$ be a θ-Anosov subgroup and $v \in \operatorname{int} \mathcal{L}_\theta$. Then there exists $\kappa_v > 0$ such that for any $f_1, f_2 \in C_c(\Omega_\theta)$,*

$$\lim_{t\to\infty} t^{\frac{\#\theta-1}{2}} \int_{\Omega_\theta} f_1(x) f_2(x \exp(tv)) d\mathsf{m}_v(x) = \kappa_v \mathsf{m}_v(f_1)\mathsf{m}_v(f_2).$$

In particular, for any $v \in \operatorname{int} \mathcal{L}_\theta$, m_v is v-balanced.

Corollary 10.2. *For any $v \in \operatorname{int} \mathcal{L}_\theta$ and any bounded Borel subset $Q \subset \tilde{\Omega}_\theta$ with non-empty interior, we have for any $T > 2$,*

$$\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}_v(Q \cap \gamma Q \exp(-tv)) dt \asymp \int_1^T t^{\frac{1-\#\theta}{2}} dt.$$

Proof. Given a bounded Borel subset $Q \subset \tilde{\Omega}_\theta$ with non-empty interior, we choose $\tilde{f}_1, \tilde{f}_2 \in C_c(\tilde{\Omega}_\theta)$ so that $0 \le \tilde{f}_1 \le \mathbb{1}_Q \le \tilde{f}_2$ and $\tilde{\mathsf{m}}_v(\tilde{f}_1) > 0$. For each $i = 1, 2$, we define the function $f_i \in C_c(\Omega_\theta)$ by $f_i(\Gamma[g]) = \sum_{\gamma\in\Gamma} \tilde{f}_i(\gamma g)$. By Theorem 10.1, for each $i = 1, 2$, we have that for all $t \ge 1$,

$$\begin{aligned}\int_{\tilde{\Omega}_\theta} \sum_{\gamma\in\Gamma} \tilde{f}_i(\gamma[g] \exp(tv)) \tilde{f}_i([g]) d\tilde{\mathsf{m}}_v([g]) &= \int_{\Omega_\theta} f_i(x \exp(tv)) f_i(x) d\mathsf{m}_v(x) \\ &\asymp t^{\frac{1-\#\theta}{2}}.\end{aligned}$$

$\square$

By Corollary 7.14 and Corollary 10.2, we get:

Proposition 10.3. *Let $v \in \operatorname{int} \mathcal{L}_\theta$ and $\delta = \psi_v(v)$. For all sufficiently large $r > 0$, we have that for all $T > 2$,*

$$\left(\int_1^T t^{\frac{1-\#\theta}{2}} dt\right)^{1/2} \ll \sum_{\substack{\gamma\in\Gamma_{v,r} \\ \psi_v(\mu_\theta(\gamma))\le\delta T}} e^{-\psi_v(\mu_\theta(\gamma))} \ll \int_1^T t^{\frac{1-\#\theta}{2}} dt. \tag{10.1}$$

Theorem 10.4. *For any $v \in \operatorname{int} \mathcal{L}_\theta$ and $u \in \mathfrak{a}_\theta^+ - \{0\}$, the following are equivalent:*

1. *$\#\theta \le 3$ and $\mathbb{R}u = \mathbb{R}v$;*
2. *$\sum_{\gamma\in\Gamma_{u,r}} e^{-\psi_v(\mu_\theta(\gamma))} = \infty$ for some $r > 0$.*

Proof. Note that $\int_1^\infty t^{\frac{1-\#\theta}{2}}dt = \infty$ if and only if $\#\theta \le 3$. Hence (1) implies (2) by Proposition 10.3. To show the implication (2) $\Rightarrow$ (1), suppose that $\sum_{\gamma\in\Gamma_{u,r}} e^{-\psi_v(\mu_\theta(\gamma))} = \infty$ for some $r > 0$. By Lemma 5.7, $\psi_v(u) = \psi_\Gamma^\theta(u)$. It follows from the strict concavity of ψ_Γ^θ [22, Thm. 12.2] that ψ_v can be tangent to ψ_Γ^θ only in the direction $\mathbb{R}v$. Therefore $\mathbb{R}u = \mathbb{R}v$. Now $\#\theta \le 3$ follows from Proposition 10.3. □

Here is the special case of Theorem 1.6 for $\dim W = 1$:

Theorem 10.5. *Let $\Gamma < G$ be a Zariski dense θ-Anosov subgroup. For any $u \in \operatorname{int}\mathcal{L}_\theta$, the following are equivalent:*

(1) $\#\theta \le 3$ *(resp.* $\#\theta \ge 4$*);*
(2) $\nu_u(\Lambda_\theta^u) = 1$ *(resp.* $\nu_u(\Lambda_\theta^u) = 0$*);*
(3) $(\Omega_\theta, A_u, \mathsf{m}_u)$ *is ergodic and completely conservative (resp. non-ergodic and completely dissipative);*
(4) $\sum_{\gamma\in\Gamma_{u,R}} e^{-\psi_u(\mu_\theta(\gamma))} = \infty$ *for some* $R > 0$ *(resp.* $\sum_{\gamma\in\Gamma_{u,R}} e^{-\psi_u(\mu_\theta(\gamma))} < \infty$ *for all* $R > 0$*).*

Proof. Since m_u is u-balanced by Theorem 10.1, the equivalences between (2)-(4) follow from Theorem 1.1. By Theorem 10.4, we have (1) $\Leftrightarrow$ (4). □

Codimension dichotomy for Anosov subgroups. We now deduce Theorem 1.6. We use the notation from Theorem 1.6 and set $\psi = \psi_u$. As in Section 9, we consider the quotient space $W^\diamond = \mathfrak{a}_\theta/(W \cap \ker\psi)$ and set $\Omega_{W^\diamond} = \Gamma\backslash\Lambda_\theta^{(2)} \times W^\diamond$ (see (9.1)). We denote by m_u' the A_θ-invariant Radon measure on $\Omega_{W^\diamond}$ such that $\mathsf{m}_u = \mathsf{m}_u' \otimes \mathrm{Leb}_{W\cap\ker\psi}$. As before, $\Omega_{W^\diamond}$ is a vector bundle over a *compact* metric space Ω_ψ with fiber $\mathbb{R}^{\dim W^\diamond - 1}$, and the local mixing theorem for the $\{a_{tu}\}$-flow on $\Omega_{W^\diamond}$ [34, Thm. 2.5.2] says that there exists $\kappa_u > 0$ such that for any $f_1, f_2 \in C_c(\Omega_{W^\diamond})$,

$$\lim_{t\to\infty} t^{\frac{\dim W^\diamond - 1}{2}} \int_{\Omega_{W^\diamond}} f_1(x) f_2(x a_{tu})\, d\mathsf{m}_u'(x) = \kappa_u \mathsf{m}_u'(f_1)\mathsf{m}_u'(f_2). \tag{10.2}$$

We then obtain the following version of Proposition 10.3, using Corollary 9.10 and 10.2:

Proposition 10.6. *For $\delta = \psi(u) > 0$ and all sufficiently large $R > 0$, we have that*

$$\left(\int_1^T t^{\frac{1-\dim W^\diamond}{2}} dt\right)^{1/2} \ll \sum_{\substack{\gamma\in\Gamma_{W,R}\\ \psi(\mu_\theta(\gamma))\le\delta T}} e^{-\psi(\mu_\theta(\gamma))} \ll \int_1^T t^{\frac{1-\dim W^\diamond}{2}} dt \tag{10.3}$$

where the implied constants are independent of $T > 2$.

Since $\dim W^\diamond - 1 = \operatorname{codim} W$ and hence $\dim W^\diamond \le 3 \Leftrightarrow \operatorname{codim} W \le 2$, the following is immediate from Proposition 10.6:

Proposition 10.7. *If Γ is a Zariski dense θ-Anosov subgroup of G, then*

$$\operatorname{codim} W \leq 2 \iff \sum_{\gamma \in \Gamma_{W,R}} e^{-\psi(\mu_\theta(\gamma))} = \infty \text{ for some } R > 0.$$

Hence the equivalence (1) $\Leftrightarrow$ (4) in Theorem 1.6 follows. Since the local mixing for $(\Omega_{W^\diamond}, \{a_{tu}\}, \mathsf{m}'_u)$ implies that m'_u is u-balanced, and hence m_u is W-balanced, we can apply Theorem 1.3 to obtain the equivalences (2)-(4) in Theorem 1.6. Therefore Theorem 1.6 follows.

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