

STRICT CONCAVITY OF THE GROWTH INDICATOR FUNCTION FOR RELATIVELY ANOSOV GROUPS

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ABSTRACT. Let Γ be a discrete subgroup of a connected semisimple real algebraic group of higher rank. The growth indicator function ψ_Γ records the directional exponential growth of the Cartan projections of elements of Γ in the positive Weyl chamber $\mathfrak{a}^+$. We prove that if Γ is a non-elementary relatively Borel Anosov group, then ψ_Γ is strictly concave on non-collinear directions. We prove this by establishing the C^1 -smoothness of the Manhattan hypersurface, defined as the unit level set of the critical-exponent map $\phi \mapsto \delta^\phi(\Gamma)$. More generally, for a non-elementary θ -transverse group, we prove local C^1 -regularity near every point of the θ -Manhattan hypersurface that is positive on the θ -limit cone and has a critical gap at infinity. In particular, the θ -Manhattan hypersurface is globally C^1 for relatively θ -Anosov groups, and their θ -growth indicator functions are strictly concave on non-collinear directions.

1. INTRODUCTION

Let G be a connected semisimple real algebraic group. The main result of this paper is a sufficient condition for the θ -growth indicator function of a Zariski dense θ -transverse group of G to be strictly concave in a neighborhood of a point. This result implies that the θ -growth indicator function of every Zariski dense relatively θ -Anosov group of G is strictly concave. We establish our main result by proving local C^1 -regularity of the θ -Manhattan hypersurface of a non-elementary θ -transverse group at every point that is positive on the θ -limit cone and has a critical gap at infinity. We begin by introducing the notation needed to state this result precisely.

Let $A < G$ be a maximal real split torus, and set $\mathfrak{a} := \text{Lie } A$. Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and let $\Delta \subset \mathfrak{a}^*$ denote the corresponding set of simple restricted roots. We denote by

$$\kappa : G \rightarrow \mathfrak{a}^+$$

the Cartan projection.

Fix a nonempty subset $\theta \subset \Delta$ which is invariant under the opposition involution of $\mathfrak{a}$ (see (2.1)). Consider the partial Cartan subspace

$$\mathfrak{a}_\theta := \bigcap_{\alpha \in \Delta \setminus \theta} \ker \alpha.$$

Let

$$\pi_\theta : \mathfrak{a} \rightarrow \mathfrak{a}_\theta$$

be the canonical projection invariant under every Weyl group element that fixes $\mathfrak{a}_\theta$ pointwise, and set

$$\kappa_\theta := \pi_\theta \circ \kappa : G \rightarrow \mathfrak{a}_\theta.$$

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A subgroup $\Gamma < \mathbf{G}$ is called θ -discrete if the restriction

$$\kappa_\theta|_\Gamma : \Gamma \rightarrow \mathfrak{a}_\theta$$

is proper. Equivalently, for every $T > 0$, the set $\{\gamma \in \Gamma : \|\kappa_\theta(\gamma)\| \leq T\}$ is finite, where $\|\cdot\|$ is any norm on $\mathfrak{a}_\theta$. While every discrete subgroup of $\mathbf{G}$ is Δ -discrete, the θ -discreteness hypothesis is required to introduce the following θ -growth indicator function. The θ -growth indicator function of a θ -discrete subgroup Γ , denoted by

$$\psi_\Gamma^\theta : \mathfrak{a}_\theta \rightarrow [0, +\infty) \cup \{-\infty\},$$

records the directional exponential growth of the set $\kappa_\theta(\Gamma)$. More precisely, for $u \in \mathfrak{a}_\theta \setminus \{0\}$, define

$$\psi_\Gamma^\theta(u) := \|u\| \inf_{\mathcal{C} \ni u} \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \# \{\gamma \in \Gamma : \kappa_\theta(\gamma) \in \mathcal{C}, \|\kappa_\theta(\gamma)\| \leq T\},$$

where the infimum is taken over all open cones $\mathcal{C} \subset \mathfrak{a}_\theta$ containing u , and set

$$\psi_\Gamma^\theta(0) := 0.$$

This definition is independent of the choice of norm. When $\theta = \Delta$, the growth indicator function was introduced by Quint [Qui02b]; the definition for general θ was introduced by the first two named authors and Wang [KOW25]. In rank one, it reduces to the usual critical exponent.

The growth indicator function is positively homogeneous, concave, and upper semicontinuous. Its effective support is the θ -limit cone

$$\mathcal{L}_\theta(\Gamma) := \{u \in \mathfrak{a}_\theta : t_n \kappa_\theta(\gamma_n) \rightarrow u \text{ for some } \{\gamma_n\} \subset \Gamma \text{ and } t_n \searrow 0\}.$$

More precisely,

$$\{u \in \mathfrak{a}_\theta : \psi_\Gamma^\theta(u) \geq 0\} = \mathcal{L}_\theta(\Gamma),$$

and $\psi_\Gamma^\theta = -\infty$ outside $\mathcal{L}_\theta(\Gamma)$.

The principal class of θ -discrete subgroups considered in this paper consists of transverse groups. Let $\mathbf{P}_\theta < \mathbf{G}$ be the parabolic subgroup associated with θ , and let

$$\mathcal{F}_\theta := \mathbf{G} / \mathbf{P}_\theta$$

be the corresponding partial flag manifold. We denote by

$$\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$$

the θ -limit set of Γ . Throughout the paper, every discrete subgroup Γ is assumed to be *non-elementary* in the sense that $\Lambda_\theta(\Gamma)$ is infinite. A discrete subgroup $\Gamma < \mathbf{G}$ is called θ -transverse if the following two conditions hold:

- for every $\alpha \in \theta$ and every sequence of distinct elements $\{\gamma_n\} \subset \Gamma$,

$$\alpha(\kappa(\gamma_n)) \rightarrow +\infty;$$

- any two distinct points $x, y \in \Lambda_\theta(\Gamma)$ are transverse; equivalently, the diagonal $\mathbf{G}$ -orbit

$$\mathbf{G} \cdot (x, y) \subset \mathcal{F}_\theta \times \mathcal{F}_\theta$$

is open.

For such a group, the action of Γ on $\Lambda_\theta(\Gamma)$ is a non-elementary convergence action. There are two important subclasses of transverse groups. A θ -transverse group is called θ -Anosov if its action on $\Lambda_\theta(\Gamma)$ is a uniform convergence action, and *relatively θ -Anosov* if this action is a geometrically finite convergence action. These classes generalize, respectively, convex cocompact and geometrically finite subgroups of rank-one Lie groups.

1.1. Strict concavity of the growth indicator function. The main purpose of this paper is to prove the strict concavity of the growth indicator function for relatively Anosov groups. Since ψ_Γ^θ is positively homogeneous, it is linear along each ray. Thus the strongest natural form of concavity one can expect is strict concavity between non-collinear directions. We say that ψ_Γ^θ is *strictly concave* if, whenever $v, w \in \mathfrak{a}_\theta$ are non-collinear and both $\psi_\Gamma^\theta(v)$ and $\psi_\Gamma^\theta(w)$ are finite, one has

$$\psi_\Gamma^\theta(v + w) > \psi_\Gamma^\theta(v) + \psi_\Gamma^\theta(w).$$

For θ -Anosov groups, work of Quint [Qui03] and Sambarino [Sam24] (see also [Sam14], [PS17]) implies that ψ_Γ^θ is strictly concave, differentiable on the relative interior of $\mathcal{L}_\theta(\Gamma)$, and has infinite slope at every nonzero point of its relative boundary. Their arguments rely on thermodynamic formalism (e.g. [BCLS15]) and do not directly extend to relatively θ -Anosov groups.

For relatively θ -Anosov groups, the results of Canary, Zhang, and the third named author, together with Quint's duality, yield the analogous differentiability and infinite-slope properties [CZZ25]. Thus strict concavity was the remaining missing part of this regularity picture.

Our first main theorem fills this gap.

Theorem 1.1 (Strict concavity of the growth indicator function). *Let $\Gamma < \mathbf{G}$ be a Zariski dense relatively θ -Anosov group. Then its θ -growth indicator function ψ_Γ^θ is strictly concave.*

Remark 1.2. Reyes–Wang [RW26] have independently established Theorem 1.1.

We record two closely related consequences of Theorem 1.1. A linear form $\phi \in \mathfrak{a}_\theta^*$ is said to be *tangent* to ψ_Γ^θ at a nonzero vector $u \in \mathfrak{a}_\theta$ if

$$\phi \geq \psi_\Gamma^\theta \quad \text{and} \quad \phi(u) = \psi_\Gamma^\theta(u).$$

Since both ϕ and ψ_Γ^θ are positively homogeneous, tangency at u depends only on the ray $\mathbb{R}_{>0} u$, which is necessarily in $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$.

Corollary 1.3 (Tangent directions). *Let $\Gamma < \mathbf{G}$ be Zariski dense relatively θ -Anosov group. For every $\phi \in \mathfrak{a}_\theta^*$, there is at most one ray in $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$ along which ϕ is tangent to ψ_Γ^θ . In addition, the tangent linear forms in $\mathfrak{a}_\theta^*$ are in one-to-one correspondence with the rays contained in $\text{int } \mathcal{L}_\theta(\Gamma)$.*

Indeed, strict concavity gives the uniqueness of the tangent ray. When Γ is Zariski dense, the limit cone $\mathcal{L}_\theta(\Gamma)$ has nonempty interior in $\mathfrak{a}_\theta$ [Ben97]. The differentiability result of [CZZ25] associates a unique tangent linear form to every ray in $\text{int } \mathcal{L}_\theta(\Gamma)$, while the infinite slope result ensures that the tangent ray of every tangent linear form lies in this interior. Thus the correspondence is bijective. The correspondence is given by

$$\mathbb{R}_{>0} u \mapsto d\psi_\Gamma^\theta|_u.$$

For a linear form $\phi \in \mathfrak{a}_\theta^*$ that is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$, this uniqueness has a natural normalized formulation. The section

$$S_\phi := \{u \in \mathcal{L}_\theta(\Gamma) : \phi(u) = 1\}$$

is compact and convex. Since ψ_Γ^θ is upper semicontinuous, it attains its maximum on S_ϕ . Moreover, strict concavity implies that this maximum is attained at a unique point. We denote it by u_ϕ ; thus

$$\psi_\Gamma^\theta(u_\phi) = \max_{u \in \mathcal{L}_\theta(\Gamma), \phi(u)=1} \psi_\Gamma^\theta(u).$$

Thus u_ϕ is the unique dominant asymptotic Cartan direction for the weighted counting problem determined by ϕ .

The uniqueness in Corollary 1.3 was needed in the local mixing result [KOP26] to identify a direction arising from thermodynamic formalism with the direction characterized by tangency to the growth indicator function. This application was one of the main motivations for the present work, together with the intrinsic geometric question of whether growth indicator functions are strictly concave.

1.2. The Manhattan hypersurface. Our approach to Theorem 1.1 is through a convex set dual to the growth indicator function. For $\phi \in \mathfrak{a}_\theta^*$, define the extended critical exponent

$$\delta^\phi(\Gamma) := \inf \left\{ s > 0 : \sum_{\gamma \in \Gamma} e^{-s\phi(\kappa_\theta(\gamma))} < +\infty \right\} \in [0, +\infty],$$

where the infimum of the empty set is understood to be $+\infty$.

Whenever $\phi \circ \kappa_\theta$ is bounded below and proper on Γ , this agrees with the counting exponent

$$\delta^\phi(\Gamma) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \# \{ \gamma \in \Gamma : \phi(\kappa_\theta(\gamma)) \leq T \}.$$

In particular, this applies when ϕ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$.

Following Burger [Bur93] and Quint [Qui02b, Qui03], define

$$\mathcal{Q}_\theta(\Gamma) := \{ \phi \in \mathfrak{a}_\theta^* : \delta^\phi(\Gamma) \leq 1 \}.$$

Its boundary

$$\partial \mathcal{Q}_\theta(\Gamma) = \{ \phi \in \mathfrak{a}_\theta^* : \delta^\phi(\Gamma) = 1 \}$$

is called the θ -Manhattan hypersurface of Γ .

Quint's duality identifies $\mathcal{Q}_\theta(\Gamma)$ with the set of linear forms dominating the growth indicator function:

$$\mathcal{Q}_\theta(\Gamma) = \{ \phi \in \mathfrak{a}_\theta^* : \phi \geq \psi_\Gamma^\theta \}.$$

Consequently,

$$\partial \mathcal{Q}_\theta(\Gamma) = \{ \phi \in \mathfrak{a}_\theta^* : \phi \text{ is tangent to } \psi_\Gamma^\theta \}.$$

If ϕ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$, the same duality gives the variational formula

$$\delta^\phi(\Gamma) = \max_{u \in \mathcal{L}_\theta(\Gamma), \phi(u)=1} \psi_\Gamma^\theta(u).$$

Thus the geometry of the Manhattan hypersurface is dual to the geometry of the growth indicator function.

Our second main theorem establishes the regularity of this dual hypersurface.

Theorem 1.4 (Smoothness of the Manhattan hypersurface). *Let $\Gamma < \mathbf{G}$ be a non-elementary relatively θ -Anosov group. Then its θ -Manhattan hypersurface $\partial \mathcal{Q}_\theta(\Gamma)$ is a $\mathcal{C}^1$ hypersurface.*

Remark 1.5. Reyes–Wang [RW26] have independently established Theorem 1.4.

Theorem 1.4 implies Theorem 1.1 by Quint’s duality [Qui03].

1.3. The general transverse case. Theorem 1.4 is a consequence of a local regularity theorem for general transverse groups. The additional hypothesis is formulated in terms of growth at infinity.

Motivated by the work of Pit–Schapira [PS18] in negative curvature, Wen [Wen26] introduced the *critical-gap-at-infinity* condition, also called the *strongly positively recurrent* condition, for transverse groups. We recall its precise definition in Section 4.3. Roughly speaking, this condition requires that the exponential growth arising outside every compact part of the associated flow space be strictly smaller than the full exponential growth. Wen proved that a critical gap at infinity implies finiteness of the associated Bowen–Margulis–Sullivan measure.

Our most general result is the following.

Theorem 1.6. *Let $\Gamma < \mathbf{G}$ be a non-elementary θ -transverse group. Suppose that*

$$\phi \in \partial \mathcal{Q}_\theta(\Gamma)$$

is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$ and has a critical gap at infinity. Then $\partial \mathcal{Q}_\theta(\Gamma)$ is a $\mathcal{C}^1$ hypersurface in a neighborhood of ϕ .

If, in addition, Γ is Zariski dense, then $\partial \mathcal{Q}_\theta(\Gamma)$ is strictly convex in a neighborhood of ϕ .

Canary, Zhang, and the third named author proved that, if Γ is relatively θ -Anosov, then every $\phi \in \partial \mathcal{Q}_\theta(\Gamma)$ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$ [CZZ25]. Wen [Wen26] further proved that every such ϕ has a critical gap at infinity. Therefore, for relatively θ -Anosov groups, Theorem 1.6 applies at every point of the Manhattan hypersurface and yields Theorem 1.4.

1.4. Comments on the proof. The proof of Theorem 1.6 is inspired by the work of Cantrell–Tanaka [CT25] on Manhattan curves associated with pairs of premetrics on hyperbolic groups. Their argument gives the $\mathcal{C}^1$ -regularity of the Manhattan hypersurface for Anosov groups and relies on the uniform hyperbolicity of the underlying group.

For relatively Anosov groups, this strategy can be adapted using the reparametrizations of flow spaces developed in [ZZ22, KO25]. For general transverse groups, however, such a reparametrization is unavailable because there is no sufficiently controlled global structure outside compact subsets of the flow space.

To overcome this, we construct a bounded continuous map

$$\mathbf{f} : \mathcal{G} \rightarrow \mathfrak{a}_\theta,$$

which we call the *Cartan displacement observable*. Its orbit integrals coarsely recover the partial Cartan displacement of recurrent orbit segments. More precisely, for orbit segments whose initial and terminal points lie over a fixed compact part of the quotient,

$$\int_0^T \mathbf{f}(g_t v) dt$$

differs from the corresponding partial Cartan projection by a uniformly bounded amount. The construction of the above observable is the main novelty of this paper.

This observable allows us to identify the derivative of a local convex graph parametrizing the Manhattan hypersurface, at every point where the derivative exists, with ratios of averages against the corresponding Bowen–Margulis–Sullivan measure. As we will show, the critical-gap-at-infinity condition is stable under small perturbations of the linear form and gives uniform control of the mass of these measures outside compact subsets. We use this to prove that the relevant averages depend continuously on the linear form.

The local graphing function of the Manhattan hypersurface is convex and hence differentiable almost everywhere. Its derivative agrees almost everywhere with a continuous function obtained from the Bowen–Margulis–Sullivan averages. A standard convexity argument then shows that the graphing function is everywhere differentiable with continuous derivative, proving the $\mathcal{C}^1$ -regularity of the Manhattan hypersurface. In the Zariski dense case, the local strict-convexity assertion follows from the known rigidity of equality in the convexity inequality for critical exponents.

1.5. Notation. Given two functions $f, g : X \rightarrow (0, +\infty)$, we write $f \ll g$, or equivalently $g \gg f$, if there exists a constant $C > 0$ such that

$$f(x) \leq Cg(x)$$

for every $x \in X$. If $f \ll g$ and $g \ll f$, we write $f \asymp g$.

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2. SEMISIMPLE ALGEBRAIC GROUPS

Throughout the paper, G is a connected semisimple real algebraic group. Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be a fixed Cartan decomposition of its Lie algebra where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces of a fixed Cartan involution of $\mathfrak{g}$ respectively. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, and fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. We denote by $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots and by $\Delta \subset \mathfrak{a}^*$ the corresponding set of simple restricted roots corresponding to the choice of $\mathfrak{a}^+$. Then we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Let Σ^+ (resp. Σ^-) denote the restricted roots which are nonnegative (respectively, nonpositive) linear combinations of elements of Δ .

2.1. Cartan projection. Let $K < G$ denote the maximal compact subgroup with Lie algebra $\mathfrak{k}$. By the Cartan decomposition, for any $g \in G$ there exists a unique $\kappa(g) \in \mathfrak{a}^+$ such that

$$g \in K(\exp \kappa(g))K.$$

This defines the map $\kappa : G \rightarrow \mathfrak{a}^+$, called the *Cartan projection*.

Fix a representative $w_0 \in K$ of the longest Weyl element that has order two. The *opposition involution* is the map

$$(2.1) \quad \mathbf{i} := -\text{Ad}_{w_0} : \mathfrak{a} \rightarrow \mathfrak{a}.$$

It satisfies

$$\kappa(g^{-1}) = \mathbf{i}\kappa(g) \quad \text{for all } g \in G.$$

The adjoint $\mathbf{i}^* : \mathfrak{a}^* \rightarrow \mathfrak{a}^*$ of the opposition involution preserves the set of simple roots. For a subset $\theta \subset \Delta$, we define

$$\mathbf{i}^*\theta := \{\mathbf{i}^*\alpha : \alpha \in \theta\}.$$

2.2. Parabolic subgroups and flag manifolds. Given a nonempty $\theta \subset \Delta$, the associated parabolic subgroup P_θ is the stabilizer under the adjoint action of the Lie algebra

$$\mathfrak{u}_\theta^+ := \bigoplus_{\alpha \in \Sigma_\theta^+} \mathfrak{g}_\alpha$$

where $\Sigma_\theta^+ := \Sigma^+ \setminus \text{span}(\Delta \setminus \theta)$.

The *Furstenberg boundary* and, more generally, the θ -*boundary* are the quotient spaces

$$\mathcal{F}_\Delta := G/P_\Delta \quad \text{and} \quad \mathcal{F}_\theta := G/P_\theta.$$

We also call $\mathcal{F}_\theta$ a *partial flag manifold*. Two elements $x \in \mathcal{F}_\theta$ and $y \in \mathcal{F}_{\mathbf{i}^*\theta}$ are *transverse* if there exists $g \in G$ such that

$$x = gP_\theta \quad \text{and} \quad y = gw_0P_{\mathbf{i}^*\theta},$$

Equivalently, the pair (x, y) is contained in the unique open G -orbit in $\mathcal{F}_\theta \times \mathcal{F}_{\mathbf{i}^*\theta}$.

2.3. The partial Iwasawa cocycle. We denote by $N < P_\Delta$ the unipotent radical of P_Δ . Then we have the Iwasawa decomposition $G = KAN$, and the product map $K \times A \times N \rightarrow G$ is a diffeomorphism.

The *Iwasawa cocycle* $B_\Delta : G \times \mathcal{F}_\Delta \rightarrow \mathfrak{a}$ is defined as follows: for $g \in G$ and $x \in \mathcal{F}_\Delta$, $B_\Delta(g, x) \in \mathfrak{a}$ is the unique element such that

$$gk \in K(\exp B_\Delta(g, x))N$$

where $k \in K$ satisfies $kP_\Delta = x$ in $\mathcal{F}_\Delta$.

For general $\theta \subset \Delta$, let

$$\mathfrak{a}_\theta := \bigcap_{\alpha \notin \theta} \ker \alpha.$$

For $\alpha \in \Delta$, let ω_α denote the (restricted) fundamental weight associated with α . Then $\{\omega_\alpha|_{\mathfrak{a}_\theta}\}_{\alpha \in \theta}$ is a basis for the dual space $\mathfrak{a}_\theta^*$. Thus, we can identify $\mathfrak{a}_\theta^*$ with the span of $\{\omega_\alpha\}_{\alpha \in \theta}$ and, in particular, view $\mathfrak{a}_\theta^*$ as a subspace of $\mathfrak{a}^*$. Moreover, there exists a unique projection

$$\pi_\theta : \mathfrak{a} \rightarrow \mathfrak{a}_\theta$$

satisfying

$$(2.2) \quad \omega_\alpha(\pi_\theta(H)) = \omega_\alpha(H) \quad \text{for all } H \in \mathfrak{a} \text{ and } \alpha \in \theta.$$

The *partial Iwasawa cocycle* $B_\theta : \mathbf{G} \times \mathcal{F}_\theta \rightarrow \mathfrak{a}_\theta$ is defined as

$$(2.3) \quad B_\theta(g, x) := \pi_\theta(B_\Delta(g, \tilde{x}))$$

for any $\tilde{x} \in \mathcal{F}_\Delta$ that projects to $x \in \mathcal{F}_\theta$ under the canonical projection $\mathcal{F}_\Delta \rightarrow \mathcal{F}_\theta$. The above definition is independent of the choice of $\tilde{x}$ [Qui02a, Lemma 6.1].

2.4. The limit cone. The *limit cone* of $\Gamma < \mathbf{G}$ is defined as the asymptotic cone of $\kappa(\Gamma) \subset \mathfrak{a}^+$:

$$\mathcal{L}(\Gamma) := \{X \in \mathfrak{a}^+ : \exists \{\gamma_n\} \subset \Gamma \text{ and } t_n \searrow 0 \text{ such that } t_n \kappa(\gamma_n) \rightarrow X\}.$$

Denoting by $\kappa_\theta := \pi_\theta \circ \kappa : \mathbf{G} \rightarrow \mathfrak{a}_\theta^+$, the θ -*limit cone* $\mathcal{L}_\theta(\Gamma)$ is the asymptotic cone of $\kappa_\theta(\Gamma) \subset \mathfrak{a}_\theta^+$. We have

$$\mathcal{L}_\theta(\Gamma) = \pi_\theta(\mathcal{L}(\Gamma)).$$

2.5. Patterson–Sullivan measures. Let $\Gamma < \mathbf{G}$ be a non-elementary θ -discrete subgroup. The θ -limit set $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$ consists of all $\xi \in \mathcal{F}_\theta$ for which there exists a sequence of distinct elements $\{\gamma_n\} \subset \Gamma$ such that

$$\min_{\alpha \in \theta} \alpha(\kappa(\gamma_n)) \rightarrow +\infty \quad \text{and} \quad k_n \mathbf{P}_\theta \rightarrow \xi,$$

where $\gamma_n = k_n \exp(\kappa(\gamma_n)) \ell_n$ with $k_n, \ell_n \in \mathbf{K}$. When Γ is Zariski dense, this is equivalently the set of $\xi \in \mathcal{F}_\theta$ for which $(\gamma_n)_* \text{Leb}_\theta$ converges to the Dirac mass at ξ for some sequence $\{\gamma_n\} \subset \Gamma$ where Leb_θ denotes the $\mathbf{K}$ -invariant probability measure on $\mathcal{F}_\theta$. Moreover, $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$ is the unique closed Γ -minimal subset of $\mathcal{F}_\theta$ [Ben97].

For $\phi \in \mathfrak{a}_\theta^*$ and $\delta \geq 0$, a Borel probability measure μ on $\mathcal{F}_\theta$ is called a δ -*dimensional ϕ -Patterson–Sullivan measure* for Γ on $\mathcal{F}_\theta$ if, for every $\gamma \in \Gamma$, the measure $\gamma_* \mu$ is absolutely continuous with respect to μ and

$$\frac{d\gamma_* \mu}{d\mu}(x) = e^{-\delta \phi(B_\theta(\gamma^{-1}, x))} \quad \text{for } \mu\text{-a.e. } x \in \mathcal{F}_\theta.$$

If $\phi \in \mathfrak{a}_\theta^*$ is tangent to ψ_Γ^θ at an interior direction of $\mathfrak{a}_\theta^+$, then there exists a one-dimensional ϕ -Patterson–Sullivan measure for Γ supported on $\Lambda_\theta(\Gamma)$ ([Qui02a] for $\theta = \Delta$, [KOW25, Proposition 5.12] in general).

We call Γ θ -*divergent* or θ -*regular* if, for any $\alpha \in \theta$ and any sequence $\{\gamma_n\} \subset \Gamma$ of distinct elements,

$$\alpha(\kappa(\gamma_n)) \rightarrow +\infty \quad \text{as } n \rightarrow +\infty.$$

For $\phi \in \mathfrak{a}_\theta^*$, we set

$$\delta^\phi(\Gamma) := \limsup_{T \rightarrow +\infty} \frac{\log \#\{\gamma \in \Gamma : \phi(\kappa(\gamma)) \leq T\}}{T} \in [0, +\infty].$$

If $\phi > 0$ on $\mathcal{L}_\theta - \{0\}$, then $0 < \delta^\phi(\Gamma) < +\infty$ [KOW25, Lemmas 3.5, 4.3].

Lemma 2.1. *If $\Gamma < \mathbf{G}$ is $\{\alpha\}$ -discrete for $\alpha \in \Delta$, then $0 < \delta^{\omega_\alpha}(\Gamma) < +\infty$.*

Proof. Set $\theta_\alpha := \{\alpha\}$. The space $\mathfrak{a}_{\theta_\alpha}$ is one-dimensional, and $\mathfrak{a}_{\theta_\alpha}^+$ is a ray. Writing $\omega_\alpha = \sum_{\beta \in \Delta} c_\beta \beta$, the coefficient c_α is positive and $c_\beta \geq 0$. Since $\beta|_{\mathfrak{a}_{\theta_\alpha}} = 0$ for all $\beta \neq \alpha$,

$$\omega_\alpha(H) = c_\alpha \alpha(H) > 0 \quad \text{for every } H \in \mathfrak{a}_{\theta_\alpha}^+ \setminus \{0\}.$$

Since $\mathcal{L}_{\theta_\alpha}(\Gamma) = \mathfrak{a}_{\theta_\alpha}^+$, it follows that ω_α is positive on $\mathcal{L}_{\theta_\alpha}(\Gamma) \setminus \{0\}$. Therefore, [KOW25, Lemmas 3.5, 4.3], applied with $\theta = \theta_\alpha$, gives $0 < \delta^{\omega_\alpha}(\Gamma) < +\infty$. $\square$

In the following proposition, the linear form $\delta^\phi(\Gamma)\phi$ is always tangent to ψ_Γ^θ [KOW25, Lemma 4.5], but the tangent direction is allowed to belong to the boundary of $\mathfrak{a}_\theta^+$.

Proposition 2.2 ([CZZ24, Proposition 3.2]). *Suppose that $\Gamma < \mathbf{G}$ is θ -divergent. If $\phi \in \mathfrak{a}_\theta^*$ satisfies $\delta^\phi(\Gamma) < +\infty$, then there exists a $\delta^\phi(\Gamma)$ -dimensional ϕ -Patterson–Sullivan measure for Γ supported on $\Lambda_\theta(\Gamma)$.*

Next, we recall some consequences of an ergodic dichotomy for transverse groups established in [CZZ24] and [KOW25]. To state the result, we need a few definitions. Given $\phi \in \mathfrak{a}_\theta^*$, the pair (Γ, ϕ) is of *divergence type* if $\delta^\phi(\Gamma) < +\infty$ and

$$\sum_{\gamma \in \Gamma} e^{-\delta^\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty.$$

Also, recall that Γ acts on $\Lambda_\theta(\Gamma)$ as a convergence group and hence there is a well-defined notion of conical limit point: $x \in \Lambda_\theta(\Gamma)$ is a *conical limit point* if there exist a sequence $\{\gamma_n\} \subset \Gamma$ of distinct elements and distinct points $a, b \in \Lambda_\theta(\Gamma)$ such that $\gamma_n^{-1}x \rightarrow a$ and $\gamma_n^{-1}y \rightarrow b$ for all $y \in \Lambda_\theta(\Gamma) \setminus \{x\}$. In this case, we say that $\{\gamma_n\} \subset \Gamma$ *converges conically to x* . The set of all conical limit points is called the *conical limit set*.

Theorem 2.3 ([CZZ24], [KOW25]). *Suppose $\Gamma < \mathbf{G}$ is a non-elementary θ -transverse group and $\phi \in \mathfrak{a}_\theta^*$ satisfies $\delta^\phi(\Gamma) < +\infty$. If (Γ, ϕ) is of divergence type, then:*

- (1) *There is a unique $\delta^\phi(\Gamma)$ -dimensional ϕ -Patterson–Sullivan measure for Γ supported on $\Lambda_\theta(\Gamma)$; denote it by μ .*
- (2) *If, in addition, Γ is Zariski dense, then there is a unique $\delta^\phi(\Gamma)$ -dimensional ϕ -Patterson–Sullivan measure for Γ supported on $\mathcal{F}_\theta$.*
- (3) *The action of Γ on $(\Lambda_\theta(\Gamma), \mu)$ is ergodic.*
- (4) *The conical limit set in $\Lambda_\theta(\Gamma)$ has full μ -measure.*

3. PROJECTIVELY VISIBLE MODELS OF TRANSVERSE GROUPS

In [CZZ24] and [CZZ26], projectively visible models were constructed for transverse groups. In this section, we recall this construction.

3.1. Background on convex real projective geometry. We recall some terminology from real projective geometry.

Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a *properly convex domain*, i.e., Ω is a bounded convex open subset of some affine chart of $\mathbb{P}(\mathbb{R}^d)$. Its *automorphism group* is

$$\text{Aut}(\Omega) := \{g \in \text{PGL}(d, \mathbb{R}) : g\Omega = \Omega\}.$$

For a subgroup $H < \text{Aut}(\Omega)$, its *limit set* is the set $\Lambda_\Omega(H) \subset \partial\Omega$ of points $x \in \partial\Omega$ such that there exist $o \in \Omega$ and a sequence $\{h_n\} \subset H$ with $h_n o \rightarrow x$.

Given $x, y \in \bar{\Omega}$, we let $[x, y]$ denote the projective line segment contained in $\bar{\Omega}$ joining x to y . We further define $(x, y) := [x, y] \setminus \{x, y\}$, $[x, y) := [x, y] \setminus \{y\}$, and $(x, y] := [x, y] \setminus \{x\}$.

The *Hilbert metric* d_Ω on Ω is defined as follows: for distinct $p, q \in \Omega$, let $a, b \in \partial\Omega$ be the unique points with $p, q \in (a, b)$ and in the order a, p, q, b , and define

$$d_\Omega(p, q) := \frac{1}{2} \log \frac{\|a - q\| \|b - p\|}{\|a - p\| \|b - q\|}$$

where $\|\cdot\|$ is any norm on any affine chart containing $\overline{\Omega}$. The Hilbert metric is a proper geodesic metric on which $\text{Aut}(\Omega)$ acts by isometries. Moreover, for $p, q \in \Omega$, the line segment $[p, q]$ can be parametrized as a geodesic.

Directly from the definition of the Hilbert metric and convexity, we have the following useful estimate on the distance between two geodesic rays with a common endpoint.

Lemma 3.1. *Let $p, q \in \Omega$ and $x \in \partial\Omega$. For every $p' \in [p, x)$, we have*

$$d_\Omega(p', [q, x)) \leq d_\Omega(p, q).$$

An element $W \in \text{Gr}_{d-1}(\mathbb{R}^d)$ is a *supporting hyperplane* at $x \in \partial\Omega$ if $x \in \mathbb{P}(W)$ and $\mathbb{P}(W) \cap \Omega = \emptyset$. Convexity of Ω implies that every boundary point is contained in at least one supporting hyperplane, and we say that $\partial\Omega$ is $\mathcal{C}^1$ -smooth at $x \in \partial\Omega$ if x is contained in a unique supporting hyperplane.

We end this section by recording a result of Tholozan. Given a discrete group $\Gamma_0 < \text{Aut}(\Omega)$, the *Hilbert metric critical exponent* is

$$\delta_\Omega(\Gamma_0) := \limsup_{R \rightarrow +\infty} \frac{\log \#\{\gamma \in \Gamma_0 : d_\Omega(\gamma o, o) \leq R\}}{R}$$

where $o \in \Omega$ is any fixed point. It follows from Tholozan's work that this critical exponent has a uniform upper bound, depending only on the dimension d .

Theorem 3.2 ([Tho17]). *If $\Gamma_0 < \text{Aut}(\Omega)$ is discrete, then $\delta_\Omega(\Gamma_0) \leq d - 2$.*

3.2. Projectively visible groups. Next we define a special class of groups acting on properly convex domains, the projectively visible groups, and their relation with transverse groups.

Definition 3.3. Let $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ be a properly convex domain. A discrete subgroup $\Gamma_0 < \text{Aut}(\Omega)$ is *projectively visible* if

- $(x, y) \subset \Omega$ for every $x, y \in \Lambda_\Omega(\Gamma_0)$,
- every $x \in \Lambda_\Omega(\Gamma_0)$ is a $\mathcal{C}^1$ -smooth point of $\partial\Omega$.

Under some mild conditions on $\mathbf{G}$ and $\mathbf{P}_\theta$, every transverse group can be identified with a projectively visible subgroup. We fix any norm $\|\cdot\|$ on $\mathfrak{a}_\theta$.

Theorem 3.4 ([CZZ24, Theorem 6.2]). *Suppose that $\mathbf{G}$ has trivial center, that $\theta \subset \Delta$ is a nonempty subset with $\theta = \mathbf{i}^*\theta$, and that $\mathbf{P}_\theta$ contains no simple factors of $\mathbf{G}$. If $\Gamma < \mathbf{G}$ is θ -transverse, then there exist $d \in \mathbb{N}$, a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, a projectively visible subgroup $\Gamma_0 < \text{Aut}(\Omega)$, an isomorphism $\rho : \Gamma_0 \rightarrow \Gamma$, and a ρ -equivariant homeomorphism $\xi : \Lambda_\Omega(\Gamma_0) \rightarrow \Lambda_\theta(\Gamma)$.*

Moreover, if $o \in \Omega$, then there exists $C_0 > 1$ such that for all $\gamma \in \Gamma_0$,

$$C_0^{-1} d_\Omega(o, \gamma o) - C_0 \leq \|\kappa_\theta(\rho(\gamma))\| \leq C_0 d_\Omega(o, \gamma o) + C_0.$$

Proof. The “moreover” assertion is not stated explicitly in [CZZ24, Theorem 6.2] so we briefly explain why it follows from the proof in [CZZ24, Appendix B]. In particular, there exists an irreducible faithful representation $\Phi : \mathbf{G} \rightarrow \text{PSL}(d, \mathbb{R})$ such that $\Phi \circ \rho = \text{id}_{\Gamma_0}$. Further, there exist positive integers $\{m_\alpha\}_{\alpha \in \theta}$ such that

$$\log \|\Phi(g)\| = \sum_{\alpha \in \theta} m_\alpha \omega_\alpha(\kappa_\theta(g))$$

for all $g \in \mathbf{G}$, where $\|\Phi(g)\|$ is the operator norm of $\Phi(g)$. Hence

$$\log(\|\Phi(g)\| \|\Phi(g)^{-1}\|) = \sum_{\alpha \in \theta} (m_\alpha + m_{i^*\alpha}) \omega_\alpha(\kappa_\theta(g))$$

for all $g \in \mathbf{G}$. By [DGK24, Proposition 10.1], there exists $c_1 > 0$ such that

$$|\mathrm{d}_\Omega(o, \gamma o) - \log(\|\Phi(\rho(\gamma))\| \|\Phi(\rho(\gamma))^{-1}\|)| \leq c_1$$

for all $\gamma \in \Gamma_0$. Since $\sum_{\alpha \in \theta} (m_\alpha + m_{i^*\alpha}) \omega_\alpha$ is positive on $\mathfrak{a}_\theta^+ \setminus \{0\}$, there exists a constant $c_2 > 1$ such that for all $g \in \mathbf{G}$,

$$c_2^{-1} \|\kappa_\theta(g)\| \leq \sum_{\alpha \in \theta} (m_\alpha + m_{i^*\alpha}) \omega_\alpha(\kappa_\theta(g)) \leq c_2 \|\kappa_\theta(g)\|.$$

Finally, using the fact that $\Phi \circ \rho = \mathrm{id}_{\Gamma_0}$, we obtain for all $\gamma \in \Gamma_0$,

$$c_2^{-1} \mathrm{d}_\Omega(o, \gamma o) - c_2^{-1} c_1 \leq \|\kappa_\theta(\rho(\gamma))\| \leq c_2 \mathrm{d}_\Omega(o, \gamma o) + c_2 c_1.$$

□

For general $\mathbf{G}$, by replacing $\mathbf{G}$ with a quotient, one can always assume that $\mathbf{G}$ has trivial center and $\mathbf{P}_\theta$ contains no simple factors of $\mathbf{G}$, see [CZZ24, Section 2.4]. Hence, we can always find $(\Omega, \Gamma_0, \rho, \xi)$ satisfying Theorem 3.4.

Definition 3.5. Given a θ -transverse group $\Gamma < \mathbf{G}$, a *projectively visible model* of Γ is a choice of $(\Omega, \Gamma_0, \rho, \xi)$ that satisfies the conclusion of Theorem 3.4.

3.3. Shadows. We assume $\theta = i^*\theta$. Suppose that $\Gamma < \mathbf{G}$ is a non-elementary θ -transverse group. Choose a projectively visible model $(\Omega, \Gamma_0, \rho, \xi)$ of Γ . Since $\xi : \Lambda_\Omega(\Gamma_0) \rightarrow \Lambda_\theta(\Gamma)$ is a ρ -equivariant homeomorphism, we may regard measures on $\Lambda_\theta(\Gamma)$ as measures on $\Lambda_\Omega(\Gamma_0)$ by pulling them back through ξ .

Fix a basepoint $o \in \Omega$. For $R > 0$ and $\gamma \in \Gamma_0$, we define the shadow of the Hilbert R -ball centered at γo by

$$\mathcal{O}_R(\gamma) := \{x \in \partial\Omega : \mathrm{d}_\Omega(\gamma o, [o, x]) < R\}.$$

We also write

$$\widehat{\mathcal{O}}_R(\rho(\gamma)) := \xi(\mathcal{O}_R(\gamma) \cap \Lambda_\Omega(\Gamma_0)) \subset \Lambda_\theta(\Gamma).$$

The following shadow Lemma estimates the size of a shadow with respect to a Patterson–Sullivan measure.

Proposition 3.6 (Shadow Lemma, [CZZ24, Proposition 7.1]). *Let $\phi \in \mathfrak{a}_\theta^*$ and $\delta \geq 0$. Let μ be a δ -dimensional ϕ -Patterson–Sullivan measure for Γ on $\Lambda_\theta(\Gamma)$. Then for all sufficiently large $R > 0$, there exists $C = C(R) > 1$ such that for all $\gamma \in \Gamma$,*

$$C^{-1} e^{-\delta\phi(\kappa(\gamma))} \leq \mu(\widehat{\mathcal{O}}_R(\gamma)) \leq C e^{-\delta\phi(\kappa(\gamma))}.$$

An important property of shadows is their uniform multiplicity. Fix a linear form $\phi \in \mathfrak{a}_\theta^*$ such that there exists a δ -dimensional ϕ -Patterson–Sullivan measure for Γ on $\Lambda_\theta(\Gamma)$. Then for $n \in \mathbb{N}$, we set

$$S_n^\phi := \{\gamma \in \Gamma : \phi(\kappa(\gamma)) \in [n, n+1)\}.$$

Lemma 3.7 (Uniform multiplicity, [CZZ24, Lemma 8.2]). *Suppose $\phi \in \mathfrak{a}_\theta^*$ is as above. Then for any $R > 0$,*

$$\sup_{n \in \mathbb{N}} \sup_{x \in \Lambda_\theta(\Gamma)} \# \left\{ \gamma \in S_n^\phi : x \in \widehat{\mathcal{O}}_R(\gamma) \right\} < +\infty.$$

3.4. Conical limit points. We continue to suppose that $\Gamma < \mathbf{G}$ is a non-elementary θ -transverse group, $(\Omega, \Gamma_0, \rho, \xi)$ is a projectively visible model of Γ , and $o \in \Omega$ is a fixed basepoint.

Recall that the convergence action of Γ on $\Lambda_\theta(\Gamma)$ has a well-defined notion of conical limit point and hence a conical limit set. The action of Γ_0 on Ω gives an equivalent characterization in terms of geodesic rays. The next result records this equivalence.

Proposition 3.8. *If $x \in \Lambda_\Omega(\Gamma_0)$ and $\{\gamma_n\} \subset \Gamma_0$ is a sequence of distinct elements, then the following are equivalent:*

- (1) $\{\rho(\gamma_n)\} \subset \Gamma$ converges conically to $\xi(x) \in \Lambda_\theta(\Gamma)$ (in the convergence group sense).
- (2) There exist $p \in \Omega$ and $R > 0$ such that $\gamma_n p \rightarrow x$ and

$$\sup_{n \geq 1} d_\Omega(\gamma_n p, [p, x]) < R.$$

- (3) There exist $R > 0$ such that

$$x \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n).$$

Proof. The equivalence between (1) and (2) is due to [CZZ26, Lemma 3.6]. Thus it suffices to show that (2) and (3) are equivalent.

Suppose (2) holds. Let $R_1 := d_\Omega(p, o)$. For each n , choose $p'_n \in [p, x]$ with $d_\Omega(\gamma_n p, p'_n) < R$. Then, by Lemma 3.1,

$$d_\Omega(p'_n, [o, x]) \leq d_\Omega(p, o) = R_1.$$

Therefore,

$$d_\Omega(\gamma_n o, [o, x]) \leq d_\Omega(\gamma_n o, \gamma_n p) + d_\Omega(\gamma_n p, p'_n) + d_\Omega(p'_n, [o, x]) < R + 2R_1.$$

Thus

$$x \in \bigcap_{n \geq 1} \mathcal{O}_{R+2R_1}(\gamma_n)$$

and hence (3) holds.

Suppose (3) holds. Then clearly

$$\sup_{n \geq 1} d_\Omega(\gamma_n o, [o, x]) < R.$$

and so we just need to prove that $\gamma_n o \rightarrow x$. Let $d_{\mathbb{P}}$ be a distance on $\mathbb{P}(\mathbb{R}^d)$ induced by a Riemannian metric. By [CZZ24, Lemma 12.5], we have $\text{diam } \mathcal{O}_R(\gamma_n) \rightarrow 0$ with respect to this distance. For each n , we can fix $x_n \in \partial\Omega$ such that $\gamma_n o \in [o, x_n]$. Then $x_n \in \mathcal{O}_R(\gamma_n)$. Since $\{\gamma_n\}$ are distinct and Γ_0 acts properly on Ω , we have $d_{\mathbb{P}}(x_n, \gamma_n o) \rightarrow 0$. Since $x, x_n \in \mathcal{O}_R(\gamma_n)$ and $\text{diam } \mathcal{O}_R(\gamma_n) \rightarrow 0$ we have $d_{\mathbb{P}}(x_n, x) \rightarrow 0$. Thus $\gamma_n o \rightarrow x$. $\square$

4. CRITICAL GAP AT INFINITY

Suppose that $\theta = i^*\theta$, that $\Gamma < \mathbf{G}$ is a non-elementary θ -transverse group, and that $(\Omega, \Gamma_0, \rho, \xi)$ is a projectively visible model of Γ .

In this section, we consider the flow space given by the projectively visible model and Bowen–Margulis–Sullivan measures on it, and discuss their relation to the critical gap at infinity, following Wen’s work [Wen26].

4.1. Flow spaces. We denote by $T^1\Omega$ the unit tangent bundle over Ω with respect to the infinitesimal Hilbert metric, and let $g_t : T^1\Omega \rightarrow T^1\Omega$, $t \in \mathbb{R}$, denote the geodesic flow. For $v \in T^1\Omega$, let $v^\pm = \lim_{t \rightarrow \pm\infty} g_tv \in \partial\Omega$ be the forward and backward endpoints of v , respectively.

We consider the flow space

$$\mathcal{G} := \{v \in T^1\Omega : v^+, v^- \in \Lambda_\Omega(\Gamma_0)\}.$$

Let $\Lambda_\Omega^{(2)}(\Gamma_0) \subset \Lambda_\Omega(\Gamma_0) \times \Lambda_\Omega(\Gamma_0)$ denote the set of ordered pairs of distinct points. Since Γ_0 is projectively visible, $\mathcal{G}$ is homeomorphic to $\Lambda_\Omega^{(2)}(\Gamma_0) \times \mathbb{R}$. Using the Hopf parametrization for $\mathcal{G}$, we can make this identification explicit.

Since each point in $\Lambda_\Omega(\Gamma_0)$ is $\mathcal{C}^1$ -smooth, for $\zeta \in \Lambda_\Omega(\Gamma_0)$ and $a, b \in \Omega$, the following limit exists (see [Bra20, Lemma 3.2]):

$$\beta_\zeta(a, b) := \lim_{\Omega \ni p \rightarrow \zeta} d_\Omega(p, a) - d_\Omega(p, b).$$

Then fixing a base point $o \in \Omega$, the map

$$v \in \mathcal{G} \mapsto (v^-, v^+, \beta_{v^+}(o, \pi(v)))$$

is a homeomorphism and the natural Γ_0 -action on $\mathcal{G}$ corresponds to the Γ_0 -action on $\Lambda_\Omega^{(2)}(\Gamma_0) \times \mathbb{R}$ given by

$$\gamma \cdot (x, y, t) = (\gamma x, \gamma y, t + \beta_y(\gamma^{-1}o, o)).$$

For the rest of the paper, we identify

$$\mathcal{G} = \Lambda_\Omega^{(2)}(\Gamma_0) \times \mathbb{R}$$

using this map. Under this identification, the Hilbert geodesic flow is given by

$$g_t(x, y, s) = (x, y, s + t) \quad \text{for } t \in \mathbb{R}.$$

4.2. BMS measures. Suppose $\phi \in \mathfrak{a}_\theta^*$ and (Γ, ϕ) is of divergence type. We define the Bowen–Margulis–Sullivan (BMS) measure on $\mathcal{G}$ associated with ϕ .

For $x, y \in \Lambda_\theta(\Gamma)$, the $\mathfrak{a}_\theta$ -valued Gromov product is defined as

$$\langle x, y \rangle_\theta := -(B_\theta(g^{-1}, x) + iB_\theta(g^{-1}, y))$$

for $g \in \mathbf{G}$ with $gP_\theta = x$ and $gw_0P_\theta = y$, noting that we are assuming $\theta = i^*\theta$. This is well defined and independent of the choice of g .

Notice that $\delta := \delta^\phi(\Gamma) = \delta^{i^*\phi}(\Gamma)$. So by Theorem 2.3, there exist unique δ -dimensional ϕ - and $i^*\phi$ -Patterson–Sullivan measures μ and μ_i for Γ on $\Lambda_\theta(\Gamma)$, respectively. Then the Radon measure on $\Lambda_\theta^{(2)}(\Gamma)$ defined by

$$e^{\delta\phi(\langle x, y \rangle_\theta)} d\mu(x) d\mu_i(y)$$

is Γ -invariant. Hence, the Radon measure $\tilde{m}_\phi$ on $\mathcal{G} = \Lambda_\Omega^{(2)}(\Gamma_0) \times \mathbb{R}$ defined by

$$d\tilde{m}_\phi(v) := e^{\delta\phi(\langle \xi(v^+), \xi(v^-) \rangle_\theta)} d\mu(\xi(v^+)) d\mu_i(\xi(v^-)) dt$$

is invariant under the Γ_0 -action and the Hilbert geodesic flow, where dt denotes the Lebesgue measure on the $\mathbb{R}$ -component. Since Γ_0 acts properly on $\mathcal{G}$, this measure descends to a geodesic-flow-invariant measure m_ϕ on the quotient $\Gamma_0 \backslash \mathcal{G}$ called the *BMS measure associated with ϕ* .

This measure has the following dynamical properties.

Theorem 4.1 ([CZZ24, KOW25, BCZZ24a]). *The geodesic flow on $\Gamma_0 \backslash \mathcal{G}$ is ergodic with respect to m_ϕ . Moreover, if $\|m_\phi\| < +\infty$, then the geodesic flow on $\Gamma_0 \backslash \mathcal{G}$ is mixing with respect to m_ϕ .*

Proof. Ergodicity follows from the ergodic dichotomy in [CZZ24, KOW25], as we are assuming that (Γ, ϕ) is of divergence type. The Hilbert length spectrum $\{\ell_\Omega(\gamma) : \gamma \in \Gamma_0\}$, where ℓ_Ω denotes the Hilbert translation length, generates a dense additive subgroup of $\mathbb{R}$ by [BZ23, Lemma 2.8]. Then mixing follows from the proof of mixing for continuous GPS systems in [BCZZ24a, Theorem 4.2]. $\square$

4.3. Critical gap at infinity. In this section, we recall Wen’s results in [Wen26] showing that a “critical gap at infinity” implies the finiteness of the BMS measure.

Suppose that θ is i -invariant and that $\Gamma < \mathbf{G}$ is a non-elementary θ -transverse group. Fix $\phi \in \mathfrak{a}_\theta^*$. We suppose that (Γ, ϕ) is of divergence type, and that $(\Omega, \Gamma_0, \rho, \xi)$ is a projectively visible model of Γ .

For a compact subset $K \subset \mathcal{G}$, let $\Gamma_K \subset \Gamma_0$ be the subset of elements $\gamma \in \Gamma_0$ for which there exist $v \in K$ and $t \geq 0$ such that

$$g_t v \in \gamma K \quad \text{and} \quad g_{[0,t]} v \cap \Gamma_0 K \subset K \cup \gamma K.$$

Then for $\phi \in \mathfrak{a}_\theta^*$, let

$$\delta^\phi(K) := \inf \left\{ s > 0 : \sum_{\gamma \in \Gamma_K} e^{-s\phi(\kappa(\rho(\gamma)))} < +\infty \right\}.$$

By [Wen26, Lemma 4.2], there exists a compact set $K_0 \subset \mathcal{G}$ such that if $\gamma \in \Gamma_0$, then $g_{[0,\infty)} v \cap \gamma K_0 \neq \emptyset$ for some $v \in K_0$. For subsets containing K_0 , Wen established the following monotonicity property.

Lemma 4.2 ([Wen26, Lemma 4.4]). *Suppose $K_1, K_2 \subset \mathcal{G}$ are compact. If $K_0 \subset K_1 \subset \text{int}(K_2)$, then*

$$\delta^\phi(K_2) \leq \delta^\phi(K_1).$$

The *entropy at infinity* for $\phi \in \mathfrak{a}_\theta^*$ is

$$\delta_\infty^\phi(\Gamma) := \inf \{ \delta^\phi(K) : K \subset \mathcal{G} \text{ compact and } K_0 \subset K \}.$$

Then ϕ has a *critical gap at infinity* for Γ if

$$\delta_\infty^\phi(\Gamma) < \delta^\phi(\Gamma).$$

This is also referred to as the *strongly positively recurrent* (SPR) condition.

Recall the θ -limit cone $\mathcal{L}_\theta(\Gamma)$ from Section 2.4.

Theorem 4.3 ([Wen26, Theorems 4.8 and 4.9]). *If $\phi \in \mathfrak{a}_\theta^*$ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$ and has a critical gap at infinity for Γ , then (Γ, ϕ) is of divergence type and the BMS measure on $\Gamma_0 \backslash \mathcal{G}$ associated with ϕ is finite.*

Proof. Wen’s results are stated in the context of abstract GPS systems, and in Appendix A, we state Wen’s general result and explain why it implies the above theorem. $\square$

5. FLOW OBSERVABLES ENCODING CARTAN DISPLACEMENT

In this section, we continue to assume that $\theta = i^*\theta$, that $\Gamma < \mathbf{G}$ is a non-elementary θ -transverse group, and that $(\Omega, \Gamma_0, \rho, \xi)$ is a projectively visible model of Γ . The main construction of the paper is given in Proposition 5.1, where we construct Cartan displacement observables on $\mathcal{G}$ whose orbit integrals coarsely re-code the Cartan displacement of recurrent geodesic flow. These observables will be a key ingredient in the proof of Theorem 1.6.

Proposition 5.1 (Cartan displacement observable). *There exists a Γ_0 -invariant continuous bounded map*

$$\mathbf{f} : \mathcal{G} \rightarrow \mathfrak{a}_\theta,$$

with the following property: For every compact set $K \subset \mathcal{G}$, there exists $C_K > 0$ such that, if $v \in K$, $\gamma \in \Gamma_0$, and $g_T v \in \gamma K$ for some $T \geq 0$, then

$$\left\| \int_0^T \mathbf{f}(g_t v) dt - \kappa_\theta(\rho(\gamma)) \right\| \leq C_K.$$

The associated additive Cartan displacement cocycle is

$$\mathbf{F}(v, T) := \int_0^T \mathbf{f}(g_t v) dt.$$

It satisfies

$$\mathbf{F}(v, S + T) = \mathbf{F}(v, S) + \mathbf{F}(g_S v, T),$$

and Proposition 5.1 says that $\mathbf{F}(v, T)$ coarsely records the partial Cartan displacement of every orbit segment whose initial and terminal points lie over a fixed compact part of the quotient.

For $\phi \in \mathfrak{a}_\theta^*$, define the associated ϕ -Cartan displacement observable by

$$f_\phi := \phi \circ \mathbf{f}.$$

Then the map $(\phi, v) \mapsto f_\phi(v)$ is continuous, the map $\phi \mapsto f_\phi$ is linear, and

$$\sup_{v \in \mathcal{G}} |f_\phi(v)| \leq C_0 \|\phi\|$$

where $C_0 = \sup_{v \in \mathcal{G}} \|\mathbf{f}(v)\|$. Moreover, for every compact set $K \subset \mathcal{G}$,

$$(5.1) \quad \left| \int_0^T f_\phi(g_t v) dt - \phi(\kappa_\theta(\rho(\gamma))) \right| \leq C_K \|\phi\|$$

whenever $v \in K$ and $g_T v \in \gamma K$.

The function f_ϕ descends to a bounded continuous function

$$\hat{f}_\phi : \Gamma_0 \backslash \mathcal{G} \rightarrow \mathbb{R}.$$

Before proving Proposition 5.1, we give one application. Fix $\psi \in \mathfrak{a}_\theta^*$ such that (Γ, ψ) is of divergence type and $\|m_\psi\| < +\infty$. Let μ_ψ denote the unique $\delta^\psi(\Gamma)$ -dimensional ψ -Patterson–Sullivan measure for Γ on $\Lambda_\theta(\Gamma)$, given by Theorem 2.3.

Corollary 5.2. *There exists a set $A \subset \Lambda_\theta(\Gamma)$ with full μ_ψ -measure with the following property: if $x \in A$, $\{\gamma_n\} \subset \Gamma$ converges to x conically, $\phi_1, \phi_2 \in \mathfrak{a}_\theta^*$, and $\int \hat{f}_{\phi_2} dm_\psi \neq 0$, then*

$$\lim_{n \rightarrow \infty} \frac{\phi_1(\kappa(\gamma_n))}{\phi_2(\kappa(\gamma_n))} = \frac{\int \hat{f}_{\phi_1} dm_\psi}{\int \hat{f}_{\phi_2} dm_\psi}.$$

Proof. By Theorem 4.1, the geodesic flow on $(\Gamma_0 \backslash \mathcal{G}, m_\psi)$ is ergodic. Fix a set $A_0 \subset \Gamma_0 \backslash \mathcal{G}$ of full m_ψ -measure such that the Birkhoff ergodic theorem holds on A_0 for each function in the finite set $\{\hat{f}_{\omega_\alpha}\}_{\alpha \in \theta}$. By linearity of $\phi \mapsto \hat{f}_\phi$, the Birkhoff ergodic theorem holds on A_0 for each function in $\{\hat{f}_\phi\}_{\phi \in \mathfrak{a}_\theta^*}$.

Let $A'_0 \subset \mathcal{G}$ denote the lift of A_0 . Then A'_0 has full $\tilde{m}_\psi$ -measure and hence

$$A := \{x \in \Lambda_\theta(\Gamma) : x = \xi(v^+) \text{ for some } v \in A'_0\}$$

has full μ_ψ -measure.

Now fix $x \in A$, a sequence $\{\gamma_n\} \subset \Gamma$ converging conically to x , and $\phi_1, \phi_2 \in \mathfrak{a}_\theta^*$ such that $\int \hat{f}_{\phi_2} dm_\psi \neq 0$.

Fix $v \in A'_0$ such that $x = \xi(v^+)$. Let $\eta_n := \rho^{-1}(\gamma_n)$. Since $\xi : \Lambda_\Omega(\Gamma_0) \rightarrow \Lambda_\theta(\Gamma)$ is a homeomorphism, $\{\eta_n\} \subset \Gamma_0$ converges conically to v^+ . Then, by Proposition 3.8, there exist a compact subset $K \subset \mathcal{G}$ and a sequence $\{t_n\} \subset [0, +\infty)$ with $t_n \rightarrow +\infty$ such that $g_{t_n}v \in \eta_n K$ for all $n \geq 1$. Then, by the Birkhoff ergodic theorem and (5.1),

$$\frac{\int \hat{f}_{\phi_1} dm_\psi}{\int \hat{f}_{\phi_2} dm_\psi} = \lim_{n \rightarrow \infty} \frac{\int_0^{t_n} f_{\phi_1}(g_tv) dt}{\int_0^{t_n} f_{\phi_2}(g_tv) dt} = \lim_{n \rightarrow \infty} \frac{\phi_1(\kappa(\gamma_n))}{\phi_2(\kappa(\gamma_n))}. \quad \square$$

Remark 5.3. We remark that when $\|m_\psi\| = +\infty$, one may try to apply the Hopf ratio ergodic theorem to prove a similar result. However, in this case, $\hat{f}_\phi$ may not be in $L^1(\Gamma_0 \backslash \mathcal{G}, m_\psi)$ and hence the Hopf ratio ergodic theorem may not apply.

Next, we verify that linear forms that are positive on the θ -limit cone have positive integrals.

Lemma 5.4. *If $\phi \in \mathfrak{a}_\theta^*$ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$, then*

$$\int \hat{f}_\phi dm_\psi > 0.$$

Proof. Since μ_ψ -a.e. point in $\Lambda_\theta(\Gamma)$ is conical (see Theorem 2.3), arguing as in the preceding proof, we can fix $v \in \mathcal{G}$ such that

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f_\phi(g_tv) dt = \frac{1}{\|m_\psi\|} \int \hat{f}_\phi dm_\psi$$

and there exist a compact subset $K \subset \mathcal{G}$, a sequence $\{t_n\} \subset [0, +\infty)$ with $t_n \rightarrow +\infty$, and a sequence $\{\eta_n\} \subset \Gamma_0$ such that $g_{t_n}v \in \eta_n K$ for all $n \geq 1$. Then there exists $C_1 > 0$ such that

$$|d_\Omega(o, \eta_n o) - t_n| \leq C_1.$$

Then by (5.1),

$$\lim_{n \rightarrow \infty} \frac{\phi(\kappa(\rho(\eta_n)))}{d_\Omega(o, \eta_n o)} = \frac{1}{\|m_\psi\|} \int \hat{f}_\phi dm_\psi.$$

Since ϕ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$, there exists $C_2 > 1$ such that

$$\phi(\kappa(\rho(\eta_n))) \geq C_2^{-1} \|\kappa_\theta(\rho(\eta_n))\| - C_2$$

and so the “moreover” part of Theorem 3.4 implies positivity of the integral. $\square$

5.1. Proof of Proposition 5.1. We begin by bounding the partial Iwasawa cocycle in terms of the Hilbert distance.

Lemma 5.5. *There exist $a, b > 0$ such that, for every $\alpha \in \theta$, $x \in \mathcal{F}_\theta$, and $\gamma \in \Gamma_0$,*

$$\omega_\alpha(B_\theta(\rho(\gamma), x)) \leq a \, d_\Omega(o, \gamma o) + b.$$

Proof. For every $\alpha \in \theta$, there exists $C_\alpha > 0$ such that

$$\omega_\alpha(B_\theta(h, x)) \leq C_\alpha \omega_\alpha(\kappa_\theta(h))$$

for every $h \in \mathbf{G}$ and $x \in \mathcal{F}_\theta$; see, for instance, [KZ26, Lemma 4.1]. Hence

$$\omega_\alpha(B_\theta(\rho(\gamma), x)) \leq C_\alpha \|\omega_\alpha\| \|\kappa_\theta(\rho(\gamma))\|.$$

Since θ is finite, the constants on the right-hand side may be chosen uniformly in α . The desired estimate now follows from the “moreover” part of Theorem 3.4. $\square$

Let d be the dimension of the ambient vector space containing Ω , so that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, and set

$$\lambda := d + a,$$

where a is given by Lemma 5.5. For every $\alpha \in \theta$ and $v \in \mathcal{G}$, define

$$P_\alpha(v) := \sum_{\gamma \in \Gamma_0} e^{-\lambda d_\Omega(\gamma o, \pi(v))} e^{\omega_\alpha(B_\theta(\rho(\gamma)^{-1}, \xi(v^+)))},$$

where $\pi : T^1\Omega \rightarrow \Omega$ is the basepoint projection.

We first verify that this series converges locally uniformly. Let $L \subset \mathcal{G}$ be compact and set

$$R_L := \sup_{v \in L} d_\Omega(o, \pi(v)).$$

For $v \in L$, the triangle inequality and Lemma 5.5, applied to γ^{-1} , give

$$\begin{aligned} & e^{-\lambda d_\Omega(\gamma o, \pi(v))} e^{\omega_\alpha(B_\theta(\rho(\gamma)^{-1}, \xi(v^+)))} \\ & \leq e^{\lambda R_L + b} e^{-(\lambda - a) d_\Omega(o, \gamma o)} = e^{\lambda R_L + b} e^{-d d_\Omega(o, \gamma o)}. \end{aligned}$$

By Theorem 3.2,

$$\sum_{\gamma \in \Gamma_0} e^{-d d_\Omega(o, \gamma o)} < +\infty.$$

It follows that P_α is positive and continuous.

We next record the equivariance of P_α . If $\eta \in \Gamma_0$, then the cocycle identity, the ρ -equivariance of ξ , and the change of variables $\gamma \mapsto \eta^{-1}\gamma$ give

$$\begin{aligned} & e^{\omega_\alpha(B_\theta(\rho(\eta), \xi(v^+)))} P_\alpha(\eta v) \\ & = \sum_{\gamma \in \Gamma_0} e^{-\lambda d_\Omega(\gamma o, \eta \pi(v))} e^{\omega_\alpha(B_\theta(\rho(\gamma)^{-1}, \xi(\eta v^+)) + B_\theta(\rho(\eta), \xi(v^+)))} \\ & = \sum_{\gamma \in \Gamma_0} e^{-\lambda d_\Omega(\eta^{-1}\gamma o, \pi(v))} e^{\omega_\alpha(B_\theta(\rho(\eta^{-1}\gamma)^{-1}, \xi(v^+)))} \\ & = P_\alpha(v). \end{aligned}$$

Thus

$$(5.2) \quad e^{\omega_\alpha(B_\theta(\rho(\eta), \xi(v^+)))} P_\alpha(\eta v) = P_\alpha(v).$$

Since the basepoint of g_tv is at Hilbert distance $|t|$ from the basepoint of v , we also have

$$(5.3) \quad e^{-\lambda|t|}P_\alpha(v) \leq P_\alpha(g_tv) \leq e^{\lambda|t|}P_\alpha(v)$$

for every $v \in \mathcal{G}$ and $t \in \mathbb{R}$.

Since $\{\omega_\alpha|_{\mathfrak{a}_\theta}\}_{\alpha \in \theta}$ is a basis of $\mathfrak{a}_\theta^*$, there exists a unique continuous map

$$\mathbf{Q} : \mathcal{G} \rightarrow \mathfrak{a}_\theta$$

such that for every $\alpha \in \theta$,

$$(5.4) \quad \omega_\alpha(\mathbf{Q}(v)) = \log P_\alpha(v).$$

By (5.2), we have

$$(5.5) \quad \mathbf{Q}(\eta v) = \mathbf{Q}(v) - B_\theta(\rho(\eta), \xi(v^+))$$

for every $\eta \in \Gamma_0$ and $v \in \mathcal{G}$.

Choose $c_\theta > 0$ such that for every $H \in \mathfrak{a}_\theta$,

$$\|H\| \leq c_\theta \max_{\alpha \in \theta} |\omega_\alpha(H)|.$$

By (5.3),

$$|\omega_\alpha(\mathbf{Q}(g_tv) - \mathbf{Q}(v))| \leq \lambda|t|.$$

Consequently,

$$(5.6) \quad \|\mathbf{Q}(g_tv) - \mathbf{Q}(v)\| \leq C_0|t|, \quad C_0 := c_\theta\lambda.$$

We now define

$$\mathbf{f}(v) := \mathbf{Q}(v) - \mathbf{Q}(g_1v).$$

The map $\mathbf{f}$ is continuous, and (5.6) gives

$$\sup_{v \in \mathcal{G}} \|\mathbf{f}(v)\| \leq C_0.$$

Moreover, since the Γ_0 -action commutes with the geodesic flow and $(g_1v)^+ = v^+$, (5.5) gives

$$\begin{aligned} \mathbf{f}(\eta v) &= \mathbf{Q}(\eta v) - \mathbf{Q}(g_1\eta v) \\ &= \mathbf{Q}(v) - B_\theta(\rho(\eta), \xi(v^+)) - \mathbf{Q}(g_1v) + B_\theta(\rho(\eta), \xi((g_1v)^+)) \\ &= \mathbf{f}(v). \end{aligned}$$

Thus $\mathbf{f}$ is Γ_0 -invariant.

It remains to verify the Cartan displacement estimate. First observe that, for every $T \geq 0$,

$$\int_0^T \mathbf{f}(g_tv) dt = \int_0^T (\mathbf{Q}(g_tv) - \mathbf{Q}(g_{t+1}v)) dt = \int_0^1 \mathbf{Q}(g_tv) dt - \int_T^{T+1} \mathbf{Q}(g_tv) dt.$$

Hence, by (5.6),

$$(5.7) \quad \left\| \int_0^T \mathbf{f}(g_tv) dt - (\mathbf{Q}(v) - \mathbf{Q}(g_Tv)) \right\| \leq C_0.$$

Now fix a compact set $K \subset \mathcal{G}$, and suppose that $v \in K$ and $g_Tv \in \gamma K$. Then $\gamma^{-1}g_Tv \in K$. Applying (5.5) with $\eta = \gamma^{-1}$ gives

$$\mathbf{Q}(\gamma^{-1}g_Tv) = \mathbf{Q}(g_Tv) - B_\theta(\rho(\gamma)^{-1}, \xi(v^+)).$$

It follows that

$$(5.8) \quad \left\| \mathbf{Q}(v) - \mathbf{Q}(g_T v) + B_\theta(\rho(\gamma)^{-1}, \xi(v^+)) \right\| = \left\| \mathbf{Q}(v) - \mathbf{Q}(\gamma^{-1} g_T v) \right\| \leq 2 \sup_{w \in K} \|\mathbf{Q}(w)\|.$$

Set

$$D := 1 + \text{diam}(\{o\} \cup \pi(K)),$$

where the diameter is taken with respect to d_Ω . Since $\gamma^{-1} g_T v \in K$,

$$d_\Omega(\gamma o, \pi(g_T v)) = d_\Omega(o, \pi(\gamma^{-1} g_T v)) < D.$$

Moreover, by Lemma 3.1, the point $\pi(g_T v) \in [\pi(v), v^+]$ is within distance D of the ray $[o, v^+]$. Therefore,

$$B_\Omega(\gamma o, 2D) \cap [o, v^+] \neq \emptyset.$$

By [CZZ24, Lemma 7.3], there exists $C_D > 0$, depending only on D , such that

$$(5.9) \quad \left\| B_\theta(\rho(\gamma)^{-1}, \xi(v^+)) + \kappa_\theta(\rho(\gamma)) \right\| \leq C_D.$$

Combining (5.8) and (5.9), we obtain

$$\left\| \mathbf{Q}(v) - \mathbf{Q}(g_T v) - \kappa_\theta(\rho(\gamma)) \right\| \leq 2 \sup_{w \in K} \|\mathbf{Q}(w)\| + C_D.$$

Finally, (5.7) yields

$$\left\| \int_0^T \mathbf{f}(g_t v) dt - \kappa_\theta(\rho(\gamma)) \right\| \leq C_0 + 2 \sup_{w \in K} \|\mathbf{Q}(w)\| + C_D.$$

This proves the proposition with

$$C_K := C_0 + 2 \sup_{w \in K} \|\mathbf{Q}(w)\| + C_D.$$

□

6. SMOOTH POINTS OF THE MANHATTAN HYPERSURFACE

Suppose that $\theta = i^* \theta$ and that $\Gamma < \mathbf{G}$ is a non-elementary θ -transverse group. Fix $\alpha_0 \in \theta$ and set $\theta_0 := \theta \setminus \{\alpha_0\}$ so that we have a subspace $\mathfrak{a}_{\theta_0}^* \subset \mathfrak{a}_\theta^*$. We also set $\psi_0 := \sum_{\alpha \in \theta} \omega_\alpha$, which is positive on $\mathfrak{a}_{\theta_0}^+$. Notice that

$$\mathfrak{a}_{\theta_0}^* \oplus \mathbb{R} \cdot \psi_0 = \mathfrak{a}_\theta^*.$$

Since $\mathcal{Q}_\theta(\Gamma)$ is convex and $\psi_0 > 0$ on $\mathfrak{a}_\theta^+ \setminus \{0\}$, there exists a convex function $\Phi : \mathfrak{a}_{\theta_0}^* \rightarrow \mathbb{R}$ so that

$$\partial \mathcal{Q}_\theta(\Gamma) = \{u + \Phi(u) \psi_0 : u \in \mathfrak{a}_{\theta_0}^*\}.$$

In other words, for each $u \in \mathfrak{a}_{\theta_0}^*$, $\Phi(u)$ is the abscissa of convergence, in the parameter s , of the series

$$\sum_{\gamma \in \Gamma} e^{-(u + s \psi_0)(\kappa(\gamma))}.$$

Since Φ is convex, it is differentiable almost everywhere, and we prove the following.

Proposition 6.1. *Suppose Φ is differentiable at $u \in \mathfrak{a}_{\theta_0}^*$, $\phi := u + \Phi(u)\psi_0$, and μ_ϕ is a 1-dimensional ϕ -Patterson–Sullivan measure for Γ on $\Lambda_\theta(\Gamma)$.*

There exists a set $A \subset \Lambda_\theta(\Gamma)$ with full μ_ϕ -measure with the following property: if $x \in A$, $\{\gamma_n\} \subset \Gamma$ converges to x conically, and $\psi \in \mathfrak{a}_{\theta_0}^$, then*

$$\lim_{n \rightarrow \infty} \frac{\psi(\kappa(\gamma_n))}{\psi_0(\kappa(\gamma_n))} = - \left. \frac{d}{dt} \right|_{t=0} \Phi(u + t\psi).$$

Proof. By the ergodic dichotomy (see [CZZ24, Theorem 1.4]), the conical limit set either has full or zero μ_ϕ -measure. In the case when the conical limit set has zero measure, the proposition is vacuously true, so we can assume the conical limit set has full μ_ϕ -measure.

By replacing $\mathbf{G}$ with a quotient, we can assume that $\mathbf{G}$ has trivial center and $\mathbf{P}_\theta$ contains no simple factors of $\mathbf{G}$, see [CZZ24, Section 2.4]. Then, using Theorem 3.4, we can fix a projectively visible model $(\Omega, \Gamma_0, \rho, \xi)$ of Γ .

By Proposition 3.8, if $x \in \Lambda_\theta(\Gamma)$ and $\{\gamma_n\} \subset \Gamma$ converges to x conically, then there exists $R > 0$ such that

$$x \in \bigcap_{n=1}^{\infty} \widehat{\mathcal{O}}_R(\gamma_n).$$

By linearity, it suffices to prove the assertion when $\psi = \omega_\alpha$, $\alpha \in \theta_0$. Fix $\alpha \in \theta_0$ and write

$$r := - \left. \frac{d}{dt} \right|_{t=0} \Phi(u + t\omega_\alpha) \quad \text{and} \quad \delta := \delta^{\omega_\alpha}(\Gamma)$$

for simplicity. Since Γ is $\{\alpha\}$ -discrete for every $\alpha \in \theta$, by Lemma 2.1, $\delta < +\infty$. For $s \neq 0$, let

$$f(s) := \frac{\Phi(s\delta\omega_\alpha + u) - \Phi(u)}{s\delta} + r.$$

Then $f(s) \rightarrow 0$ as $s \rightarrow 0$.

It also suffices to fix $\epsilon > 0$ and construct a full μ_ϕ -measure subset $A_\epsilon \subset \Lambda_\theta(\Gamma)$ with the following property: if $x \in A_\epsilon$ and $\{\gamma_j\} \subset \Gamma$ converges to x conically, then

$$(6.1) \quad 2\epsilon + \liminf_{j \rightarrow \infty} \frac{\omega_\alpha(\kappa(\gamma_j))}{\psi_0(\kappa(\gamma_j))} \geq r \geq -2\epsilon + \limsup_{j \rightarrow \infty} \frac{\omega_\alpha(\kappa(\gamma_j))}{\psi_0(\kappa(\gamma_j))}.$$

Indeed, to finish the proof we can take $A := \bigcap_{n \in \mathbb{N}} A_{1/n}$.

Fix $\epsilon > 0$ and then fix $s > 0$ such that $\epsilon - f(s) > 0$ and $\epsilon + f(-s) > 0$. Let

$$\begin{aligned} \phi_s &:= (s\delta\omega_\alpha + u) + \Phi(s\delta\omega_\alpha + u)\psi_0 \quad \text{and} \\ \phi_{-s} &:= (-s\delta\omega_\alpha + u) + \Phi(-s\delta\omega_\alpha + u)\psi_0. \end{aligned}$$

Let μ be a δ -dimensional ω_α -Patterson–Sullivan measure, let μ_{ϕ_s} be a 1-dimensional ϕ_s -Patterson–Sullivan measure, and let $\mu_{\phi_{-s}}$ be a 1-dimensional ϕ_{-s} -Patterson–Sullivan measure for Γ on $\Lambda_\theta(\Gamma)$. Fix $R_0 \in \mathbb{N}$ sufficiently large so that the Shadow Lemma (Proposition 3.6) holds for μ , μ_ϕ , μ_{ϕ_s} , and $\mu_{\phi_{-s}}$ whenever $R \geq R_0$.

We first show that for μ_ϕ -a.e. $x \in \Lambda_\theta(\Gamma)$, if $\{\gamma_j\} \subset \Gamma$ converges to x conically, then

$$\liminf_{j \rightarrow \infty} \frac{\omega_\alpha(\kappa(\gamma_j))}{\psi_0(\kappa(\gamma_j))} \geq r - 2\epsilon.$$

Fix $R \geq R_0$. For each $n \in \mathbb{N}$, recall

$$S_n^{\psi_0} = \{\gamma \in \Gamma : \psi_0(\kappa(\gamma)) \in [n, n+1)\}.$$

and let

$$E_{R,n} := \bigcup \left\{ \widehat{\mathcal{O}}_R(\gamma) : \gamma \in S_n^{\psi_0} \text{ and } \mu \left(\widehat{\mathcal{O}}_R(\gamma) \right) \geq e^{-(r-\epsilon)\delta\psi_0(\kappa(\gamma))} \right\}.$$

Then by definition,

$$\mathbf{1}_{E_{R,n}} \leq \sum_{\gamma \in S_n^{\psi_0}} \mu \left(\widehat{\mathcal{O}}_R(\gamma) \right)^s e^{s(r-\epsilon)\delta\psi_0(\kappa(\gamma))} \mathbf{1}_{\widehat{\mathcal{O}}_R(\gamma)}$$

where $\mathbf{1}$ denotes the indicator function, and hence

$$\mu_\phi(E_{R,n}) \leq \sum_{\gamma \in S_n^{\psi_0}} \mu \left(\widehat{\mathcal{O}}_R(\gamma) \right)^s e^{s(r-\epsilon)\delta\psi_0(\kappa(\gamma))} \mu_\phi \left(\widehat{\mathcal{O}}_R(\gamma) \right).$$

Then, by the Shadow Lemma (Proposition 3.6)

$$\begin{aligned} \mu_\phi(E_{R,n}) &\ll \sum_{\gamma \in S_n^{\psi_0}} e^{-s\delta\omega_\alpha(\kappa(\gamma))} e^{s(r-\epsilon)\delta\psi_0(\kappa(\gamma))} e^{-\phi(\kappa(\gamma))} \\ &\asymp e^{s(r-\epsilon)\delta n - \Phi(u)n} \sum_{\gamma \in S_n^{\psi_0}} e^{-(s\delta\omega_\alpha + u)(\kappa(\gamma))}. \end{aligned}$$

Here and below, the implicit constants depend on R but not on n .

Notice that by the Shadow Lemma (Proposition 3.6) and the uniform multiplicity of shadows in the sets $S_n^{\psi_0}$ (Lemma 3.7),

$$\sum_{\gamma \in S_n^{\psi_0}} e^{-((s\delta\omega_\alpha + u) + \Phi(s\delta\omega_\alpha + u)\psi_0)(\kappa(\gamma))} = \sum_{\gamma \in S_n^{\psi_0}} e^{-\phi_s(\kappa(\gamma))} \asymp \sum_{\gamma \in S_n^{\psi_0}} \mu_{\phi_s} \left(\widehat{\mathcal{O}}_R(\gamma) \right) \ll 1.$$

Hence,

$$\sum_{\gamma \in S_n^{\psi_0}} e^{-(s\delta\omega_\alpha + u)(\kappa(\gamma))} \ll e^{\Phi(s\delta\omega_\alpha + u)n},$$

and therefore

$$\mu_\phi(E_{R,n}) \ll e^{s(r-\epsilon)\delta n - \Phi(u)n} e^{\Phi(s\delta\omega_\alpha + u)n}.$$

Then

$$\mu_\phi(E_{R,n}) \ll \left(e^{s(r-\epsilon)\delta - rs\delta + f(s)s\delta} \right)^n = \left(e^{-s\delta(\epsilon - f(s))} \right)^n.$$

Since $s > 0$ satisfies $\epsilon - f(s) > 0$, we have

$$\sum_{n=1}^{\infty} \mu_\phi(E_{R,n}) < +\infty.$$

By the Borel–Cantelli lemma, this implies that for μ_ϕ -a.e. $x \in \Lambda_\theta(\Gamma)$,

$$x \notin E_{R,n} \text{ for all sufficiently large } n.$$

Recall that $R \geq R_0$ was arbitrary. Let

$$E := \bigcup_{R \in [R_0, \infty) \cap \mathbb{N}} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_{R,n}.$$

Then E is μ_ϕ -null. The following claim finishes the proof of the first inequality in (6.1).

Claim: If $x \in \Lambda_\theta(\Gamma) \setminus E$ and $\{\gamma_j\} \subset \Gamma$ converges to x conically, then

$$\liminf_{j \rightarrow \infty} \frac{\omega_\alpha(\kappa(\gamma_j))}{\psi_0(\kappa(\gamma_j))} \geq r - 2\epsilon.$$

Proof of Claim: By Proposition 3.8, there exists $R \in [R_0, \infty) \cap \mathbb{N}$ such that

$$x \in \bigcap_{j \geq 1} \widehat{\mathcal{O}}_R(\gamma_j).$$

Hence, in order to have $x \notin E_{R,n}$ for all sufficiently large $n \in \mathbb{N}$, we must have

$$\mu \left(\widehat{\mathcal{O}}_R(\gamma_j) \right) < e^{-(r-\epsilon)\delta\psi_0(\kappa(\gamma_j))} \quad \text{for all sufficiently large } j \in \mathbb{N}.$$

Then the Shadow Lemma (Proposition 3.6) implies that

$$\frac{\omega_\alpha(\kappa(\gamma_j))}{\psi_0(\kappa(\gamma_j))} \geq r - 2\epsilon \quad \text{for all sufficiently large } j \in \mathbb{N}.$$

This completes the proof of the claim.

A similar argument also shows the second inequality in (6.1). We present this for the sake of completeness. Fix $R \geq R_0$ and consider the set

$$F_{R,n} := \bigcup \left\{ \widehat{\mathcal{O}}_R(\gamma) : \gamma \in S_n^{\psi_0} \text{ and } \mu \left(\widehat{\mathcal{O}}_R(\gamma) \right) \leq e^{-(r+\epsilon)\delta\psi_0(\kappa(\gamma))} \right\}.$$

Then by definition,

$$\mathbf{1}_{F_{R,n}} \leq \sum_{\gamma \in S_n^{\psi_0}} \mu \left(\widehat{\mathcal{O}}_R(\gamma) \right)^{-s} e^{-s(r+\epsilon)\delta\psi_0(\kappa(\gamma))} \mathbf{1}_{\widehat{\mathcal{O}}_R(\gamma)}$$

and so

$$\mu_\phi(F_{R,n}) \leq \sum_{\gamma \in S_n^{\psi_0}} \mu \left(\widehat{\mathcal{O}}_R(\gamma) \right)^{-s} e^{-s(r+\epsilon)\delta\psi_0(\kappa(\gamma))} \mu_\phi \left(\widehat{\mathcal{O}}_R(\gamma) \right).$$

Then, by the Shadow Lemma (Proposition 3.6)

$$\begin{aligned} \mu_\phi(F_{R,n}) &\ll \sum_{\gamma \in S_n^{\psi_0}} e^{s\delta\omega_\alpha(\kappa(\gamma))} e^{-s(r+\epsilon)\delta\psi_0(\kappa(\gamma))} e^{-\phi(\kappa(\gamma))} \\ &\asymp e^{-s(r+\epsilon)\delta n - \Phi(u)n} \sum_{\gamma \in S_n^{\psi_0}} e^{-(s\delta\omega_\alpha + u)(\kappa(\gamma))}. \end{aligned}$$

Here and below, the implicit constants depend on R but not on n .

Notice that by the Shadow Lemma (Proposition 3.6) and the uniform multiplicity of shadows in the sets $S_n^{\psi_0}$ (Lemma 3.7),

$$\sum_{\gamma \in S_n^{\psi_0}} e^{-((-s\delta\omega_\alpha + u) + \Phi(-s\delta\omega_\alpha + u)\psi_0)(\kappa(\gamma))} = \sum_{\gamma \in S_n^{\psi_0}} e^{-\phi_{-s}(\kappa(\gamma))} \asymp \sum_{\gamma \in S_n^{\psi_0}} \mu_{\phi_{-s}} \left(\widehat{\mathcal{O}}_R(\gamma) \right) \ll 1.$$

Hence,

$$\sum_{\gamma \in S_n^{\psi_0}} e^{-(s\delta\omega_\alpha + u)(\kappa(\gamma))} \ll e^{\Phi(-s\delta\omega_\alpha + u)n},$$

and therefore

$$\mu_\phi(F_{R,n}) \ll e^{-s(r+\epsilon)\delta n - \Phi(u)n} e^{\Phi(-s\delta\omega_\alpha + u)n}.$$

Then

$$\mu_\phi(F_{R,n}) \ll \left(e^{-s(r+\epsilon)\delta + rs\delta - f(-s)s\delta} \right)^n = \left(e^{-s\delta(\epsilon + f(-s))} \right)^n.$$

Since $\epsilon + f(-s) > 0$,

$$\sum_{n=1}^{\infty} \mu_\phi(F_{R,n}) < +\infty.$$

By the Borel–Cantelli lemma, this implies that for μ_ϕ -a.e. $x \in \Lambda_\theta(\Gamma)$,

$$x \notin F_{R,n} \quad \text{for all sufficiently large } n.$$

Recall that $R \geq R_0$ was arbitrary. Hence the set

$$F := \bigcup_{R \in [R_0, \infty) \cap \mathbb{N}} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_{R,n}$$

is μ_ϕ -null. As in the preceding claim, one verifies that if $x \in \Lambda_\theta(\Gamma) \setminus F$ and $\{\gamma_j\} \subset \Gamma$ converges to x conically, then

$$\limsup_{j \rightarrow \infty} \frac{\omega_\alpha(\kappa(\gamma_j))}{\psi_0(\kappa(\gamma_j))} \leq r + 2\epsilon.$$

Then $A_\epsilon := \Lambda_\theta(\Gamma) \setminus (E \cup F)$ is our desired full μ_ϕ -measure subset. $\square$

7. CONTINUITY PROPERTIES OF BMS MEASURES

Suppose $\theta = i^*\theta$, $\Gamma < \mathbf{G}$ is θ -transverse, and $(\Omega, \Gamma_0, \rho, \xi)$ is a projectively visible model of Γ .

We prove the following continuity result for BMS measures associated to linear forms that are positive on the θ -limit cone and have a critical gap at infinity.

Theorem 7.1. *Suppose $\phi \in \mathfrak{a}_\theta^*$ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$ and has a critical gap at infinity for Γ . Then the following statements hold:*

- (1) *There exists a neighborhood $\mathcal{U} \subset \mathfrak{a}_\theta^*$ of ϕ such that, for every $\psi \in \mathcal{U}$,*
 - (1-a) *ψ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$ and*
 - (1-b) *ψ has a critical gap at infinity for Γ .*
- (2) *If m_ψ denotes the BMS measure for $\psi \in \mathcal{U}$ and $f : \Gamma_0 \setminus \mathcal{G} \rightarrow \mathbb{R}$ is a bounded continuous function, then*

$$\psi \in \mathcal{U} \mapsto \int f dm_\psi \in \mathbb{R}$$

is continuous.

The rest of the section is devoted to the proof of the theorem. Fix $0 < \epsilon < 1$ such that

$$(7.1) \quad \left(\frac{1+\epsilon}{1-\epsilon} \right) \delta_\infty^\phi(\Gamma) < \delta^\phi(\Gamma).$$

Let $\mathbb{S}$ denote the unit sphere in $\mathfrak{a}_\theta$ with respect to some norm. By compactness of $\mathbb{S} \cap \mathcal{L}_\theta(\Gamma)$ and positivity of ϕ on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$, we can fix a neighborhood $\mathcal{U}$ of ϕ such that

$$(1-\epsilon)\phi(v) \leq \psi(v) \leq (1+\epsilon)\phi(v)$$

for all $\psi \in \mathcal{U}$ and $v \in \mathbb{S} \cap \mathcal{L}_\theta(\Gamma)$. By linearity, these inequalities extend to all of $\mathcal{L}_\theta(\Gamma)$. Thus (1-a) follows. Further,

$$(7.2) \quad \delta_\infty^\psi(\Gamma) \leq \left(\frac{1}{1-\epsilon} \right) \delta_\infty^\phi(\Gamma) \quad \text{and} \quad \delta^\psi(\Gamma) \geq \left(\frac{1}{1+\epsilon} \right) \delta^\phi(\Gamma)$$

for all $\psi \in \mathcal{U}$. Then by our choice of $\epsilon > 0$, every linear form $\psi \in \mathcal{U}$ has a critical gap at infinity for Γ . This shows (1-b).

Now Theorems 4.3 and 4.1 imply that for each $\psi \in \mathcal{U}$, the BMS measure m_ψ has finite mass and the Hilbert geodesic flow on $\Gamma_0 \setminus \mathcal{G}$ is ergodic with respect to m_ψ .

Since $\psi \mapsto \delta^\psi(\Gamma)$ is convex (see, e.g., [CZZ24, Theorem 13.1]), the critical exponent depends continuously on the linear form. Then the construction of the BMS measures and uniqueness of Patterson–Sullivan measures (see Theorem 2.3) imply that for any compactly supported continuous function $f : \Gamma_0 \backslash \mathcal{G} \rightarrow \mathbb{R}$, the map

$$(7.3) \quad \psi \in \mathcal{U} \mapsto \int f dm_\psi$$

is continuous. Thus, part (2) of the theorem holds for compactly supported continuous functions. To upgrade to bounded continuous functions, we use the following lemma.

Lemma 7.2. *After shrinking $\mathcal{U}$, for any $r > 0$ there exists a compact set $K \subset \Gamma_0 \backslash \mathcal{G}$ such that*

$$\sup_{\psi \in \mathcal{U}} m_\psi(K^c) < r.$$

Assuming the lemma, we complete the proof of (2) in the theorem. Fix $\psi_n \rightarrow \psi$ in $\mathcal{U}$. Then fix $r > 0$ and a compact set $K \subset \Gamma_0 \backslash \mathcal{G}$ such that

$$\sup_{\psi \in \mathcal{U}} m_\psi(K^c) < r.$$

Fix a compactly supported continuous function $\chi : \Gamma_0 \backslash \mathcal{G} \rightarrow [0, 1]$ with $\chi|_K \equiv 1$. Then by the continuity in (7.3),

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left| \int f dm_{\psi_n} - \int f dm_\psi \right| &\leq \limsup_{n \rightarrow \infty} \left| \int \chi f dm_{\psi_n} - \int \chi f dm_\psi \right| + 2 \|f\|_\infty r \\ &= 2 \|f\|_\infty r. \end{aligned}$$

Since $r > 0$ was arbitrary, $\int f dm_{\psi_n} \rightarrow \int f dm_\psi$. $\square$

It remains to prove the lemma. The argument is similar to the proof of [Wen26, Theorem 3.3], which is itself similar to the arguments in [PS18, Section 5.3].

Proof of Lemma 7.2. Fix $R > 0$ sufficiently large and let

$$K_R := \{v \in \mathcal{G} : d_\Omega(\pi(v), o) \leq R\}$$

where $\pi : T^1\Omega \rightarrow \Omega$ is the basepoint projection. Lemma 4.2 implies that

$$\lim_{R \rightarrow \infty} \delta^\phi(K_R) = \delta_\infty^\phi(\Gamma).$$

Using (7.1), we can increase $R > 0$ and assume that

$$\left(\frac{1+\epsilon}{1-\epsilon} \right) \delta^\phi(K_R) < \delta^\phi(\Gamma).$$

Then (7.2) implies that for all $\psi \in \mathcal{U}$,

$$(7.4) \quad \delta^\phi(K_R) < (1-\epsilon)\delta^\psi(\Gamma).$$

Let $q : \mathcal{G} \rightarrow \Gamma_0 \backslash \mathcal{G}$ be the quotient map and let $\hat{K}_R := q(K_R)$. Using the continuity in (7.3), increasing R , and replacing $\mathcal{U}$ with a smaller neighborhood, we can assume that for all $\psi \in \mathcal{U}$,

$$m_\psi(\hat{K}_R) > 0.$$

For $\gamma \in \Gamma_{K_R}$, let

$$A_\gamma := \{v \in \mathcal{G} : \{g_t v\}_{t \leq 0} \cap K_R \neq \emptyset \text{ and } \{g_t v\}_{t \geq 0} \cap \gamma K_R \neq \emptyset\}.$$

Since the flow on $\Gamma_0 \setminus \mathcal{G}$ is ergodic with respect to each m_ψ (Theorems 4.3 and 4.1), almost every flow line visits $\widehat{K}_R$ at an unbounded set of times. Hence if

$$\mathcal{A} := \bigcup_{\gamma \in \Gamma_{K_R}} A_\gamma,$$

then $q(\mathcal{A})$ has full m_ψ -measure for all $\psi \in \mathcal{U}$.

Claim: There exists $C = C(R) > 0$ such that: If $\gamma \in \Gamma_{K_R}$, then

$$A_\gamma \subset \Lambda_\Omega(\Gamma_0) \times \mathcal{O}_{2R}(\gamma) \times [-C, C + d_\Omega(o, \gamma o)].$$

Proof of Claim: We use the Hopf coordinates on $\mathcal{G}$. Let

$$C := \max\{|s| : (w^-, w^+, s) \in K_R\}.$$

Then

$$\max\{|s| : (w^-, w^+, s) \in \gamma K_R\} \leq C + d_\Omega(o, \gamma o).$$

Now fix $v = (v^-, v^+, t) \in A_\gamma$. By the definition of A_γ , we can choose $a \leq 0$ with $g_a v \in K_R$ and choose $b \geq 0$ with $g_b v \in \gamma K_R$. Then

$$t \geq t + a \geq -C \quad \text{and} \quad t \leq t + b \leq C + d_\Omega(o, \gamma o).$$

So

$$t \in [-C, C + d_\Omega(o, \gamma o)].$$

By the definition of A_γ , $[\pi(g_a v), v^+]$ intersects $B_\Omega(\gamma o, R)$ and $\pi(g_a v) \in B_\Omega(o, R)$, where B_Ω denotes a d_Ω -ball. So Lemma 3.1 implies that $[o, v^+]$ intersects $B_\Omega(\gamma o, 2R)$. Hence $v^+ \in \mathcal{O}_{2R}(\gamma)$. This completes the proof of the claim.

Notice that $\max_{v \in K_R} \|\langle \xi(v^-), \xi(v^+) \rangle_\theta\|$ is finite. So by the Shadow Lemma (Proposition 3.6), if $\gamma \in \Gamma_{K_R}$, then

$$\tilde{m}_\psi(A_\gamma) \ll (d_\Omega(o, \gamma o) + 2C)e^{-\delta^\psi(\Gamma)\psi(\kappa(\rho(\gamma)))}.$$

Since ϕ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$, there exists $c_1 > 0$ such that for all $\gamma \in \Gamma_0$,

$$\phi(\kappa(\rho(\gamma))) = \phi(\kappa_\theta(\rho(\gamma))) \geq c_1^{-1} \|\kappa_\theta(\rho(\gamma))\| - c_1.$$

Then by Theorem 3.4, there exists $c_2, c'_2 > 0$ such that for all $\gamma \in \Gamma_0$,

$$d_\Omega(o, \gamma o) \leq c_2 \phi(\kappa(\rho(\gamma))) + c'_2.$$

We also have that for all $\gamma \in \Gamma_0$ and $\psi \in \mathcal{U}$,

$$\begin{aligned} \psi(\kappa(\rho(\gamma))) &\geq \phi(\kappa(\rho(\gamma))) - \|\phi - \psi\| \|\kappa_\theta(\rho(\gamma))\| \\ &\geq (1 - c_1 \|\phi - \psi\|) \phi(\kappa(\rho(\gamma))) - c_1^2 \|\phi - \psi\|. \end{aligned}$$

So by replacing $\mathcal{U}$ with a smaller neighborhood, we can assume that

$$\psi(\kappa(\rho(\gamma))) \geq (1 - \epsilon/2) \phi(\kappa(\rho(\gamma))) - 1$$

for all $\gamma \in \Gamma_0$ and $\psi \in \mathcal{U}$. Since ϕ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$ and thus the set

$$\{\phi(\kappa(\rho(\gamma))) : \gamma \in \Gamma_0\}$$

is bounded below, we obtain that for all $\gamma \in \Gamma_0$ and $\psi \in \mathcal{U}$,

$$\tilde{m}_\psi(A_\gamma) \ll (\phi(\kappa(\rho(\gamma))) + (2C + c'_2)/c_2)e^{-\delta^\psi(\Gamma)(1-\epsilon/2)\phi(\kappa(\rho(\gamma)))}.$$

Hence

$$\tilde{m}_\psi(A_\gamma) \ll e^{-\delta^\psi(\Gamma)(1-\epsilon)\phi(\kappa(\rho(\gamma)))}.$$

Choose s such that

$$\delta^\phi(K_R) < s < \left(\frac{1-\epsilon}{1+\epsilon} \right) \delta^\phi(\Gamma).$$

By (7.2), we have $s < (1-\epsilon)\delta^\psi(\Gamma)$ for every $\psi \in \mathcal{U}$. Since the set $\{\phi(\kappa(\rho(\gamma))) : \gamma \in \Gamma_0\}$ is bounded below, the preceding estimate implies, with a constant independent of $\psi \in \mathcal{U}$, that

$$\tilde{m}_\psi(A_\gamma) \ll e^{-s\phi(\kappa(\rho(\gamma)))}.$$

The series $\sum_{\gamma \in \Gamma_{K_R}} e^{-s\phi(\kappa(\rho(\gamma)))}$ converges by the definition of $\delta^\phi(K_R)$. Hence the tails of

$$\sum_{\gamma \in \Gamma_{K_R}} \tilde{m}_\psi(A_\gamma)$$

tend to zero uniformly in $\psi \in \mathcal{U}$. Therefore, we can fix a finite set $F \subset \Gamma_{K_R}$ such that for all $\psi \in \mathcal{U}$,

$$\sum_{\gamma \in \Gamma_{K_R} \setminus F} \tilde{m}_\psi(A_\gamma) < r.$$

Then

$$K := q \left(\bigcup_{\gamma \in F} A_\gamma \right)$$

is a compact subset of $\Gamma_0 \setminus \mathcal{G}$ and

$$\sup_{\psi \in \mathcal{U}} m_\psi(K^c) \leq \sum_{\gamma \in \Gamma_{K_R} \setminus F} \tilde{m}_\psi(A_\gamma) < r.$$

□

8. PROOF OF THEOREM 1.6

We now combine our previous results to deduce Theorem 1.6. Suppose that $\theta = i^*\theta$ and that $\Gamma < \mathbf{G}$ is a non-elementary θ -transverse group. By replacing $\mathbf{G}$ with a quotient, we can assume that $\mathbf{G}$ has trivial center and $\mathbf{P}_\theta$ contains no simple factors of $\mathbf{G}$, see [CZZ24, Section 2.4]. Then, using Theorem 3.4, we can fix a projectively visible model $(\Omega, \Gamma_0, \rho, \xi)$ of Γ .

Fix $\phi_0 \in \partial \mathcal{Q}_\theta(\Gamma)$ such that ϕ_0 is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$ and has a critical gap at infinity for Γ . By Theorem 7.1, there exists an open neighborhood $\mathcal{U}$ of ϕ_0 in $\mathfrak{a}_\theta^*$ such that every $\psi \in \mathcal{U}$ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$ and has a critical gap at infinity for Γ .

As in Section 6, we can view $\partial \mathcal{Q}_\theta(\Gamma)$ as the graph of a convex function $\Phi : \mathfrak{a}_{\theta_0}^* \rightarrow \mathbb{R}$. Next fix an open set $\mathcal{D} \subset \mathfrak{a}_{\theta_0}^*$ such that $\{u + \Phi(u)\psi_0 : u \in \mathcal{D}\}$ is an open neighborhood of ϕ_0 in $\partial \mathcal{Q}_\theta(\Gamma)$ and is contained in $\mathcal{U}$.

For $\phi \in \mathfrak{a}_\theta^*$, let $\hat{f}_\phi : \Gamma_0 \setminus \mathcal{G} \rightarrow \mathbb{R}$ be the function in Section 5. By Proposition 6.1, Lemma 5.4, and Corollary 5.2, if $u \in \mathcal{D}$, Φ is differentiable at u , and $\phi := u + \Phi(u)\psi_0$, then

$$\left. \frac{d}{dt} \right|_{t=0} \Phi(u + t\psi) = - \frac{\int \hat{f}_\psi dm_\phi}{\int \hat{f}_{\psi_0} dm_\phi}$$

for all $\psi \in \mathfrak{a}_{\theta_0}^*$. Theorem 7.1 implies that the right-hand side depends continuously on u . Since Φ is convex, its derivative exists almost everywhere and agrees there with a continuous function. A standard convexity argument therefore implies that Φ is differentiable everywhere on $\mathcal{D}$.

It remains to prove the strict-convexity assertion when Γ is Zariski dense. Choose a convex neighborhood $\mathcal{V}$ of ϕ_0 whose closure is contained in $\mathcal{U}$, and restrict the boundary patch to $\mathcal{V}$. Suppose $\phi_1 \neq \phi_2$ lie in this patch, let $t \in (0, 1)$, and set $\phi_t := t\phi_1 + (1-t)\phi_2$. Since $\phi_t \in \mathcal{U}$, Theorem 4.3 implies that (Γ, ϕ_t) is of divergence type. Hence [CZZ24, Corollary 13.2] gives $\delta^{\phi_t}(\Gamma) < 1$. The critical exponent is continuous on $\mathcal{U}$, so ϕ_t lies in the interior of $\mathcal{Q}_\theta(\Gamma)$. Thus every nontrivial open segment between two points of the boundary patch lies in the interior, proving local strict convexity. $\square$

APPENDIX A. GPS SYSTEMS AND WEN'S FINITENESS RESULT

In this appendix, we recall the definition of a GPS system, state Wen's finiteness result for BMS measures [Wen26], and then explain why Wen's result applies in our context.

A.1. GPS systems and expanding cocycles. In [BCZZ24b], Blayac, Canary, Zhu, and the third named author introduced the notion of a Gromov–Patterson–Sullivan (GPS) system, consisting of a convergence group action, two coarse cocycles, and an associated Gromov product. We briefly recall the special case of continuous GPS systems.

Suppose that M is a compact metrizable space and $\Gamma < \text{Homeo}(M)$ is a non-elementary convergence group. Recall that a continuous cocycle $\sigma : \Gamma \times M \rightarrow \mathbb{R}$ is a continuous function satisfying

$$\sigma(\gamma_1\gamma_2, x) = \sigma(\gamma_1, \gamma_2x) + \sigma(\gamma_2, x) \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma \text{ and } x \in M.$$

Let $M^{(2)} := \{(x, y) \in M^2 : x \neq y\}$. Denoting by $\gamma^+, \gamma^- \in M$ the attracting and repelling fixed points, respectively, of a loxodromic $\gamma \in \Gamma$, a continuous cocycle σ is called *proper* if for any sequence $\{\gamma_n\} \subset \Gamma$ of distinct loxodromic elements such that $\{(\gamma_n^+, \gamma_n^-)\}$ is precompact in $M^{(2)}$, we have

$$\sigma(\gamma_n, \gamma_n^+) \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

We now recall the definition of continuous GPS systems from [BCZZ24b].

Definition A.1. A *continuous GPS system* consists of a non-elementary convergence group $\Gamma < \text{Homeo}(M)$, continuous proper cocycles $\sigma, \bar{\sigma} : \Gamma \times M \rightarrow \mathbb{R}$, and a continuous function $G : M^{(2)} \rightarrow \mathbb{R}$ such that

$$G(\gamma x, \gamma y) = G(x, y) + \bar{\sigma}(\gamma, x) + \sigma(\gamma, y) \quad \text{for all } \gamma \in \Gamma \text{ and } (x, y) \in M^{(2)}.$$

The function G is called a *Gromov product*.

Gerasimov [Ger12, Proposition 8.3.1] showed that M can be used to compactify Γ . In particular, $\Gamma \sqcup M$ has a unique metrizable compact topology such that $\Gamma, M \hookrightarrow \Gamma \sqcup M$ are embeddings and the induced Γ -action on $\Gamma \sqcup M$ is a convergence action (see also [BCZZ24b, Proposition 2.4]). A metric on $\Gamma \sqcup M$ generating this topology is called a *compatible metric*.

Given a cocycle $\sigma : \Gamma \times M \rightarrow \mathbb{R}$, a function $\|\cdot\|_\sigma : \Gamma \rightarrow \mathbb{R}$ is called a σ -*magnitude* if for any $\epsilon > 0$, there exists $C > 0$ such that for any $\gamma \in \Gamma$ and $x \in M$ with $d(\gamma^{-1}, x) > \epsilon$, we have

$$\|\gamma\|_\sigma - C \leq \sigma(\gamma, x) \leq \|\gamma\|_\sigma + C.$$

When a σ -magnitude exists for a proper cocycle σ , we call σ *expanding*. The cocycles in a GPS system are expanding [BCZZ24b, Proposition 3.4].

Given an expanding cocycle σ on M and $\delta \geq 0$, a probability measure μ on M is a δ -dimensional Patterson–Sullivan measure for σ if, for every $\gamma \in \Gamma$, the measure $\gamma_*\mu$ is absolutely continuous with respect to μ and

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-\delta\sigma(\gamma^{-1}, x)} \quad \mu\text{-a.e.}$$

The *critical exponent* associated to σ is given by

$$\delta^\sigma(\Gamma) := \inf \left\{ s > 0 : \sum_{\gamma \in \Gamma} e^{-s\|\gamma\|_\sigma} < +\infty \right\} \in [0, +\infty]$$

and σ is of *divergent type* if

$$\sum_{\gamma \in \Gamma} e^{-\delta^\sigma(\Gamma)\|\gamma\|_\sigma} = +\infty.$$

We will mainly consider a Patterson–Sullivan measure whose dimension is equal to the critical exponent of the cocycle.

One can also define an *entropy at infinity* $\delta_\infty^\sigma(\Gamma)$ as in Section 4.3 by replacing the exponent $\phi(\kappa(\gamma))$ with $\|\gamma\|_\sigma$.

A.2. BMS measures and Wen’s finiteness result. Given an expanding cocycle σ' on M , we can define a Γ -action on $M^{(2)} \times \mathbb{R}$ by

$$\gamma \cdot (x, y, s) = (\gamma x, \gamma y, s + \sigma'(\gamma, y)).$$

This action is proper [BCZZ24b, Proposition 10.2], and this space has a natural flow defined by

$$\psi^t(x, y, s) = (x, y, s + t)$$

which descends to a flow on the quotient $\Gamma \backslash (M^{(2)} \times \mathbb{R})$, also denoted by ψ^t .

Now suppose $(\sigma, \bar{\sigma}, G)$ is a GPS system such that $\delta^\sigma(\Gamma) = \delta^{\bar{\sigma}}(\Gamma) =: \delta$. Suppose also that $\mu, \bar{\mu}$ are Patterson–Sullivan measures of dimension δ for σ and $\bar{\sigma}$, respectively. Notice that the cocycle σ' defining the action of Γ on $M^{(2)} \times \mathbb{R}$ may differ from the cocycles in the GPS system.

The Radon measure $\tilde{m}_\sigma$ on $M^{(2)} \times \mathbb{R}$ defined by

$$d\tilde{m}_\sigma(x, y, t) := e^{\delta G(x, y)} d\bar{\mu}(x) d\mu(y) dt$$

is invariant under the Γ -action and the flow $\{\psi^t\}$. Since Γ acts properly on $M^{(2)} \times \mathbb{R}$, this measure descends to a flow-invariant measure m_σ on the quotient $\Gamma \backslash (M^{(2)} \times \mathbb{R})$ called a $(\sigma, \bar{\sigma}, G)$ -BMS measure.

Theorem A.2 ([Wen26, Theorems 4.8 and 4.9]). *With the notation above, suppose in addition that:*

- $\delta_\infty^\sigma(\Gamma) < \delta^\sigma(\Gamma) < +\infty$,
- For every $\epsilon > 0$ there exists $C_\epsilon > 1$ such that for all $\gamma \in \Gamma$,

$$\|\gamma\|_{\sigma'} \leq C_\epsilon e^{\epsilon \|\gamma\|_\sigma}.$$

Then $\sigma, \bar{\sigma}$ are of divergent type and the $(\sigma, \bar{\sigma}, G)$ -BMS measure on $\Gamma \backslash (M^{(2)} \times \mathbb{R})$ is finite.

A.3. Proof of Theorem 4.3. We now explain why Theorem 4.3 is a consequence of Theorem A.2.

By [BCZZ24a, Proposition 10.3], the cocycle

$$\sigma(\gamma, x) := \phi \circ B_\theta(\rho(\gamma), \xi(x))$$

on $\Lambda_\Omega(\Gamma_0)$ is part of a continuous GPS system with magnitude $\|\gamma\|_\sigma := \phi(\kappa(\rho(\gamma)))$.

Since Γ_0 is projectively visible, the action of Γ_0 on $\Lambda_\Omega(\Gamma_0)$ is a convergence group action; see [CZZ26, Proposition 3.15]. By [Bra20, Lemma 3.2],

$$\sigma'(\gamma, x) := \beta_x(\gamma^{-1}o, o)$$

defines a continuous cocycle for the action of Γ_0 on $\Lambda_\Omega(\Gamma_0)$.

Theorem 4.3 is a consequence of Theorem A.2 and the next two lemmas.

Lemma A.3. *σ' is an expanding cocycle, and $\|\gamma\|_{\sigma'} := d_\Omega(o, \gamma o)$, $\gamma \in \Gamma_0$, is a σ' -magnitude.*

Proof. We first show that σ' is proper. Suppose not. Then there exists a sequence $\{\gamma_n\} \subset \Gamma_0$ of distinct loxodromic elements (in the convergence group sense) such that two sequences $\{\gamma_n^+\}$ and $\{\gamma_n^-\}$ converge to two distinct points in $\Lambda_\Omega(\Gamma_0)$ as $n \rightarrow \infty$ while we have $\sup_{n \in \mathbb{N}} \beta_{\gamma_n^+}(o, \gamma_n o) < +\infty$.

On the other hand, there exists $R > 0$ such that $d_\Omega(o, (\gamma_n^+, \gamma_n^-)) < R$ for all $n \in \mathbb{N}$. Since each (γ_n^+, γ_n^-) is invariant under γ_n , we also have that $d_\Omega(\gamma_n o, (\gamma_n^+, \gamma_n^-)) < R$ for all $n \in \mathbb{N}$. This implies

$$\left| \beta_{\gamma_n^+}(o, \gamma_n o) - d_\Omega(o, \gamma_n o) \right| < 2R \quad \text{for all } n \in \mathbb{N}.$$

Hence, the sequence $\{\gamma_n o\} \subset \Omega$ is bounded, which is a contradiction. This shows the properness of σ' .

Next we show that $\|\gamma\|_{\sigma'} := d_\Omega(o, \gamma o)$ defines a σ' -magnitude. Fix a compatible metric d on $\Gamma_0 \sqcup \Lambda_\Omega(\Gamma_0)$. Suppose, to the contrary, that there exist $\epsilon > 0$ and sequences $\{\gamma_n\} \subset \Gamma_0$ and $\{x_n\} \subset \Lambda_\Omega(\Gamma_0)$ with $d(\gamma_n^{-1}, x_n) > \epsilon$ such that $|\beta_{x_n}(\gamma_n^{-1}o, o) - d_\Omega(o, \gamma_n o)| > n$ for all $n \in \mathbb{N}$.

After passing to a subsequence, we may assume that $x_n \rightarrow x \in \Lambda_\Omega(\Gamma_0)$ and $\gamma_n^{-1}o \rightarrow y \in \Lambda_\Omega(\Gamma_0)$ as $n \rightarrow \infty$. Then $x \neq y$ by hypothesis. Since Γ_0 is projectively visible, $(x, y) \subset \Omega$, and hence $\sup_{n \in \mathbb{N}} d_\Omega(o, [\gamma_n^{-1}o, x_n]) < +\infty$. This implies that $\sup_{n \in \mathbb{N}} |\beta_{x_n}(\gamma_n^{-1}o, o) - d_\Omega(\gamma_n^{-1}o, o)| < +\infty$, which is a contradiction. This finishes the proof. $\square$

Lemma A.4. *For every $\epsilon > 0$ there exists $C_\epsilon > 1$ such that for all $\gamma \in \Gamma_0$,*

$$\|\gamma\|_{\sigma'} \leq C_\epsilon e^{\epsilon \|\gamma\|_\sigma}.$$

Proof. Since ϕ is positive on $\mathcal{L}_\theta(\Gamma) \setminus \{0\}$, there exists $c_1 > 0$ such that

$$\|\gamma\|_\sigma = \phi(\kappa_\theta(\rho(\gamma))) \geq c_1^{-1} \|\kappa_\theta(\rho(\gamma))\| - c_1$$

for all $\gamma \in \Gamma_0$. Then the “moreover” part of Theorem 3.4 implies the desired estimate. $\square$

REFERENCES

- [BCLS15] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino. The pressure metric for Anosov representations. *Geom. Funct. Anal.*, 25(4):1089–1179, 2015.

- [BCZZ24a] Pierre-Louis Blayac, Richard Canary, Feng Zhu, and Andrew Zimmer. Counting, mixing and equidistribution for GPS systems with applications to relatively Anosov groups. *arXiv e-prints*, page arXiv:2404.09718, April 2024.
- [BCZZ24b] Pierre-Louis Blayac, Richard Canary, Feng Zhu, and Andrew Zimmer. Patterson–Sullivan theory for coarse cocycles. *arXiv e-prints*, page arXiv:2404.09713, April 2024.
- [Ben97] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.*, 7(1):1–47, 1997.
- [Bra20] Harrison Bray. Ergodicity of Bowen-Margulis measure for the Benoist 3-manifolds. *J. Mod. Dyn.*, 16:305–329, 2020.
- [Bur93] Marc Burger. Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank 2. *Internat. Math. Res. Notices*, (7):217–225, 1993.
- [BZ23] Pierre-Louis Blayac and Feng Zhu. Ergodicity and equidistribution in Hilbert geometry. *J. Mod. Dyn.*, 19:879–945, 2023.
- [CT25] Stephen Cantrell and Ryokichi Tanaka. The Manhattan curve, ergodic theory of topological flows and rigidity. *Geom. Topol.*, 29(4):1851–1907, 2025.
- [CZZ24] Richard Canary, Tengren Zhang, and Andrew Zimmer. Patterson-Sullivan measures for transverse subgroups. *J. Mod. Dyn.*, 20:319–377, 2024.
- [CZZ25] Richard Canary, Tengren Zhang, and Andrew Zimmer. Patterson-Sullivan measures for relatively Anosov groups. *Math. Ann.*, 392(2):2309–2363, 2025.
- [CZZ26] Richard Canary, Tengren Zhang, and Andrew Zimmer. Entropy rigidity for cusped Hitchin representations. *J. Topol.*, 19(1):Paper No. e70064, 2026.
- [DGK24] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Convex cocompact actions in real projective geometry. *Ann. Sci. Éc. Norm. Supér. (4)*, 57(6):1753–1843, 2024.
- [Ger12] Victor Gerasimov. Floyd maps for relatively hyperbolic groups. *Geom. Funct. Anal.*, 22(5):1361–1399, 2012.
- [KO25] Dongryul M. Kim and Hee Oh. Relatively Anosov groups: finiteness, measure of maximal entropy, and reparameterization. *J. Reine Angew. Math.*, 826:91–142, 2025.
- [KOP26] M. Dongryul Kim, Hee Oh, and Wenyu Pan. Local mixing for cusped positive representations. *Preprint*, 2026.
- [KOW25] Dongryul M. Kim, Hee Oh, and Yahui Wang. Properly discontinuous actions, growth indicators, and conformal measures for transverse subgroups. *Math. Ann.*, 393(2):2391–2450, 2025.
- [KZ26] Dongryul M. Kim and Andrew Zimmer. Vector-valued horofunction boundaries and Patterson–Sullivan measures. *arXiv e-prints*, page arXiv:2603.28693, March 2026.
- [PS17] R. Potrie and A. Sambarino. Eigenvalues and entropy of a Hitchin representation. *Invent. Math.*, 209(3):885–925, 2017.
- [PS18] Vincent Pit and Barbara Schapira. Finiteness of Gibbs measures on noncompact manifolds with pinched negative curvature. *Ann. Inst. Fourier (Grenoble)*, 68(2):457–510, 2018.
- [Qui02a] J.-F. Quint. Mesures de Patterson-Sullivan en rang supérieur. *Geom. Funct. Anal.*, 12(4):776–809, 2002.
- [Qui02b] Jean-François Quint. Divergence exponentielle des sous-groupes discrets en rang supérieur. *Comment. Math. Helv.*, 77(3):563–608, 2002.
- [Qui03] Jean-François Quint. L’indicateur de croissance des groupes de Schottky. *Ergodic Theory Dynam. Systems*, 23(1):249–272, 2003.
- [RW26] Eduardo Reyes and Tianqi Wang. Regularity of Manhattan manifolds and exact dimensionality for relatively Anosov groups. *Preprint*, 2026.
- [Sam14] A. Sambarino. Hyperconvex representations and exponential growth. *Ergodic Theory Dynam. Systems*, 34(3):986–1010, 2014.
- [Sam24] Andrés Sambarino. A report on an ergodic dichotomy. *Ergodic Theory Dynam. Systems*, 44(1):236–289, 2024.
- [Tho17] Nicolas Tholozan. Volume entropy of Hilbert metrics and length spectrum of Hitchin representations into $\mathrm{PSL}(3, \mathbb{R})$. *Duke Math. J.*, 166(7):1377–1403, 2017.
- [Wen26] Rou Wen. Finiteness of Bowen-Margulis-Sullivan Measure for Gromov-Patterson-Sullivan Systems. *arXiv e-prints*, page arXiv:2604.03982, April 2026.
- [ZZ22] Feng Zhu and Andrew Zimmer. Relatively Anosov representations via flows I: theory. *arXiv preprint arXiv:2207.14737*, 2022. To appear in Groups Geom. Dyn.

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