RELATIVELY ANOSOV GROUPS: FINITENESS, MEASURE OF MAXIMAL ENTROPY, AND REPARAMETERIZATION

DONGRYUL M. KIM AND HEE OH

Abstract. For a geometrically finite Kleinian group Γ, the Bowen-Margulis-Sullivan measure is finite and is the unique measure of maximal entropy for the geodesic flow, as shown by Sullivan and Otal-Peigné respectively. Moreover, it is strongly mixing by a result of Babillot. We obtain a higher-rank analogue of this theorem. Given a relatively Anosov subgroup Γ of a semisimple real algebraic group, there is a family of flow spaces parameterized by linear forms tangent to the growth indicator. We construct a reparameterization of each flow space by the geodesic flow on the Groves-Manning space of Γ which exhibits exponential expansion along unstable foliations. Using this reparameterization, we prove that the Bowen-Margulis-Sullivan measure of each flow space is finite and is the unique measure of maximal entropy. Moreover, it is strongly mixing.

Contents

1. Introduction

For a geometrically finite Kleinian group Γ of $\mathrm{SO}^\circ(n,1) = \mathrm{Isom}^+(\mathbb{H}^n)$, $n \geq 2$, it is a classical result of Sullivan ([29], see also [13]) that the associated Bowen-Margulis-Sullivan measure m^{BMS} on the unit tangent bundle $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ is finite, and the measure-theoretic entropy of the geodesic flow

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with respect to m^{BMS} equals the topological entropy. Hence the Bowen-Margulis-Sullivan measure is the measure of maximal entropy. Moreover, Otal-Peigné [24] showed that this measure is the unique measure of maximal entropy. It is also strongly mixing by a theorem of Babillot [1].

In this paper, we obtain higher-rank analogues of these theorems. Let G be a connected semisimple real algebraic group. Anosov subgroups and relatively Anosov subgroups of G are regarded as higher-rank generalizations of convex cocompact and geometrically finite rank-one groups, respectively. There is an even broader class of discrete subgroups called transverse subgroups, which are viewed as generalizations of rank-one discrete subgroups. For a transverse subgroup Γ, we have a family of Bowen-Margulis-Sullivan measures $\mathsf{m}_\psi^{\text{BMS}}$ parameterized by a distinguished collection of linear forms ψ. Each such measure $\mathsf{m}_\psi^{\text{BMS}}$ lives on a fibered dynamical system over a canonical one-dimensional base flow space $(\Omega_\psi, m_\psi, \phi_t)$ where the fiber is the kernel of ψ and $\mathsf{m}_\psi^{\text{BMS}}$ is equal to the product measure $m_\psi \otimes \text{Leb}_{\ker \psi}$. We refer to m_ψ as the base Bowen-Margulis-Sullivan measure on Ω_ψ.

We prove that if Γ is a relatively Anosov subgroup, then the base BMS measure m_ψ is finite and is the unique measure of maximal entropy for the flow $\{\phi_t\}$. Moreover, we show that for any transverse subgroup for which m_ψ is finite, the dynamical system $(\Omega_\psi, m_\psi, \phi_t)$ is strongly mixing. In particular, both entropy-maximization and strong mixing holds for $(\Omega_\psi, m_\psi, \phi_t)$ associated with relatively Anosov subgroups.

To formulate these results precisely, we fix a Cartan decomposition $G = KA^+K$, where K is a maximal compact subgroup of G and $A^+ = \exp \mathfrak{a}^+$ is a positive Weyl chamber of a maximal split torus A of G. We denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection defined by the condition $g \in K \exp \mu(g) K$ for $g \in G$. Let Π be the set of all simple roots for $(\operatorname{Lie} G, \mathfrak{a}^+)$. Given a non-empty subset $\theta \subset \Pi$, there is the notion of relatively Anosov and transverse subgroup. Let $\mathcal{F}_\theta = G/P_\theta$ where P_θ is the standard parabolic subgroup associated with θ. Let $\Gamma < G$ be a discrete subgroup and let Λ_θ denote the limit set of Γ in $\mathcal{F}_\theta$ as defined in (2.1), which we assume contains at least 3 points, that is, Γ is non-elementary. In the rest of the introduction, we assume that Γ is a *θ-transverse* (or simply, transverse) subgroup. This means that Γ satisfies

- *regularity*: $\liminf_{\gamma \in \Gamma} \alpha(\mu(\gamma)) = \infty$ for all $\alpha \in \theta$;
- *antipodality*: any $\xi \neq \eta \in \Lambda_{\theta \cup \mathrm{i}(\theta)}$ are in general position (see (2.3)).

Here $\mathrm{i} = -\operatorname{Ad}_{w_0} : \Pi \to \Pi$ denotes the opposition involution where w_0 is the longest Weyl element.

Fibered dynamical systems. Let $\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha$ and $A_\theta = \exp \mathfrak{a}_\theta$. The centralizer of A_θ is a Levi subgroup of P_θ which is a direct product $A_\theta S_\theta$ where S_θ is a compact central extension of a semisimple algebraic subgroup. The right translation action of A_θ on the quotient space G/S_θ is equivariantly conjugate to the $\mathfrak{a}_\theta$-translation action on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ where $\mathcal{F}_\theta^{(2)}$

consists of all pairs $(\xi,\eta)\in\mathcal{F}_\theta\times\mathcal{F}_{\mathrm{i}(\theta)}$ in general position. The left Γ-action on G/S_θ is not properly discontinuous in general. On the other hand, if we set $\Lambda_\theta^{(2)}=(\Lambda_\theta\times\Lambda_{\mathrm{i}(\theta)})\cap\mathcal{F}_\theta^{(2)}$, then it is shown in [18, Theorem 9.1] that Γ acts properly discontinuously on the following space:

$$\tilde\Omega_\Gamma:=\Lambda_\theta^{(2)}\times\mathfrak{a}_\theta\simeq\{gS_\theta\in G/S_\theta: gP_\theta\in\Lambda_\theta, gw_0P_{\mathrm{i}(\theta)}\in\Lambda_{\mathrm{i}(\theta)}\}.$$

Hence

$$\Omega_\Gamma:=\Gamma\backslash\tilde\Omega_\Gamma.$$

is a second countable locally compact Hausdorff space on which $\mathfrak{a}_\theta$ acts by translations. Moreover, for each (Γ,θ)-proper[1] linear form $\psi\in\mathfrak{a}_\theta^*$, the space Ω_Γ fibers over a one-dimensional flow space $\Omega_\psi:=\Gamma\backslash(\Lambda_\theta^{(2)}\times\mathbb{R})$.

More precisely, via the projection $(\xi,\eta,v)\mapsto(\xi,\eta,\psi(v))$, the Γ-action on $\tilde\Omega_\Gamma$ descends to a proper discontinuous action on $\tilde\Omega_\psi:=\Lambda_\theta^{(2)}\times\mathbb{R}$ [18, Theorem 9.2]. Therefore $\Omega_\psi:=\Gamma\backslash\tilde\Omega_\psi$ is a second countable locally compact Hausdorff space over which Ω_Γ is a trivial $\ker\psi$-bundle:

$$\begin{array}{c}(\Omega_\Gamma,\mathfrak{a}_\theta)\simeq\Omega_\psi\times\ker\psi\\ \big\downarrow\\ (\Omega_\psi,\mathbb{R})\end{array}$$

The translation flow $\phi_t(\xi,\eta,s)=(\xi,\eta,s+t)$ on $\tilde\Omega_\psi=\Lambda_\theta^{(2)}\times\mathbb{R}$ descends to a translation flow on Ω_ψ which we also denote by $\{\phi_t\}$ by abuse of notation. The (Γ,θ)-properness of $\psi\in\mathfrak{a}_\theta^*$ is crucial for the proper discontinuity of the Γ-action on $\tilde\Omega_\psi$. See Remark 3.2 for examples.

For a pair of a (Γ,ψ)-Patterson-Sullivan measure ν on Λ_θ and a $(\Gamma,\psi\circ\mathrm{i})$-Patterson-Sullivan measure ν_{i} on $\Lambda_{\mathrm{i}(\theta)}$, we denote by $\mathsf{m}_\psi^{\mathrm{BMS}}=\mathsf{m}_{\nu,\nu_{\mathrm{i}}}^{\mathrm{BMS}}$ the associated A_θ-invariant Bowen-Margulis-Sullivan measure on Ω_Γ, locally equivalent to the product $\nu\otimes\nu_{\mathrm{i}}\otimes\mathrm{Leb}_{\mathfrak{a}_\theta}$. Similarly, we denote by m_ψ the associated $\{\phi_t\}$-invariant *Bowen-Margulis-Sullivan measure* on Ω_ψ, locally equivalent to the product $\nu\otimes\nu_{\mathrm{i}}\otimes\mathrm{Leb}_{\mathbb{R}}$. Then $\mathsf{m}_\psi^{\mathrm{BMS}}=m_\psi\otimes\mathrm{Leb}_{\ker\psi}$. As we are not assuming the uniqueness of ν and ν_{i} for a given ψ, $\mathsf{m}_\psi^{\mathrm{BMS}}$ and m_ψ are not necessarily determined by ψ. Nevertheless, it is convenient to refer to them as BMS measures associated to ψ.

Relatively Anosov groups. A transverse subgroup $\Gamma<G$ is called *relatively Anosov* (more precisely relatively θ-Anosov) if Γ is a relatively hyperbolic group and there exists a Γ-equivariant homeomorphism between the Bowditch boundary of Γ and the limit set Λ_θ. When Γ is hyperbolic, its Bowditch boundary is the Gromov boundary of Γ, and in this case, the relatively Anosov subgroup Γ is simply an Anosov subgroup. When G has rank-one, relatively Anosov subgroups coincide with geometrically finite Kleinian groups. Recall that for a geometrically finite Kleinian group

[1] ψ is called (Γ,θ)-proper if $\psi\circ\mu:\Gamma\to[-\varepsilon,\infty)$ is a proper map for some $\varepsilon>0$.

Γ, there exists a unique Patterson-Sullivan measure of dimension equal to the critical exponent δ_Γ. In higher-rank, we consider the growth indicator ψ_Γ^θ of Γ, a generalization of the critical exponent (see (2.7) for the definition). A linear form ψ is said to be *tangent* to ψ_Γ^θ if $\psi \geq \psi_\Gamma^\theta$ and equality holds at some non-zero $u \in \mathfrak{a}_\theta$. For a relatively Anosov subgroup Γ and a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$ tangent to ψ_Γ^θ, there exists a unique (Γ, ψ)-Patterson-Sullivan measure on Λ_θ, and hence a unique BMS measure m_ψ associated with ψ (see [21], [28] for Anosov groups and [11] for relatively Anosov groups).

For Anosov subgroups, the associated base space Ω_ψ is known to be homeomorphic to the Gromov geodesic flow space and is compact ([12], [7], [28]). In fact, for a transverse subgroup, Γ is Anosov if and only if Ω_ψ is compact [18]. In particular, Ω_ψ is non-compact for relatively Anosov subgroups that are not Anosov. Analogous to the classical result on the finitness of the Bowen-Margulis-Sullivan measure for a geometrically finite Kleinian group, we prove the following:

Theorem 1.1 (Finiteness and mixing)**.** *Let Γ be a relatively Anosov subgroup of G. For any (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$ tangent to the growth indicator of Γ, the BMS measure m_ψ is finite:*

$$|m_\psi| < \infty.$$

Moreover, the system $(\Omega_\psi, m_\psi, \phi_t)$ is strongly mixing.

In fact, we establish strong mixing in a broader setting of transverse subgroups, which can be regarded as a higher-rank analogue of Babillot's mixing theorem (see Theorem 4.1).

Given the finiteness of m_ψ, the metric entropy $h_{m_\psi}(\{\phi_t\})$ of the normalized measure $m_\psi/|m_\psi|$ is well-defined. For a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$, the associated ψ-critical exponent is given by

$$\delta_\psi = \limsup_{T \to \infty} \frac{\log \#\{\gamma \in \Gamma : \psi(\mu(\gamma)) < T\}}{T} \in (0, \infty)$$

and one has $\delta_\psi = 1$ if and only if ψ is tangent to ψ_Γ^θ ([11, Theorem 10.1], [18, Theorem 4.5]).

Theorem 1.2 (Unique measure of maximal entropy)**.** *Let Γ be a relatively Anosov subgroup of G. For any (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$ tangent to the growth indicator of Γ,*

$$m_\psi \text{ is the unique measure of maximal entropy for } (\Omega_\psi, \{\phi_t\})$$

and the entropy $h_{m_\psi}(\{\phi_t\})$ is equal to $\delta_\psi = 1$.

For Anosov subgroups, this theorem is due to Sambarino ([27], [28]), as a consequence of thermodynamic formalism. Our proof, by constrast, does not use the thermodynamic formalism and thus provides an alternative argument even in the Anosov case.

Remark 1.3. The identity $\delta_\psi = 1$ follows from the normalization that ψ is tangent to ψ_Γ^θ. In rank-one, ϕ_t corresponds to the time-changed geodesic flow g_{t/δ_Γ} and m_{δ_Γ} is the unique measure of maximal entropy for g_t, satisfying $h_{m_{\delta_\Gamma}}(\{g_t\}) = \delta_\Gamma$. Hence $h_{m_{\delta_\Gamma}}(\{\phi_t\}) = h_{m_{\delta_\Gamma}}(\{g_t\})/\delta_\Gamma = 1$.

A key technical ingredient of Theorems 1.1 and 1.2 is the following coarse reparameterization theorem, which is also of independent interest. Let (X_{GM}, d_{GM}) denote the Groves-Manning cusp space of Γ and let $\mathcal{G}$ denote the space of all parameterized bi-infinite geodesics in the Groves-Manning cusp space [15]. Define the geodesic flow $\varphi_s : \mathcal{G} \to \mathcal{G}$ by $(\varphi_s\sigma)(\cdot) = \sigma(\cdot + s)$.

Theorem 1.4 (Reparameterization)**.** *There exists a continuous, surjective, proper Γ-equivariant map*

$$\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$$

together with a continuous cocycle $\tilde{\mathsf{t}} : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ such that for all $\sigma \in \mathcal{G}$ and $s \in \mathbb{R}$,

1. $\tilde{\Psi}(\varphi_s\sigma) = \phi_{\tilde{\mathsf{t}}(\sigma,s)}\tilde{\Psi}(\sigma)$;
2. $\tilde{\mathsf{t}}(\sigma, s) = -\tilde{\mathsf{t}}(\varphi_s\sigma, -s)$;
3. *there exists an absolute constant $B > 0$ such that*
$$a|s| - B \le \tilde{\mathsf{t}}(\sigma, |s|) \le a'|s| + B$$
where
$$0 < a := \liminf_{\gamma\in\Gamma} \frac{\psi(\mu(\gamma))}{d_{GM}(e, \gamma)} \quad \textit{and} \quad a' := 3 \limsup_{\gamma\in\Gamma} \frac{\psi(\mu(\gamma))}{d_{GM}(e, \gamma)} < \infty;$$
4. *all fibers $\{\sigma(0) \in X_{GM} : \sigma \in \tilde{\Psi}^{-1}(x)\}$, $x \in \tilde{\Omega}_\psi$, have uniformly bounded diameter.*

Moreover, the flow ϕ_t is exponentially expanding along unstable foliations of $\tilde{\Omega}_\psi = \Lambda_\theta^{(2)} \times \mathbb{R}$, as described in Theorem 8.1.

The map $\Psi : \Gamma\backslash\mathcal{G} \to \Omega_\psi$, induced from $\tilde{\Psi}$, provides a thick-thin decomposition of Ω_ψ that plays a crucial role in the proof of the finiteness of m_ψ (Theorem 1.1). This decomposition is used in conjunction with the work of Canary-Zhang-Zimmer [11], which analyzes the critical exponents of peripheral subgroups of Γ. The exponentially expanding property of ϕ_t is essential in constructing a measurable partition of $\tilde{\Omega}_\psi$ subordinated to unstable foliations (Proposition 10.2), a key step in the proof of Theorem 1.2 concerning the uniqueness of the measure of maximal entropy.

Remark 1.5. Recently, Blayac-Canary-Zhu-Zimmer [4] showed that for θ-transverse Γ and $\psi \in \mathfrak{a}_\theta^*$, if there exists a (Γ, θ)-Patterson-Sullivan measure on Λ_θ, then ψ must be (Γ, θ)-proper. This result implies that the (Γ, θ)-properness condition is not a genuinely restrictive assumption when studying dynamics associated to Bowen-Margulis-Sullivan measures.

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2. Preliminaries

We review some basic facts about Lie groups, following [18, Section. 2] which we refer for more details. Throughout the paper, let G be a connected semisimple real algebraic group. Let $P < G$ be a minimal parabolic subgroup with a fixed Langlands decomposition $P = MAN$ where A is a maximal real split torus of G, M is the maximal compact subgroup of P commuting with A and N is the unipotent radical of P. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebras of G and A. Fix a positive Weyl chamber $\mathfrak{a}^+ < \mathfrak{a}$ so that $\log N$ consists of positive root subspaces and set $A^+ = \exp \mathfrak{a}^+$. We fix a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. We denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection defined by the condition $g \in K \exp \mu(g) K$ for $g \in G$. Let $X = G/K$ be the associated Riemannian symmetric space and $o = [K] \in X$. Fix a K-invariant norm $\| \cdot \|$ on $\mathfrak{g}$. This induces the left G-invariant Riemannian metric d on X.

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$ denote the set of all roots, $\Phi^+ \subset \Phi$ the set of all positive roots, and $\Pi \subset \Phi^+$ the set of all simple roots. Fix a Weyl element $w_0 \in K$ of order 2 in the normalizer of A representing the longest Weyl element so that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The map

$$\mathrm{i} = -\mathrm{Ad}_{w_0} : \mathfrak{a} \to \mathfrak{a}$$

is called the opposition involution. It induces an involution $\Phi \to \Phi$ preserving Π, for which we use the same notation i, such that $\mathrm{i}(\alpha) \circ \mathrm{Ad}_{w_0} = -\alpha$ for all $\alpha \in \Phi$.

Henceforth, we fix a non-empty subset $\theta \subset \Pi$. Let P_θ denote a standard parabolic subgroup of G corresponding to θ; that is, P_θ is generated by MA and all root subgroups U_α, where α ranges over all positive roots which are not $\mathbb{Z}$-linear combinations of $\Pi - \theta$. Hence $P_\Pi = P$. Let

$$\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha, \qquad \mathfrak{a}_\theta^+ = \mathfrak{a}_\theta \cap \mathfrak{a}^+,$$

$$A_\theta = \exp \mathfrak{a}_\theta, \quad \text{and} \quad A_\theta^+ = \exp \mathfrak{a}_\theta^+.$$

Let $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ denote the projection invariant under all Weyl elements fixing $\mathfrak{a}_\theta$ pointwise. We write $\mu_\theta := p_\theta \circ \mu : G \to \mathfrak{a}_\theta^+$. The space $\mathfrak{a}_\theta^* = \mathrm{Hom}(\mathfrak{a}_\theta, \mathbb{R})$ can be identified with the subspace of $\mathfrak{a}^*$ which is p_θ-invariant: $\mathfrak{a}_\theta^* = \{\psi \in \mathfrak{a}^* : \psi \circ p_\theta = \psi\}$. We have the Levi-decomposition $P_\theta = L_\theta N_\theta$ where L_θ is the centralizer of A_θ and $N_\theta = R_u(P_\theta)$ is the unipotent radical of P_θ. We set $M_\theta = K \cap P_\theta \subset L_\theta$.

Limit set Λ_θ. We set

$$\mathcal{F}_\theta = G/P_\theta.$$

The subgroup K acts transitively on $\mathcal{F}_\theta$, and hence $\mathcal{F}_\theta \simeq K/M_\theta$.

Definition 2.1. For a sequence $g_i \in G$ and $\xi \in \mathcal{F}_\theta$, we write $\lim_{i\to\infty} g_i = \xi$ and say g_i *converges* to ξ if

- for each $\alpha \in \theta$, $\alpha(\mu(g_i)) \to \infty$ as $g_i \to \infty$;
- $\lim_{i\to\infty} \kappa_i \xi_\theta = \xi$ in $\mathcal{F}_\theta$ for some $\kappa_i \in K$ such that $g_i \in \kappa_i A^+ K$.

The *θ-limit set* of a discrete subgroup Γ can be defined as follows:

$$\Lambda_\theta = \Lambda_\theta(\Gamma) := \{\lim \gamma_i \in \mathcal{F}_\theta : \gamma_i \in \Gamma\} \tag{2.1}$$

where $\lim \gamma_i$ is defined as in Definition 2.1. If Γ is Zariski dense, this is the unique Γ-minimal subset of $\mathcal{F}_\theta$ ([2], [26]).

Jordan projections. Any $g \in G$ can be written as the commuting product $g = g_h g_e g_u$ where g_h is hyperbolic, g_e is elliptic and g_u is unipotent. The hyperbolic component g_h is conjugate to a unique element $\exp \lambda(g) \in A^+$ and $\lambda(g)$ is called the *Jordan projection* of g. We write $\lambda_\theta := p_\theta \circ \lambda$.

Theorem 2.2. [3] *For any Zariski dense subgroup $\Gamma < G$, the subgroup generated by $\{\lambda(\gamma) \in \mathfrak{a}^+ : \gamma \in \Gamma\}$ is dense in $\mathfrak{a}$.*

Busemann map and Gromov product. The *$\mathfrak{a}$-valued Busemann map* $\beta : \mathcal{F}_\Pi \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi)$$

where $\sigma(g^{-1}, \xi) \in \mathfrak{a}$ is the unique element such that $g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N$ for any $k \in K$ with $\xi = kP$. For $(\xi, g, h) \in \mathcal{F}_\theta \times G \times G$, we define

$$\beta^\theta_\xi(g, h) := p_\theta(\beta_{\xi_0}(g, h)) \tag{2.2}$$

for any $\xi_0 \in \mathcal{F}_\Pi$ projecting to ξ. This is well-defined independent of the choice of ξ_0 [26, Lemma 6.1]. Moreover, since product map $K \times A \times N \to G$ is a diffeomorphism, Busemann maps are continuous.

Two points $\xi \in \mathcal{F}_\theta$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ are said to be *in general position* if

$$\xi = gP_\theta \text{ and } \eta = gw_0 P_{\mathrm{i}(\theta)} \text{ for some } g \in G. \tag{2.3}$$

We set

$$\mathcal{F}^{(2)}_\theta = \{(\xi, \eta) \in \mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)} : \xi, \eta \text{ are in general position}\} \tag{2.4}$$

which is the unique open G-orbit in $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)}$ under the diagonal G-action.

For $(\xi, \eta) \in \mathcal{F}^{(2)}_\theta$, we define the *$\mathfrak{a}_\theta$-valued Gromov product* as

$$\langle \xi, \eta \rangle = \beta^\theta_\xi(e, g) + \mathrm{i}(\beta^{\mathrm{i}(\theta)}_\eta(e, g)) \tag{2.5}$$

where $g \in G$ satisfies $(gP_\theta, gw_0 P_{\mathrm{i}(\theta)}) = (\xi, \eta)$. This does not depend on the choice of g [18, Lemma 9.13].

Patterson-Sullivan measures. For $\psi \in \mathfrak{a}^*_\theta$, a (Γ, ψ)-*conformal measure* is a Borel probability measure on $\mathcal{F}_\theta$ such that

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{\psi(\beta^\theta_\xi(e, \gamma))} \quad \text{for all } \gamma \in \Gamma \text{ and } \xi \in \mathcal{F}_\theta \tag{2.6}$$

where $\gamma_* \nu(D) = \nu(\gamma^{-1} D)$ for any Borel subset $D \subset \mathcal{F}_\theta$ and β^θ_ξ denotes the $\mathfrak{a}_\theta$-valued Busemann map defined in (2.2). A (Γ, ψ)-conformal measure supported on Λ_θ is called a (Γ, ψ)-*Patterson Sullivan measure.*

Growth indicator. Let $\Gamma < G$ be a θ-discrete subgroup, that is, $\mu_\theta|_\Gamma$ is a proper map. The *θ-growth indicator* $\psi_\Gamma^\theta : \mathfrak{a}_\theta \to [-\infty, \infty)$ is a higher-rank version of the critical exponent, which is defined as follows: If $u \in \mathfrak{a}_\theta$ is non-zero,

$$\psi_\Gamma^\theta(u) = \|u\| \inf_{u \in \mathcal{C}} \tau_{\mathcal{C}}^\theta \tag{2.7}$$

where $\tau_{\mathcal{C}}^\theta$ is the abscissa of convergence of the series $\sum_{\gamma \in \Gamma, \mu_\theta(\gamma) \in \mathcal{C}} e^{-s\|\mu_\theta(\gamma)\|}$ and $\mathcal{C} \subset \mathfrak{a}_\theta$ ranges over all open cones containing u. Set $\psi_\Gamma^\theta(0) = 0$. This definition was given in [18], extending Quint's growth indicator [25] to a general θ.

For Γ transverse and ψ (Γ, θ)-proper, it is proved in [18] that if there exists a (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$, then

$$\psi \geq \psi_\Gamma^\theta.$$

We say that $\psi \in \mathfrak{a}_\theta^*$ is *tangent* to ψ_Γ^θ if $\psi \geq \psi_\Gamma^\theta$ and $\psi(u) = \psi_\Gamma^\theta(u)$ for some $u \in \mathfrak{a}_\theta - \{0\}$. In the rank-one case, if δ_Γ is the critical exponent of the Poincaré series $\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)}$ and $v \in \mathfrak{a}^+$ is the unique vector with $d(o, \exp vo) = 1$, then ψ_Γ^Π on $\mathfrak{a}^+ = \mathbb{R}_+ v$ is given by $\psi_\Gamma^\Pi(tv) = \delta_\Gamma t$. As ψ_Γ^Π itself is the restriction of a linear form to $\mathfrak{a}^+$, it is the unique linear form tangent to itself. In higher-rank, ψ_Γ^θ is typically non-linear but concave and there are abundant tangent linear forms in general. As in the rank-one setting, interesting geometry and dynamics occur for tangent linear forms.

3. Vector bundle structure of the non-wandering set Ω_Γ

We fix a non-empty subset θ of Π. In this section, we assume that $\Gamma < G$ is a non-elementary *θ-transverse* subgroup, that is, Γ satisfies

- (non-elementary): $\#\Lambda_\theta \geq 3$;
- (regularity): $\liminf_{\gamma \in \Gamma} \alpha(\mu(\gamma)) = \infty$ for all $\alpha \in \theta$; and
- (antipodality): any two distinct $\xi, \eta \in \Lambda_{\theta \cup \mathrm{i}(\theta)}$ are in general position as in (2.3).

We will define a locally compact Hausdorff space Ω_Γ which is the non-wandering set for the action of A_θ. Recall that the centralizer of A_θ is the direct product $A_\theta S_\theta$ where S_θ is a compact central extension of a connected semisimple real algebraic subgroup. Note that S_θ is compact if and only if $\theta = \Pi$.

The homogeneous space G/S_θ can be identified with the space $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ via the map

$$gS_\theta \mapsto (gP_\theta, gw_0 P_{\mathrm{i}(\theta)}, \beta_{gP_\theta}^\theta(e, g)),$$

recalling that $w_0 \in K$ is the longest Weyl element, and the left G-action on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ given by

$$g(\xi, \eta, v) = (g\xi, g\eta, v + \beta_\xi^\theta(g^{-1}, e))$$

makes the above identification G-equivariant. Since S_θ commutes with A_θ, the diagonal subgroup A_θ acts on G/S_θ on the right, and this action is conjugate to the action of $\mathfrak{a}_\theta$ on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ by the translation on the last component. Since S_θ is not compact in general, the action of Γ on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ is not properly discontinuous. However for Γ transverse, the Γ-action restricted to the subspace $\tilde{\Omega}_\Gamma := \Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ turns out to be properly discontinuous where $\Lambda_\theta^{(2)} = \mathcal{F}_\theta^{(2)} \cap (\Lambda_\theta \times \Lambda_{\mathrm{i}(\theta)})$ [18, Theorem 9.1]. Hence we obtain the locally compact second countable Hausdorff space

$$\Omega_\Gamma := \Gamma\backslash\tilde{\Omega}_\Gamma,$$

which is the non-wandering set for the right A_θ-action.

For each (Γ, θ)-proper form $\psi \in \mathfrak{a}_\theta^*$, Ω_Γ admits a $\ker\psi$-bundle structure over a non-wandering set Ω_ψ for a one-dimensional flow. More precisely,

Theorem 3.1. [18, Theorem 9.2] *The Γ-action on the space $\tilde{\Omega}_\psi := \Lambda_\theta^{(2)} \times \mathbb{R}$ given by*

$$\gamma(\xi, \eta, s) = (\gamma\xi, \gamma\eta, s + \psi(\beta_\xi^\theta(\gamma^{-1}, e)))$$

is properly discontinuous. Thus the space

$$\Omega_\psi := \Gamma\backslash\tilde{\Omega}_\psi = \Gamma\backslash(\Lambda_\theta^{(2)} \times \mathbb{R}) \tag{3.1}$$

is a locally compact second countable Hausdorff space equipped with the translation flow $\{\phi_t\}$ on the $\mathbb{R}$-component.

Remark 3.2. Any linear form which is positive on $\mathfrak{a}^+ \cap \mathfrak{a}_\theta - \{0\}$, e.g., any non-negative linear combination of the fundamental weights ω_α, $\alpha \in \theta$, is (Γ, θ)-proper. On the other hand, a linear form which takes negative values on some part of the θ-limit cone is never (Γ, θ)-proper (see [18]).

Explicitly, the *translation flow* $\{\phi_t\}$ is defined as follows: for $t \in \mathbb{R}$ and $(\xi, \eta, s) \in \tilde{\Omega}_\psi$,

$$\phi_t(\xi, \eta, s) = (\xi, \eta, s + t).$$

This flow $\{\phi_t\}$ on $\tilde{\Omega}_\psi$ commutes with the Γ-action, and hence induces the one-dimensional flow on Ω_ψ which we also denote by ϕ_t by abusing notations.

Consider the projection $\Omega_\Gamma \to \Omega_\psi$ induced by the Γ-equivariant projection $\tilde{\Omega}_\Gamma \to \tilde{\Omega}_\psi$ given by $(\xi, \eta, v) \mapsto (\xi, \eta, \psi(v))$. This is a principal $\ker\psi$-bundle, which is trivial since $\ker\psi$ is a vector space. It follows that there exists a $\ker\psi$-equivariant homeomorphism between Ω_Γ and $\Omega_\psi \times \ker\psi$.

$$\begin{array}{c} \Omega_\Gamma \simeq \Omega_\psi \times \ker\psi \\ \big\downarrow \\ \Omega_\psi \end{array}$$

Let ν and ν_{i} be a pair of (Γ, ψ) and $(\Gamma, \psi \circ \mathrm{i})$-Patterson-Sullivan measures on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively. The Bowen-Margulis-Sullivan measure $\mathsf{m}_\psi^{\mathrm{BMS}}$ on Ω_Γ associated with the pair (ν, ν_{i}) is the A_θ-invariant measure induced by the Γ-invariant measure $d\tilde{\mathsf{m}}_\psi^{\mathrm{BMS}}(\xi,\eta,v) := e^{\psi(\langle\xi,\eta\rangle)}d\nu(\xi)d\nu_{\mathrm{i}}(\eta)d\,\mathrm{Leb}_{\mathfrak{a}_\theta}(v)$ on $\tilde{\Omega}_\Gamma$, where $\langle\cdot,\cdot\rangle$ denotes the Gromov product (2.5) and $d\,\mathrm{Leb}_{\mathfrak{a}_\theta}$ denotes the Lebesgue measure on $\mathfrak{a}_\theta$.

We also have a $\{\phi_t\}$-invariant Radon measure m_ψ on Ω_ψ induced by the Γ-invariant measure

$$d\tilde{m}_\psi(\xi,\eta,s) := e^{\psi(\langle\xi,\eta\rangle)}d\nu(\xi)d\nu_{\mathrm{i}}(\eta)ds \tag{3.2}$$

on $\tilde{\Omega}_\psi$ where ds denotes the Lebesgue measure on $\mathbb{R}$. The measure m_ψ is also referred to as *Bowen-Margulis-Sullivan measure* on Ω_ψ associated with the pair (ν,ν_{i}). By the $\ker\psi$-equivariant homeomorphism $\Omega_\Gamma \simeq \Omega_\psi\times\ker\psi$, $\mathsf{m}_\psi^{\mathrm{BMS}}$ disintegrates over the measure m_ψ with conditional measure being the Lebesgue measure $\mathrm{Leb}_{\ker\psi}$ so that

$$\mathsf{m}_\psi^{\mathrm{BMS}} = m_\psi\otimes\mathrm{Leb}_{\ker\psi}\,.$$

4. Strong mixing for transverse groups with finite BMS measure

Let $\Gamma<G$ be a non-elementary θ-transverse subgroup. Fix a (Γ,θ)-proper form $\psi\in\mathfrak{a}_\theta^*$ and a pair (ν,ν_{i}) of (Γ,ψ) and $(\Gamma,\psi\circ\mathrm{i})$-Patterson-Sullivan measures on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively. Let Ω_ψ be as in Theorem 3.1 and $m_\psi = m_\psi(\nu,\nu_{\mathrm{i}})$ denote a BMS measure on Ω_ψ associated to a pair (ν,ν_{i}).

This section is devoted to the proof of the following:

Theorem 4.1. *If $|m_\psi|<\infty$, then $(\Omega_\psi,m_\psi,\phi_t)$ is strongly mixing. That is, for any $f_1,f_2\in L^2(\Omega_\psi,m_\psi)$,*

$$\lim_{|t|\to\infty}\int f_1(\phi_t(x))f_2(x)\ dm_\psi(x) = \frac{1}{|m_\psi|}\int f_1\ dm_\psi\int f_2\ dm_\psi.$$

We begin by observing the ergodicity of m_ψ:

Theorem 4.2. *If $|m_\psi|<\infty$, then $(\Omega_\psi,m_\psi,\phi_t)$ is ergodic.*

Proof. By the Poincaré recurrence theorem, the dynamical system $(\Omega_\psi,m_\psi,\phi_t)$ is conservative. Hence it follows from the higher-rank Hopf-Tsuji-Sullivan dichotomy [18, Theorem 10.2] that $(\Omega_\psi,m_\psi,\phi_t)$ is ergodic. $\square$

Although the flow space Ω_ψ was not considered, Theorem 4.2 can also be deduced from [10] once Ω_ψ is shown to make sense. See also [22] and [28] for Anosov cases.

θ-transitivity subgroups. For $g\in G$, we set $g^+ := gP_\theta\in\mathcal{F}_\theta$ and $g^- := gw_0P_{\mathrm{i}(\theta)}\in\mathcal{F}_{\mathrm{i}(\theta)}$. Set $N_\theta^+ = w_0N_{\mathrm{i}(\theta)}w_0^{-1}$. We use the following notion of θ-transitivity subgroup:

Definition 4.3. For $g\in G$ with $(g^+,g^-)\in\Lambda_\theta^{(2)}$, we define the subset $\mathcal{H}_\Gamma^\theta(g)$ of A_θ as follows: for $a\in A_\theta$, $a\in\mathcal{H}_\Gamma^\theta(g)$ if and only if there exist $\gamma\in\Gamma$, $s\in S_\theta$ and a sequence $n_1,\cdots,n_k\in N_\theta\cup N_\theta^+$, such that

(1) $((gn_1\cdots n_r)^+,(gn_1\cdots n_r)^-)\in\Lambda_\theta^{(2)}$ for all $1\le r\le k$; and

(2) $gn_1 \cdots n_k = \gamma gas$.

It is not hard to see that $\mathcal{H}_\Gamma^\theta(g)$ is a subgroup (cf. [31, Lemma 3.1]). We call $\mathcal{H}_\Gamma^\theta$ the *θ-transitivity subgroup* for Γ.

In the following, we prove that the θ-transitivity subgroup $\mathcal{H}_\Gamma^\theta$ contains $\exp \lambda_\theta(\Gamma_0)$ for some Schottky subgroup $\Gamma_0 < \Gamma$.

Proposition 4.4. *For any $g \in G$ such that $(g^+, g^-) \in \Lambda_\theta^{(2)}$, the subgroup $\psi(\log \mathcal{H}_\Gamma^\theta(g))$ is dense in $\mathbb{R}$.*

Proof. It was shown in [19, Proposition 8.3] that if Γ is a Zariski dense θ-transverse subgroup and if $g \in G$ is such that $(g^+, g^-) \in \Lambda_\theta^{(2)}$, then the subgroup $\mathcal{H}_\Gamma^\theta(g)$ is dense in A_θ, by proving that for a Schottky subgroup $\Gamma_0 < \Gamma$, the set of Jordan projections $\lambda_\theta(\Gamma_0)$ is contained in $\log \mathcal{H}_\Gamma^\theta(g)$. The Zariski dense hypothesis was used to guarantee that Γ_0 can be taken to be Zariski dense, and hence $\lambda_\theta(\Gamma_0)$ generates a dense subgroup in $\mathfrak{a}_\theta$ ([3], Theorem 2.2).

In general, let H be the Zariski closure of Γ and consider the Levi decomposition of H: $H = LU$ where L is a reductive algebraic subgroup and U the unipotent radical of H. Moreover, we have a Cartan decomposition $G = KA^+K$ so that $L = (K \cap L)(A^+ \cap L)(K \cap L)$ by [23]. If $\pi : H \to L$ denotes the projection, then $\pi(\Gamma)$ is Zariski dense in L and hence its Jordan projection generates a dense subgroup of $\mathfrak{a} \cap \operatorname{Lie} L$. This allows the same proof of [19, Proposition 8.3] to work within L, and hence the claim follows. $\square$

Contractions by flow on Ω_ψ. For $g \in G$, we write

$$[g] := (g^+, g^-, \psi(\beta_{g^+}^\theta(e, g))) \in \mathcal{F}_\theta^{(2)} \times \mathbb{R}.$$

We mainly consider the case when $[g] \in \tilde{\Omega}_\psi = \Lambda_\theta^{(2)} \times \mathbb{R}$, that is, when $(g^+, g^-) \in \Lambda_\theta^{(2)}$. For $[g] \in \tilde{\Omega}_\psi$, we denote by $\Gamma[g] \in \Omega_\psi$ the element of Ω_ψ obtained as the projection of $[g]$ by $\tilde{\Omega}_\psi \to \Omega_\psi$.

We set for $g \in G$ such that $[g] \in \tilde{\Omega}_\psi$,

$$\begin{aligned} \tilde{W}^+([g]) &:= \{[gn] \in \tilde{\Omega}_\psi : n \in N_\theta^+\}; \\ \tilde{W}^-([g]) &:= \{[gn] \in \tilde{\Omega}_\psi : n \in N_\theta\}. \end{aligned} \tag{4.1}$$

The elements of $\tilde{W}^\pm([g])$ can be described as follows:

Lemma 4.5. [19, Lemma 8.4] *Let $g \in G$, $n \in N_\theta^+$, and $n' \in N_\theta$. Then*

$$\begin{aligned} [gn] &= \left((gn)^+, g^-, \psi\left(\beta_{g^+}^\theta(e, g) + \langle (gn)^+, g^- \rangle - \langle (g^+, g^-) \rangle\right)\right); \\ [gn'] &= \left(g^+, (gn')^-, \psi\left(\beta_{g^+}^\theta(e, g)\right)\right). \end{aligned}$$

These are leaves of foliations $\tilde{W}^{\pm} := \{\tilde{W}^{+}([g]) : [g] \in \tilde{\Omega}_\psi\}$. For $z \in \Omega_\psi$, we set

$$W^+(z) := \Gamma\backslash\tilde{W}^+([g]), \quad \text{and} \quad W^-(z) := \Gamma\backslash\tilde{W}^-([g]) \tag{4.2}$$

where $g \in G$ is such that $\Gamma[g] = z$. The following proposition says that we may consider $W^+ := \{W^+(z) : z \in \Omega_\psi\}$ and $W^- := \{W^-(z) : z \in \Omega_\psi\}$ as *unstable and stable foliations* for the flow ϕ_t in Ω_ψ: note that since Ω_ψ is a locally compact second countable Hausdorff space by Theorem 3.1, so is its one-point compactification Ω_ψ^*. Hence Ω_ψ^* is metrizable. Therefore, we can choose a metric d on Ω_ψ which is a restriction of a metric on Ω_ψ^*. That we can use this kind of metric d to prove the following proposition was first observed in [4].

Proposition 4.6. [19, Proposition 8.6] *Let $z \in \Omega_\psi$. We have*

(1) *if $x, y \in W^+(z)$, then*
$$\mathsf{d}(\phi_{-t}(x), \phi_{-t}(y)) \to 0 \quad \text{as } t \to +\infty.$$
(2) *if $x, y \in W^-(z)$, then*
$$\mathsf{d}(\phi_{t}(x), \phi_{t}(y)) \to 0 \quad \text{as } t \to +\infty.$$

Moreover, the convergence is uniform on compact subsets.

Proof of Theorem 4.1. We are now ready to prove the strong mixing. We recall the following lemma proved by Babillot:

Lemma 4.7. [1, Lemma 1] *Let $(\mathcal{X}, m, \{T_t\}_{t\in\mathbb{R}})$ be a probability measure-preserving system. Let $f \in L^2(\mathcal{X}, m)$ be such that $\int f dm = 0$. Suppose that $f \circ T_{t_i} \not\to 0$ weakly*[2] *for some $t_i \to \infty$. Then there exists a non-constant function F such that by passing to a subsequence,*

$$f \circ T_{t_i} \to F \quad \text{and} \quad f \circ T_{-t_i} \to F \quad \text{weakly} \quad \text{as } i \to \infty.$$

The following is an easy observation in measure theory:

Lemma 4.8. *Let $(\mathcal{X}, m)$ be a probability measure space. If $f_i \to F$ weakly in $L^2(\mathcal{X}, m)$, then there exists a subsequence f_{i_j} such that the Cesaro average converges:*

$$\frac{1}{\ell^2}\sum_{j=1}^{\ell^2} f_{i_j} \to F \quad m\text{-a.e.}$$

Now going back to our setting, let $f_1, f_2 \in L^2(\Omega_\psi, m_\psi)$. We may assume that m_ψ is a probability measure. By replacing f_1 with $f_1 - \int f_1 dm_\psi$, it suffices to show that for any $f \in L^2(\Omega_\psi, m_\psi)$ with $\int f dm_\psi = 0$, we have $f \circ \phi_t \to 0$ weakly as $|t| \to \infty$. Since $C_c(\Omega_\psi)$ is dense in $L^2(\Omega_\psi, m_\psi)$, we may assume without loss of generality that f is a continuous function with compact support on Ω_ψ. Suppose that $f \circ \phi_t \not\to 0$ weakly as $t \to \infty$. By

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[2] $f_n \to 0$ weakly if and only if $\int f_n g\, dm \to 0$ for all $g \in L^2(\mathcal{X}, m)$

Lemma 4.7 and Lemma 4.8, there exists a non-constant function $F : \Omega_\psi \to \mathbb{R}$ and a subsequence $t_i \to \infty$ such that

$$\frac{1}{\ell^2}\sum_{i=1}^{\ell^2} f \circ \phi_{t_i} \to F \quad \text{and} \quad \frac{1}{\ell^2}\sum_{i=1}^{\ell^2} f \circ \phi_{-t_i} \to F \quad m_\psi\text{-a.e. as } \ell \to \infty. \tag{4.3}$$

We claim that F is invariant under the flow ϕ_t; this yields a contradiction to the ergodicity of $(\Omega_\psi, m_\psi, \phi_t)$ obtained in Theorem 4.2.

Let $W_0 = \{x \in \Omega_\psi : (4.3) \text{ holds}\}$, which is m_ψ-conull. Since f is uniformly continuous, it follows from Proposition 4.6 that if $g \in G$ and $n \in N_\theta \cup N_\theta^+$ are such that $[g], [gn] \in \tilde{\Omega}_\psi$ and $\Gamma[g], \Gamma[gn] \in W_0$, then

$$F(\Gamma[g]) = F(\Gamma[gn]).$$

Denote by $\tilde{W}_0$ and $\tilde{F}$ the Γ-invariant lifts of W_0 and F to $\tilde{\Omega}_\psi$ respectively. We set

$$W_1 := \{(\xi, \eta) : (\xi, \eta, t) \in \tilde{W}_0 \text{ for Leb-a.e. } t\}.$$

We also set

$$W = \{(\xi, \eta) \in W_1 : (\xi, \eta'), (\xi', \eta) \in W_1 \text{ for } \nu\text{-a.e. } \xi' \text{ and } \nu_{\mathrm{i}}\text{-a.e. } \eta'\}.$$

Recall that we also denote by $\{\phi_t\}$ the translation flow on $\tilde{\Omega}_\psi$. For any $\varepsilon > 0$ and $x \in \tilde{\Omega}_\psi$, let

$$F_\varepsilon(x) := \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \tilde{F}(\phi_s(x))\, ds.$$

Then F_ε is continuous on each $\{\phi_t\}$-orbit and as $\varepsilon \to 0$, we have the convergence $F_\varepsilon \to \tilde{F}$ m_ψ-a.e. Hence it suffices to show that F_ε is invariant under the flow ϕ_t.

By the definition of W and the observation on W_0 made above, we have that if $g \in G$ and $n \in N_\theta \cup N_\theta^+$ are such that $[g], [gn] \in W \times \mathbb{R} \subset \tilde{\Omega}_\psi$, then $F_\varepsilon([g]) = F_\varepsilon([gn])$. Fix $g \in G$ such that $[g] \in W \times \mathbb{R}$ and let $t_0 \in \psi(\log \mathcal{H}_\Gamma^\theta(g))$ and $a \in \mathcal{H}_\Gamma^\theta(g)$ such that $\psi(\log a) = t_0$. We then have $\phi_{t_0}([g]) = [ga]$. By the definition of the θ-transitivity subgroup, there exist $\gamma \in \Gamma$, $s \in S_\theta$, and a sequence $n_1, \cdots, n_k \in N_\theta \cup N_\theta^+$, such that

(1) $((gn_1 \cdots n_r)^+, (gn_1 \cdots n_r)^-) \in \Lambda_\theta^{(2)}$ for all $1 \le r \le k$;
(2) $gn_1 \cdots n_k = \gamma gas$.

As in the proof of [19, Proposition 8.8], there exist a sequence $a_j \in A_\theta$ and a sequence of k-tuples $(n_{1,j}, \cdots, n_{k,j}) \in \prod_{i=1}^k N_\theta \cup N_\theta^+$ converging to a and $(n_1, \cdots, n_k)$ respectively as $j \to \infty$, and such that for each $j \ge 1$, we have

$$[gn_{1,j} \cdots n_{r,j}] \in W \times \mathbb{R} \quad \text{for all } 1 \le r \le k \quad \text{and} \quad [gn_{1,j} \cdots n_{k,j}] = [\gamma g a_j].$$

Therefore, we have for each $j \ge 1$ that

$$\begin{aligned} F_\varepsilon([g]) &= F_\varepsilon([gn_{1,j}]) = \cdots = F_\varepsilon([gn_{1,j} \cdots n_{k-1,j}]) = F_\varepsilon([gn_{1,j} \cdots n_{k,j}]) \\ &= F_\varepsilon([\gamma g a_j]) = F_\varepsilon([g a_j]). \end{aligned}$$

Taking the limit $j \to \infty$, it follows from the continuity of F_ε on each $\{\phi_t\}$-orbit that

$$F_\varepsilon([g]) = F_\varepsilon([ga]) = (F_\varepsilon \circ \phi_{t_0})([g]).$$

Since $\psi(\log \mathcal{H}_\Gamma^\theta(g))$ is dense in $\mathbb{R}$ by Proposition 4.4, this implies that

$$F_\varepsilon([g]) = (F_\varepsilon \circ \phi_t)([g]) \quad \text{for all } t \in \mathbb{R}.$$

Since $[g] \in W \times \mathbb{R}$ is arbitrary and $(\nu \otimes \nu_{\mathrm{i}})(W) = 1$, this completes the proof. □

5. Relatively Anosov groups

Relatively Anosov groups are relatively hyperbolic groups as abstract groups, which we now define. Let Γ be a countable group acting on a compact metrizable space $\mathcal{X}$ by homeomorphisms. This action is called a *convergence group action* if for any sequence of distinct elements $\gamma_n \in \Gamma$, there exist a subsequence γ_{n_k} and $a, b \in \mathcal{X}$ such that as $k \to \infty$, $\gamma_{n_k}(x)$ converges to a for all $x \in \mathcal{X} - \{b\}$, uniformly on compact subsets. An element $\gamma \in \Gamma$ of infinite order fixes either exactly two points in $\mathcal{X}$ or exactly one point in $\mathcal{X}$. In the former case, we call γ *loxodromic*, and *parabolic* otherwise. An infinite subgroup $P < \Gamma$ is called *parabolic* if P fixes some point in $\mathcal{X}$ and every infinite order element of P is parabolic.

A point $\xi \in \mathcal{X}$ is called a *conical limit point* if there exist a sequence of distinct elements $\gamma_n \in \Gamma$ and distinct points $a, b \in \mathcal{X}$ such that as $n \to \infty$, $\gamma_n \xi \to a$ and $\gamma_n \eta \to b$ for all $\eta \in \mathcal{X} - \{\xi\}$. A point $\xi \in \mathcal{X}$ is called a *parabolic limit point* if ξ is fixed by a parabolic subgroup of Γ. We say that a parabolic limit point $\xi \in \mathcal{X}$ is bounded if $\mathrm{Stab}_\Gamma(x) \backslash (\mathcal{X} - \{\xi\})$ is compact. The action of Γ on $\mathcal{X}$ is called a *geometrically finite convergence group action* if every point of $\mathcal{X}$ is either conical or bounded parabolic limit point. A typical example of geometrically finite convergence group action is the action of a geometrically finite Kleinian group on its limit set.

Let Γ be a finitely generated group and $\mathcal{P}$ a finite collection of finitely generated infinite subgroups of Γ. We say that Γ is *hyperbolic relative to* $\mathcal{P}$ (or that $(\Gamma, \mathcal{P})$ is *relatively hyperbolic*), if Γ admits a geometrically finite convergence group action on some compact perfect metrizable space $\mathcal{X}$ and the collection of maximal parabolic subgroups is

$$\mathcal{P}^\Gamma := \{\gamma P \gamma^{-1} : P \in \mathcal{P}, \gamma \in \Gamma\}.$$

Bowditch [6] showed that for Γ hyperbolic relative to $\mathcal{P}$, the space $\mathcal{X}$ satisfying the above hypothesis is unique up to a Γ-equivariant homeomorphism. Hence this space is called *Bowditch boundary* and denoted by $\partial(\Gamma, \mathcal{P})$.

The Groves-Manning cusp space. Let Γ be a hyperbolic group relative to $\mathcal{P}$. The *Groves-Manning cusp space* for $(\Gamma, \mathcal{P})$ is a proper geodesic Gromov hyperbolic space constructed by Groves-Manning [15] on which Γ acts properly discontinuously and by isometries. We briefly review the construction of the Groves-Manning cusp space. We first need a notion of combinatorial horoballs: for a graph Y equipped with a simplicial distance d_Y, the combinatorial horoball $\mathcal{H}(Y)$ is the graph with the vertex set $Y^{(0)} \times \mathbb{N}$ and two types of edges: vertical edges between vertices (y, n) and $(y, n+1)$ for $y \in Y$ and $n \in \mathbb{N}$, and horizontal edges between vertices (y_1, n) and (y_2, n) for $y_1, y_2 \in Y$ and $n \in \mathbb{N}$ if $d_Y(y_1, y_2) \leq 2^{n-1}$. We also equip $\mathcal{H}(Y)$ with the simplicial distance.

Now fix a finite generating set S of Γ such that for each $P \in \mathcal{P}$, $S \cap P$ generates P. We denote by $\mathcal{C}(\Gamma, S)$ and $\mathcal{C}(P, S \cap P)$ the Cayley graphs of Γ and P with respect to S and $S \cap P$ respectively. For each $\gamma \in \Gamma$ and $P \in \mathcal{P}$, we glue the horoball $\mathcal{H}(\gamma\mathcal{C}(P, S \cap P))$ to $\mathcal{C}(\Gamma, S)$, by identifying $\gamma\mathcal{C}(P, S \cap P) \subset \mathcal{C}(\Gamma, S)$ with $\gamma\mathcal{C}(P, S \cap P) \times \{1\} \subset \mathcal{H}(\gamma\mathcal{C}(P, S \cap P))$. The resulting graph equipped with the simplicial distance is called the Groves-Manning cusp space for $(\Gamma, \mathcal{P})$ and S, which we denote by $X_{GM}(\Gamma, \mathcal{P}, S)$.

Theorem 5.1. [15, Theorem 3.25] *The space $X_{GM}(\Gamma, \mathcal{P}, S)$ is a proper geodesic Gromov hyperbolic space.*

From the construction, the natural action of Γ on the Cayley graph $\mathcal{C}(\Gamma, S)$ induces the isometric action of Γ on $X_{GM}(\Gamma, \mathcal{P}, S)$ which is properly discontinuous. Hence the induced Γ-action on the Gromov boundary $\partial X_{GM}(\Gamma, \mathcal{P}, S)$ is a convergence group action [5, Lemma 2.11], and moreover is a geometrically finite convergence group action by the construction of $X_{GM}(\Gamma, \mathcal{P}, S)$. Therefore the Gromov boundary of $X_{GM}(\Gamma, \mathcal{P}, S)$ is the Bowditch boundary:

$$\partial X_{GM}(\Gamma, \mathcal{P}, S) = \partial(\Gamma, \mathcal{P}).$$

Relatively Anosov subgroups. Let $\Gamma < G$ be a finitely generated non-elementary θ-transverse subgroup with the limit set Λ_θ and $\mathcal{P}$ a finite collection of finitely generated infinite subgroups of Γ.

Definition 5.2. We say that Γ is *θ-Anosov relative to $\mathcal{P}$* if Γ is hyperbolic relative to $\mathcal{P}$ and there exists a Γ-equivariant homeomorphism $\partial(\Gamma, \mathcal{P}) \to \Lambda_\theta$.

Let Γ be a θ-Anosov relative to $\mathcal{P}$ in the rest of the section. We denote by $X_{GM} := X_{GM}(\Gamma, \mathcal{P}, S)$ the associated Groves-Manning cusp space for some fixed generating set S. We then have the Γ-equivariant homeomorphism

$$f : \partial X_{GM} \to \Lambda_\theta,$$

which has the following property: Noting that the action of Γ is faithful on X_{GM}, we have a well-defined map $\Gamma x \to \Gamma o$ given by $\gamma x \mapsto \gamma o$ for any $x \in X_{GM}$.

Proposition 5.3. [11, Proposition 4.3] *Let $x \in X_{GM}$. Then the map $\Gamma x \to \Gamma o$ extends continuously to a unique Γ-equivariant homeomorphism $f : \partial X_{GM} \to \Lambda_\theta$.*

By the antipodality of Γ, the canonical projections $\pi_\theta : \Lambda_{\theta\cup \mathrm{i}(\theta)} \to \Lambda_\theta$ and $\pi_{\mathrm{i}(\theta)} : \Lambda_{\theta\cup \mathrm{i}(\theta)} \to \Lambda_{\mathrm{i}(\theta)}$ are Γ-equivariant homeomorphisms. This implies that being relatively θ-Anosov implies being relatively $\theta \cup \mathrm{i}(\theta)$-Anosov as well as relatively $\mathrm{i}(\theta)$-Anosov. In particular, setting the composition $f_{\mathrm{i}} := \pi_{\mathrm{i}(\theta)} \circ \pi_\theta^{-1} \circ f$, two maps

$$f : \partial X_{GM} \to \Lambda_\theta \quad \text{and} \quad f_{\mathrm{i}} : \partial X_{GM} \to \Lambda_{\mathrm{i}(\theta)}$$

have the property that if $\xi, \eta \in \partial X_{GM}$ are distinct, then $(f(\xi), f_{\mathrm{i}}(\eta)) \in \mathcal{F}_\theta^{(2)}$.

Compatibility of shadows. We first define the shadows in the symmetric space X: for $p \in X$ and $R > 0$, let $B(p, R)$ denote the metric ball $\{x \in X : d(x, p) < R\}$. For $q \in X$, the *θ-shadow* $O_R^\theta(q, p) \subset \mathcal{F}_\theta$ of $B(p, R)$ viewed from q is defined as

$$O_R^\theta(q, p) = \{gP_\theta \in \mathcal{F}_\theta : g \in G,\ go = q,\ gA^+o \cap B(p, R) \neq \emptyset\}.$$

The following two lemmas will be useful:

Lemma 5.4. [21, Lemma 5.7] *There exists $\kappa > 0$ such that for any $g, h \in G$ and $R > 0$, we have*

$$\sup_{\xi \in O_R^\theta(go, ho)} \|\beta_\xi^\theta(g, h) - \mu_\theta(g^{-1}h)\| \leq \kappa R.$$

Lemma 5.5. [18, Lemma 9.9] *Let $g_n \in G$ and $\xi_n \in \mathcal{F}_\theta$ be sequences both converging to some $\xi \in \mathcal{F}_\theta$. Suppose that there exists a sequence $\eta_n \in \mathcal{F}_{\mathrm{i}(\theta)}$ converging to some $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ such that $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$ and the sequence $g_n^{-1}(\xi_n, \eta_n)$ is precompact in $\mathcal{F}_\theta^{(2)}$. Then there exists $R > 0$ such that*

$$\xi_n \in O_R^\theta(o, g_n o) \quad \textit{for all } n \geq 1.$$

We also consider shadows in Groves-Manning cusp space. Let d_{GM} be the simplicial distance on X_{GM}.

The following theorem is obtained in [11, Theorem 10.1]; although it stated only the lower bound, the upper bound also follows from its proof:

Theorem 5.6. *For any (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$, there exists positive constants c, c' and C such that for all $\gamma \in \Gamma$,*

$$c\, d_{GM}(e, \gamma) - C \leq \psi(\mu_\theta(\gamma)) \leq c'\, d_{GM}(e, \gamma) + C.$$

For $y \in X_{GM}$ and $R > 0$, we denote the R-ball centered at y by

$$B_{GM}(y, R) := \{z \in X_{GM} : d_{GM}(y, z) < R\}.$$

For $x, y \in X_{GM}$ and $R > 0$, we define the shadow of $B_{GM}(y, R)$ viewed from x as follows:

$$O_R^{GM}(x, y) := \left\{\xi \in \partial X_{GM} : \begin{array}{c}\text{there exists a geodesic ray from } x \text{ to } \xi \\ \text{passing through } B_{GM}(y, R)\end{array}\right\}.$$

Note that $\xi \in \partial X_{GM}$ is a conical limit point if and only if there exists $R > 0$ such that $\xi \in O_R^{GM}(o, \gamma_n o)$ for an infinite sequence $\gamma_n \in \Gamma$.

We prove the following compatibility of shadows under $f : \partial X_{GM} \to \Lambda_\theta$:

Proposition 5.7. *Let $x \in X_{GM}$ and $o \in X$. For all sufficiently large $R > 1$, there exist $r_1 = r_1(R), r_2 = r_2(R) > 0$ such that for any $\gamma \in \Gamma$, we have*

$$O^{\theta}_{r_1}(o, \gamma o) \cap \Lambda_\theta \subset f(O^{GM}_{R}(x, \gamma x)) \subset O^{\theta}_{r_2}(o, \gamma o) \cap \Lambda_\theta.$$

Moreover, we can take $r_1(R) \to \infty$ as $R \to \infty$.

We begin with some lemmas:

Lemma 5.8. *For any $x \in X_{GM}$, there exists $R_0 > 0$ such that $O^{GM}_{R_0}(x, \gamma x) \neq \emptyset$ for any $\gamma \in \Gamma$.*

Proof. Suppose not. Then there exists an infinite sequence $\gamma_n \in \Gamma$ so that $O^{GM}_{n}(x, \gamma_n x) = \emptyset$, and hence $O^{GM}_{n}(\gamma_n^{-1} x, x) = \emptyset$ for all $n \geq 1$. This forces ∂X_{GM} to be a singleton, which contradicts the perfectness of ∂X_{GM}. □

Lemma 5.9. *Let $x \in X_{GM}$ and $R > 0$. Let $\gamma_n \in \Gamma$ and $\xi_n \in \partial X_{GM}$ be sequences such that $\xi_n \in O^{GM}_{R}(x, \gamma_n x)$ for all $n \geq 1$. If $\gamma_n x \to \xi \in \partial X_{GM}$ as $n \to \infty$, then $\xi_n \to \xi$ as $n \to \infty$.*

Proof. Suppose to the contrary that the sequence ξ_n, after passing to a subsequence, converges to $\xi' \in \partial X_{GM}$ distinct from ξ. Since $\gamma_n x \to \xi$ as $n \to \infty$ and X_{GM} is Gromov hyperbolic (Theorem 5.1), this implies that there exist a constant $R' > 0$ and a sequence of geodesic rays $[\gamma_n x, \xi_n]$ from $\gamma_n x$ to ξ_n such that $d_{GM}(x, [\gamma_n x, \xi_n]) < R'$ for all $n \geq 1$. On the other hand, since $\xi_n \in O^{GM}_{R}(x, \gamma_n x)$, there exists a geodesic ray $[x, \xi_n]$ from x to ξ_n and a point $c_n \in [x, \xi_n]$ such that $d_{GM}(c_n, \gamma_n x) < R$ for all $n \geq 1$. Since the distance between $\gamma_n x$ and c_n is uniformly bounded, the Hausdorff distance between two geodesic rays $[\gamma_n x, \xi_n]$ and $[c_n, \xi_n] \subset [x, \xi_n]$ is uniformly bounded, by the Gromov hyperbolicity of X_{GM} (Theorem 5.1). Since the distance $d_{GM}(x, [\gamma_n x, \xi_n])$ is uniformly bounded, this implies that the distance $d_{GM}(x, [c_n, \xi_n])$ is uniformly bounded as well. Since $[c_n, \xi_n]$ is the geodesic ray contained in the geodesic ray $[x, \xi_n]$, we have that $d_{GM}(x, c_n) = d_{GM}(x, [c_n, \xi_n])$ is uniformly bounded. Therefore, it follows from the uniform boundedness of $d_{GM}(c_n, \gamma_n x)$ that $d_{GM}(x, \gamma_n x)$ is uniformly bounded, which contradicts the hypothesis that $\gamma_n x \to \xi$ as $n \to \infty$. This finishes the proof. □

Proof of Proposition 5.7. Note that the first inclusion and the last claim follow once we show that for any $c > 0$, there exists $C > 0$ such that $O^{\theta}_{c}(o, \gamma o) \subset f(O^{GM}_{C}(x, \gamma x))$ for all $\gamma \in \Gamma$. Suppose not. Then there exist sequences $\gamma_n \in \Gamma$ and $\xi_n \in \partial X_{GM} - O^{GM}_{n}(x, \gamma_n x)$ such that $f(\xi_n) \in O^{\theta}_{c}(o, \gamma_n o)$ for all $n \geq 1$. After passing to a subsequence, we may assume that the sequence $\gamma_n^{-1} x$ converges to some point $\eta \in \partial X_{GM}$ as $n \to \infty$. Since $\gamma_n^{-1} \xi_n \notin O^{GM}_{n}(\gamma_n^{-1} x, x)$ for all $n \geq 1$, we have that

$$\lim_{n \to \infty} \gamma_n^{-1} \xi_n = \eta. \tag{5.1}$$

On the other hand, by Proposition 5.3, we have $\lim_{n\to\infty}\gamma_n^{-1} = f_{\mathrm{i}}(\eta) \in \Lambda_{\mathrm{i}(\theta)}$. Since $f(\gamma_n^{-1}\xi_n) \in O_c^\theta(\gamma_n^{-1}o, o)$ for all $n \geq 1$ and $\lim_{n\to\infty}\gamma_n^{-1} = f_{\mathrm{i}}(\eta)$, it follows from (5.1) and the continuity of higher-rank shadows on viewpoints [19, Proposition 3.4] that $f(\eta) = \lim_{n\to\infty} f(\gamma_n^{-1}\xi_n) \in \Lambda_\theta$ is in general position with $f_{\mathrm{i}}(\eta)$. This yields contradiction.

We now prove the second inclusion. Let $R_0 > 0$ be as given by Lemma 5.8 and fix $R > R_0$. Let $x \in X_{GM}$ and $o \in X$. Suppose on the contrary that there exists a sequence $\gamma_n \in \Gamma$ such that

$$f(O_R^{GM}(x, \gamma_n x)) \not\subset O_n^\theta(o, \gamma_n o) \quad \text{for all } n \geq 1.$$

This means that there exists a sequence $\xi_n \in O_R^{GM}(x, \gamma_n x)$ such that $f(\xi_n) \notin O_n^\theta(o, \gamma_n o)$ for all $n \geq 1$. After passing to a subsequence, we may assume that the sequence $\gamma_n x$ converges to a point $\xi \in \partial X_{GM}$. By Proposition 5.3, we have

$$\gamma_n \to f(\xi) \quad \text{as } n \to \infty. \tag{5.2}$$

In addition, it follows from Lemma 5.9 that $\xi_n \to \xi$ as $n \to \infty$. For each $n \geq 1$, we choose a point $\eta_n \in O_R^{GM}(\gamma_n x, x)$ which is possible by Lemma 5.8. We may assume that the sequence η_n converges to $\eta \in \partial X_{GM}$, after passing to a subsequence. Since $\gamma_n x \to \xi$ as $n \to \infty$ and $\eta_n \in O_R^{GM}(\gamma_n x, x)$ for all $n \geq 1$, we have $\xi \neq \eta$. Therefore, we have the following convergence of the sequence in $\mathcal{F}_\theta^{(2)}$:

$$(f(\xi_n), f_{\mathrm{i}}(\eta_n)) \to (f(\xi), f_{\mathrm{i}}(\eta)) \in \mathcal{F}_\theta^{(2)} \quad \text{as } n \to \infty. \tag{5.3}$$

On the other hand, we also have $\gamma_n^{-1}\xi_n \in O_R^{GM}(\gamma_n^{-1}x, x)$ and $\gamma_n^{-1}\eta_n \in O_R^{GM}(x, \gamma_n^{-1}x)$ for all $n \geq 1$. Together with the Γ-equivariance of f and f_{i}, a similar argument as above implies that

$$\text{the sequence } \gamma_n^{-1}(f(\xi_n), f_{\mathrm{i}}(\eta_n)) \text{ is precompact in } \mathcal{F}_\theta^{(2)}. \tag{5.4}$$

By (5.2), (5.3), and (5.4), we apply Lemma 5.5 and deduce that there exists $R' > 0$ so that $f(\xi_n) \in O_{R'}^\theta(o, \gamma_n o)$ for all $n \geq 1$. This contradicts to the choice of the sequence ξ_n that $f(\xi_n) \notin O_n^\theta(o, \gamma_n o)$ for all $n \geq 1$. This completes the proof. $\square$

Lemma 5.10. *Let $x \in X_{GM}$ and $R > 0$. Then there exists a compact subset $Q \subset \mathfrak{a}_\theta$ satisfying the following: if $\xi, \eta \in \partial X_{GM}$ are such that $d_{GM}(x, [\xi, \eta]) < R$ for some bi-infinite geodesic $[\xi, \eta]$, then*

$$\langle f(\xi), f_{\mathrm{i}}(\eta)\rangle \in Q$$

where $\langle \cdot, \cdot \rangle$ is the Gromov product defined in (2.5).

Proof. Suppose not. Then there exists a sequence of bi-infinite geodesics $[\xi_n, \eta_n]$ for some $\xi_n, \eta_n \in \partial X_{GM}$ such that we have $\sup_n d_{GM}(x, [\xi_n, \eta_n]) < R$ and the Gromov products $\langle f(\xi_n), f_{\mathrm{i}}(\eta_n)\rangle$ escape every compact subset of $\mathfrak{a}_\theta$ as $n \to \infty$. After passing to a subsequence, we may assume that $\xi_n \to \xi$ and $\eta_n \to \eta$ in ∂X_{GM}. The hypothesis $\sup_n d_{GM}(x, [\xi_n, \eta_n]) < R$

implies $\xi \neq \eta$, since X_{GM} is Gromov hyperbolic (Theorem 5.1). Therefore $(f(\xi), f_{\mathrm{i}}(\eta)) \in \Lambda_\theta^{(2)}$ and hence $\langle f(\xi), f_{\mathrm{i}}(\eta)\rangle \in \mathfrak{a}_\theta$ is well-defined. On the other hand, by the continuity of the Gromov product, we have $\langle f(\xi_n), f_{\mathrm{i}}(\eta_n)\rangle \to \langle f(\xi), f_{\mathrm{i}}(\eta)\rangle \in \mathfrak{a}_\theta$ as $n \to \infty$. This yields a contradiction. $\square$

6. Reparameterization for relatively Anosov groups

Let $\Gamma < G$ be a θ-Anosov subgroup relative to $\mathcal{P}$ and $X_{GM} = X_{GM}(X, \mathcal{P}, S)$ the associated Groves-Manning cusp space for a fixed generating set S. Fix a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$. Recall from section 3 the space $\tilde{\Omega}_\psi := \Lambda_\theta^{(2)} \times \mathbb{R}$ equipped with the Γ-action given by

$$\gamma(\xi, \eta, s) = (\gamma\xi, \gamma\eta, s + \psi(\beta_\xi^\theta(\gamma^{-1}, e))).$$

As stated in Theorem 3.1, the space

$$\Omega_\psi := \Gamma \backslash \tilde{\Omega}_\psi$$

is a locally compact second countable Hausdorff space. The translation flow $\{\phi_t\}$ on the $\mathbb{R}$-component of $\tilde{\Omega}_\psi$ commutes with the Γ-action, and hence it induces the translation flow on Ω_ψ which we also denote by $\{\phi_t\}$. We will relate $\tilde{\Omega}_\psi$ and Ω_ψ with the Groves-Manning cusp space X_{GM} in this section. More precisely, let

$$\mathcal{G} := \{\sigma : \mathbb{R} \to X_{GM} : \text{ bi-infinite geodesic}\}.$$

The space $\mathcal{G}$ admits the geodesic flow $\varphi_s : \mathcal{G} \to \mathcal{G}$ defined by $(\varphi_s\sigma)(\cdot) = \sigma(\cdot + s)$ for $s \in \mathbb{R}$, and the inversion $I : \mathcal{G} \to \mathcal{G}$ defined by $(I\sigma)(s) = \sigma(-s)$ for $s \in \mathbb{R}$. The canonical isometric action of Γ on $\mathcal{G}$ commutes with the geodesic flow and I, and is properly discontinuous. Hence we can also consider the locally compact Hausdorff space $\Gamma \backslash \mathcal{G}$. This section is devoted to the proof of the following reparameterization theorem:

Set

$$a = \liminf_{\gamma \in \Gamma} \frac{\psi(\mu_\theta(\gamma))}{d_{GM}(e, \gamma)} \quad \text{and} \quad a' = 3 \limsup_{\gamma \in \Gamma} \frac{\psi(\mu_\theta(\gamma))}{d_{GM}(e, \gamma)}. \tag{6.1}$$

By Theorem 5.6, we have $0 < a \leq a' < \infty$.

Theorem 6.1 (Reparameterization, Theorem 1.4(1)-(3))**.** *There exists a continuous, surjective, proper Γ-equivariant map*

$$\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi.$$

Moreover, we have a continuous cocycle $\tilde{\mathfrak{t}} : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ such that for all $\sigma \in \mathcal{G}$ and $s \in \mathbb{R}$,

1. $\tilde{\Psi}(\varphi_s\sigma) = \phi_{\tilde{\mathfrak{t}}(\sigma, s)}\tilde{\Psi}(\sigma)$;
2. $\tilde{\mathfrak{t}}(\sigma, s) = -\tilde{\mathfrak{t}}(\varphi_s\sigma, -s)$;
3. *there exists an absolute constant $B > 0$ such that*

$$a|s| - B \leq \tilde{\mathfrak{t}}(\sigma, |s|) \leq a'|s| + B.$$

In the above theorem, $\tilde{\mathsf{t}} : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ being a continuous cocycle means that it is continuous and for all $\sigma \in \mathcal{G}$ and $s_1, s_2 \in \mathbb{R}$,

$$\tilde{\mathsf{t}}(\sigma, s_1 + s_2) = \tilde{\mathsf{t}}(\sigma, s_1) + \tilde{\mathsf{t}}(\varphi_{s_1}\sigma, s_2).$$

Since $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$ in Theorem 6.1 is Γ-equivariant, this descends to the map $\Psi : \Gamma\backslash\mathcal{G} \to \Omega_\psi$. The following is immediate from Theorem 6.1.

Corollary 6.2 (Reparameterization)**.** *There exists a continuous, surjective, proper map*

$$\Psi : \Gamma\backslash\mathcal{G} \to \Omega_\psi.$$

Moreover, we have a continuous cocycle $\mathsf{t} : \Gamma\backslash\mathcal{G} \times \mathbb{R} \to \mathbb{R}$ *such that for all* $\sigma \in \mathcal{G}$ *and* $s \in \mathbb{R}$*,*

(1) $\Psi([\varphi_s\sigma]) = \phi_{\mathsf{t}(\sigma,s)}\Psi([\sigma])$*;*
(2) $\mathsf{t}(\sigma, s) = -\mathsf{t}(\varphi_s\sigma, -s)$*;*
(3) *there exists an absolute constant* $B > 0$ *such that*

$$a|s| - B \leq \mathsf{t}(\sigma, |s|) \leq a'|s| + B.$$

Thick-thin decomposition of $\mathcal{G}$. For $P \in \mathcal{P}$, let $\xi_P \in \partial X_{GM}$ be the bounded parabolic limit point fixed by P. We consider the open *horoball* $H_P \subset X_{GM}$ based at ξ_P invariant under P, defined as follows: let $H'_P \subset X_{GM}$ be the subgraph induced by the vertices $\{(g, n) : g \in P, n \geq 2\}$ and $\hat{H}_P \subset X_{GM}$ be the subgraph induced by the vertices $\{(g, 2) : g \in P\}$. We then set

$$H_P := H'_P - \hat{H}_P.$$

For $\gamma \in \Gamma$, we also set

$$H_{\gamma P\gamma^{-1}} := \gamma H_P$$

which is the open horoball based at $\xi_{\gamma P\gamma^{-1}} := \gamma\xi_P$ and invariant under $\gamma P\gamma^{-1} \in \mathcal{P}^\Gamma$. The boundary $\partial H_{\gamma P\gamma^{-1}}$ consists of the vertices $\gamma\{(g, 2) : g \in P\}$. We then have the Γ-invariant family $\{H_P : P \in \mathcal{P}^\Gamma\}$ of open horoballs with disjoint closures.

We define the following subsets of $\mathcal{G}$: for $P \in \mathcal{P}^\Gamma$, let

$$\mathcal{G}_P := \{\sigma \in \mathcal{G} : \sigma(0) \in H_P\};$$
$$\partial\mathcal{G}_P := \{\sigma \in \mathcal{G} : \sigma(0) \in \partial H_P\}.$$

We have the *thick-thin decomposition* of $\mathcal{G}$:

$$\mathcal{G}_{thin} := \bigcup_{P \in \mathcal{P}^\Gamma} \mathcal{G}_P \quad \text{and} \quad \mathcal{G}_{thick} := \mathcal{G} - \mathcal{G}_{thin}.$$

Since the Groves-Manning cusp space X_{GM} is constructed by attaching combinatorial horoballs to the Cayley graph of Γ, the Γ-action on $X_{GM} - \bigcup_{P\in\mathcal{P}^\Gamma} H_P$ is cocompact. Hence the Γ-action on $\mathcal{G}_{thick}$ which consists of bi-infinite geodesics based at $X_{GM} - \bigcup_{P\in\mathcal{P}^\Gamma} H_P$ is also cocompact.

We also introduce the following subsets of $\partial\mathcal{G}_P$ for each $P \in \mathcal{P}^\Gamma$:

$$\partial^+\mathcal{G}_P := \{\sigma \in \partial\mathcal{G}_P : \sigma(t) \in H_P \text{ for all sufficiently small } t > 0\};$$
$$\partial^-\mathcal{G}_P := \{\sigma \in \partial\mathcal{G}_P : \sigma(-t) \in H_P \text{ for all sufficiently small } t > 0\}.$$

Note that $\partial^+\mathcal{G}_P \cap \partial^-\mathcal{G}_P = \emptyset$. For $\sigma \in \partial^+\mathcal{G}_P$, we set

$$T_\sigma^+ := \min\{t \in (0,\infty] : \sigma(t) \notin H_P\},$$

and for $\sigma \in \partial^-\mathcal{G}_P$, we set

$$T_\sigma^- := \max\{t \in [-\infty, 0) : \sigma(t) \notin H_P\},$$

which are the escaping times for the horoball H_P. We then have

$$\mathcal{G}_P = \left(\bigcup_{\sigma\in\partial^+\mathcal{G}_P} \bigcup_{t\in(0,T_\sigma^+)} \varphi_t\sigma \right) \cup \left(\bigcup_{\sigma\in\partial^-\mathcal{G}_P} \bigcup_{t\in(T_\sigma^-,0)} \varphi_t\sigma \right).$$

Construction of the reparameterization. To construct the reparameterization, we consider the trivial bundle

$$\mathcal{G} \times \mathbb{R}_+ \to \mathcal{G}.$$

Given $\sigma \in \mathcal{G}$, we denote by $\sigma^+ = \sigma(\infty) \in \partial X_{GM}$ and $\sigma^- = \sigma(-\infty) \in \partial X_{GM}$ the forward and backward endpoint of the bi-infinite geodesic σ. Noting that we have Γ-equivariant homeomorphisms $f : \partial X_{GM} \to \Lambda_\theta$ and $f_{\mathrm{i}} : \partial X_{GM} \to \Lambda_{\mathrm{i}(\theta)}$, we identify ∂X_{GM}, Λ_θ, and $\Lambda_{\mathrm{i}(\theta)}$ in this section via the homeomorphisms. We define the Γ-action on $\mathcal{G} \times \mathbb{R}_+$ as follows: for $\gamma \in \Gamma$ and $(\sigma, v) \in \mathcal{G} \times \mathbb{R}_+$,

$$\gamma(\sigma, v) = \left(\gamma\sigma, ve^{\psi\left(\beta^\theta_{\sigma^+}(\gamma^{-1},e)\right)}\right).$$

This action makes the following projection Γ-equivariant:

$$\begin{aligned} \Psi_0 : \mathcal{G} \times \mathbb{R}_+ &\longrightarrow \qquad \tilde{\Omega}_\psi \\ (\sigma, v) \quad &\longmapsto (\sigma^+, \sigma^-, \log v). \end{aligned}$$

We construct the reparameterization $\Psi : \Gamma\backslash\mathcal{G} \to \Omega_\psi$ in Theorem 6.1 by constructing a nice Γ-equivariant section $u : \mathcal{G} \to \mathcal{G} \times \mathbb{R}_+$ of the trivial bundle so that we obtain a Γ-equivariant map $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$ as follows, with the desired properties:

$$\begin{tikzcd}
\mathcal{G} \times \mathbb{R}_+ \arrow[d] \arrow[dr, "\Psi_0"] & \\
\mathcal{G} \arrow[u, dashed, bend left, "u"] \arrow[r, dashed, "\tilde{\Psi}"'] & \tilde{\Omega}_\psi
\end{tikzcd}$$

Norms on fibers. To construct a section of the trivial bundle $\mathcal{G} \times \mathbb{R}_+ \to \mathcal{G}$, we define a continuous family of Γ-equivariant norms on fibers. More precisely, we define a Γ-invariant continuous function

$$\| \cdot \| : \mathcal{G} \times \mathbb{R}_+ \to \mathbb{R}_+$$

such that for each $\sigma \in \mathcal{G}$, $\|(\sigma, \cdot)\|$ is the restriction of a norm on $\mathbb{R}$ to $\mathbb{R}_+$. We simply write

$$\| \cdot \|_\sigma := \|(\sigma, \cdot)\| \quad \text{for each } \sigma \in \mathcal{G}.$$

Once we define the norm, we will define a section $u : \mathcal{G} \to \mathcal{G} \times \mathbb{R}_+$ by $u(\sigma) = (\sigma, v_\sigma)$ where $v_\sigma \in \mathbb{R}_+$ is the unique unit vector with respect to the norm $\| \cdot \|_\sigma$, i.e., $\|v_\sigma\|_\sigma = 1$. The Γ-equivariance and the continuity of the norms imply that the section u is also Γ-equivariant and continuous. To make the reparameterization $\tilde{\Psi} = \Psi_0 \circ u$ satisfy the conditions in Theorem 6.1, our norms should have a property that the contraction rate along the geodesic flow is bounded from both *above* and *below* by uniform exponential functions.

Our construction of the family of norms is motivated by [32] which considered flat bundles for relatively Anosov subgroups of $\mathrm{SL}(n, \mathbb{R})$ with respect to a maximal parabolic subgroup. Our proof of the contraction property is motivated by ([9], [32]) where the upper bound of the contraction rate of norms on flat bundles for relatively Anosov subgroups of $\mathrm{SL}(n, \mathbb{R})$ with respect to a maximal parabolic subgroup was proved. We also remark that the contraction property was earlier studied in ([30], [12]) for Anosov subgroups.

We now define a family of norms as follows (compare to a similar construction in [32]): first we fix a continuous family of Γ-equivariant norms $\| \cdot \|_\sigma$ for $\sigma \in \mathcal{G}_{thick}$ such that $\| \cdot \|_\sigma = \| \cdot \|_{I\sigma}$ for all $\sigma \in \mathcal{G}_{thick}$. Let $\sigma \in \mathcal{G}_{thin}$. Then $\sigma \in \mathcal{G}_P$ for some $P \in \mathcal{P}^\Gamma$. Let

$$c > 0 \tag{6.2}$$

be the constant given by Theorem 5.6. There are two cases indicated by the Figures 1 and 2:

Case 1. If $\sigma = \varphi_t \sigma_0$ for some $\sigma_0 \in \partial^+ \mathcal{G}_P$ and $t \in (0, T^+_{\sigma_0})$, we write $T := T^+_{\sigma_0}$ and

- if $t \in \left(0, \frac{1}{3}T\right]$, we set
$$\| \cdot \|_\sigma := e^{-ct} \| \cdot \|_{\sigma_0}.$$
- if $t \in \left[\frac{2}{3}T, T\right)$, we set
$$\| \cdot \|_\sigma := e^{c(T-t)} \| \cdot \|_{\varphi_T \sigma_0}.$$
- if $t \in \left(\frac{1}{3}T, \frac{2}{3}T\right)$, we set
$$\| \cdot \|_\sigma := \| \cdot \|_{\varphi_{T/3}\sigma_0}^{2 - \frac{3}{T}t} \| \cdot \|_{\varphi_{2T/3}\sigma_0}^{\frac{3}{T}t - 1}.$$

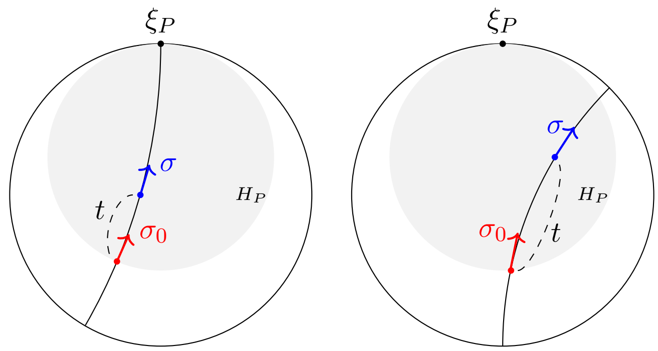

FIGURE 1. Two possible configurations of $\sigma \in \mathcal{G}_P$ in Case 1 depending on whether $T^+_{\sigma_0} = \infty$ or not. Only the first item in Case 1 applies to the left figure.

Case 2. If $\sigma = \varphi_s \tilde{\sigma}_0$ for some $\tilde{\sigma}_0 \in \partial^- \mathcal{G}_P$ and $s \in (T^-_{\tilde{\sigma}_0}, 0)$, we write $T := T^-_{\tilde{\sigma}_0}$ and

- if $s \in \left[\frac{1}{3}T, 0\right)$, we set

$$\| \cdot \|_\sigma := e^{-cs} \| \cdot \|_{\tilde{\sigma}_0}.$$

- if $s \in \left(T, \frac{2}{3}T\right]$, we set

$$\| \cdot \|_\sigma := e^{c(T-s)} \| \cdot \|_{\varphi_T \tilde{\sigma}_0}.$$

- if $s \in \left(\frac{2}{3}T, \frac{1}{3}T\right)$, we set

$$\| \cdot \|_\sigma := \| \cdot \|^{\frac{3}{T}s - 1}_{\varphi_{2T/3}\tilde{\sigma}_0} \| \cdot \|^{2 - \frac{3}{T}s}_{\varphi_{T/3}\tilde{\sigma}_0}.$$

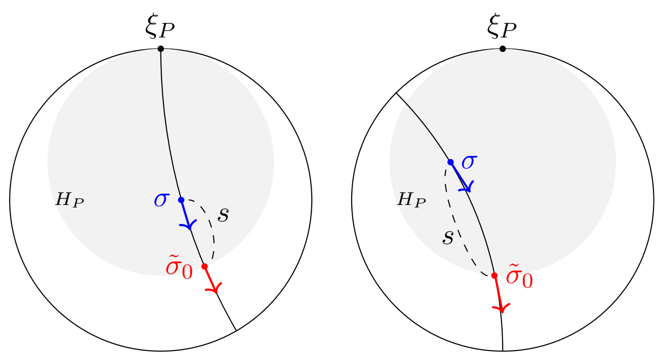

FIGURE 2. Two possible configurations of $\sigma \in \mathcal{G}_P$ in Case 2 depending on whether $T^-_{\tilde{\sigma}_0} = -\infty$ or not. Only the first item in Case 2 applies to the left figure.

Note that both cases can happen at the same time, and in that case two definitions coincide. The resulting family of norms is continuous and Γ-equivariant.

Contraction rate along geodesic flow. For $\sigma \in \mathcal{G}$, there exists a unique $v_\sigma \in \mathbb{R}_+$ such that $\|v_\sigma\|_\sigma = 1$. For $t \in \mathbb{R}$, we define

$$\kappa_t(\sigma) := \|v_\sigma\|_{\varphi_t \sigma}; \tag{6.3}$$

this measures the contraction rates of norms under the geodesic flow. It is easy to see that for $\sigma \in \mathcal{G}$ and $t, s \in \mathbb{R}$, we have

$$v_{\varphi_t \sigma} = \frac{v_\sigma}{\|v_\sigma\|_{\varphi_t \sigma}} \quad \text{and} \quad \kappa_{t+s}(\sigma) = \kappa_s(\varphi_t \sigma)\kappa_t(\sigma). \tag{6.4}$$

Moreover, $\kappa_t(\cdot)$ is Γ-invariant.

Lemma 6.3. *For $\sigma \in \mathcal{G}$, $t \in \mathbb{R}$, and $\gamma \in \Gamma$, we have*

$$\kappa_t(\gamma\sigma) = \kappa_t(\sigma).$$

Proof. By the Γ-equivariance of the norm, we have

$$1 = \|v_\sigma\|_\sigma = \left\| v_\sigma e^{\psi(\beta^\theta_{\sigma^+}(\gamma^{-1}, e))} \right\|_{\gamma\sigma}.$$

This implies

$$v_{\gamma\sigma} = v_\sigma e^{\psi(\beta^\theta_{\sigma^+}(\gamma^{-1}, e))}. \tag{6.5}$$

Since $\varphi_t \gamma\sigma = \gamma\varphi_t\sigma$, we have

$$\begin{aligned} \kappa_t(\gamma\sigma) &= \|v_{\gamma\sigma}\|_{\varphi_t\gamma\sigma} = \|v_\sigma\|_{\gamma\varphi_t\sigma} e^{\psi(\beta^\theta_{\sigma^+}(\gamma^{-1}, e))} \\ &= \left\| v_\sigma e^{\psi(\beta^\theta_{\gamma\sigma^+}(\gamma, e))} \right\|_{\varphi_t\sigma} e^{\psi(\beta^\theta_{\sigma^+}(\gamma^{-1}, e))} \\ &= \|v_\sigma\|_{\varphi_t\sigma} = \kappa_t(\sigma) \end{aligned}$$

as desired. ∎

The following is the desired estimate on the contraction rate:

Theorem 6.4. *There exists $b > 1$ such that for all $\sigma \in \mathcal{G}$ and $t \geq 0$, we have*

$$\frac{1}{b} e^{-a't} \leq \kappa_t(\sigma) \leq b e^{-at}$$

where $a = \liminf_{\gamma \in \Gamma} \frac{\psi(\mu_\theta(\gamma))}{d_{GM}(e,\gamma)}$ and $a' = 3 \limsup_{\gamma \in \Gamma} \frac{\psi(\mu_\theta(\gamma))}{d_{GM}(e,\gamma)}$.

We begin by observing that the recurrence to a compact subset implies the exponential contraction:

Lemma 6.5. *For any compact subset $Q \subset X_{GM}$, there exists $C_Q > 1$ such that if $\sigma \in \mathcal{G}$, $t \geq 0$, and $\gamma \in \Gamma$ satisfy $\sigma(0), \gamma^{-1}\sigma(t) \in Q$, then*

$$\frac{1}{C_Q} e^{-\psi(\mu_\theta(\gamma))} \leq \kappa_t(\sigma) \leq C_Q e^{-\psi(\mu_\theta(\gamma))}.$$

Proof. Suppose not. Then there exist sequences $\sigma_n \in \mathcal{G}$, $t_n \geq 0$, and $\gamma_n \in \Gamma$ such that $\sigma_n(0), \gamma_n^{-1}\sigma_n(t_n) \in Q$ for all $n \geq 1$ while the sequence

$$\log\left(\kappa_{t_n}(\sigma_n)e^{\psi(\mu_\theta(\gamma_n))}\right) = \psi(\mu_\theta(\gamma_n)) + \log \kappa_{t_n}(\sigma_n) \quad \text{is unbounded.} \tag{6.6}$$

In particular, γ_n is an infinite sequence and $t_n \to \infty$ as $n \to \infty$.

By the hypothesis that $\sigma_n(0), \gamma_n^{-1}\sigma_n(t_n) \in Q$, there exist $q \in Q$ and $R > 0$ depending on Q so that we have $\sigma_n^+ \in O_R^{GM}(q, \gamma_n q)$ for all $n \geq 1$. It follows from Proposition 5.7 that for some $r > 0$, we have $\sigma_n^+ \in O_r^\theta(o, \gamma_n o)$ for all $n \geq 1$. By Lemma 5.4, we deduce from (6.6) that the sequence

$$\psi\left(\beta^\theta_{\sigma_n^+}(e, \gamma_n)\right) + \log \kappa_{t_n}(\sigma_n) \quad \text{is unbounded.} \tag{6.7}$$

On the other hand, by the Γ-equivariance of the norms $\|\cdot\|$, we have

$$\begin{aligned}\kappa_{t_n}(\sigma_n) = \|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n} &= \left\| v_{\sigma_n} e^{\psi\left(\beta^\theta_{\sigma_n^+}(\gamma_n, e)\right)} \right\|_{\gamma_n^{-1}\varphi_{t_n}\sigma_n} \\ &= e^{\psi\left(\beta^\theta_{\sigma_n^+}(\gamma_n, e)\right)} \|v_{\sigma_n}\|_{\gamma_n^{-1}\varphi_{t_n}\sigma_n}\end{aligned}$$

and therefore

$$\psi\left(\beta^\theta_{\sigma_n^+}(e, \gamma_n)\right) + \log \kappa_{t_n}(\sigma_n) = \log \|v_{\sigma_n}\|_{\gamma_n^{-1}\varphi_{t_n}\sigma_n}. \tag{6.8}$$

Since both $\sigma_n(0)$ and $\gamma_n^{-1}\sigma_n(t_n) = (\gamma_n^{-1}\varphi_{t_n}\sigma_n)(0)$ belong to the compact subset Q for all $n \geq 1$, there exists a compact subset of $\mathcal{G}$ containing σ_n and $\gamma_n^{-1}\varphi_{t_n}\sigma_n$ for all $n \geq 1$. Therefore, the sequence (6.8) is uniformly bounded, which contradicts (6.7). Hence the claim follows. □

We obtain the following estimate of the contraction rate between the entrance and exit of a horoball.

Corollary 6.6. *There exists a constant $c_0 \geq 1$ such that if $\sigma \in \partial^+\mathcal{G}_P$ for some $P \in \mathcal{P}^\Gamma$ with $T_\sigma^+ < \infty$, then*

$$\frac{1}{c_0} e^{-c' T_\sigma^+} \leq \kappa_{T_\sigma^+}(\sigma) \leq c_0 e^{-c T_\sigma^+}$$

where c and c' are given by Theorem 5.6.

Proof. Let $P \in \mathcal{P}^\Gamma$ and $\sigma \in \partial^+\mathcal{G}_P$ with $T_\sigma^+ < \infty$. By Lemma 6.3, we may assume that $P \in \mathcal{P}$ and $\sigma(0) = (e, 2)$ in the combinatorial horoball attached to a Cayley graph of P. We then have $\sigma(T_\sigma^+) = (\gamma, 2)$ for some $\gamma \in P$. Setting $Q = \overline{B_{GM}(e, 1)}$ which is a compact subset of X_{GM}, we have $\sigma(0), \gamma^{-1}\sigma(T_\sigma^+) \in Q$. Hence by Lemma 6.5, we have

$$\frac{1}{C_Q} e^{-\psi(\mu_\theta(\gamma))} \leq \kappa_{T_\sigma^+}(\sigma) \leq C_Q e^{-\psi(\mu_\theta(\gamma))}$$

where C_Q is the constant therein. On the other hand, it follows from Theorem 5.6 that

$$\begin{aligned}\psi(\mu_\theta(\gamma)) &\geq c d_{GM}(e,\gamma) - C \\ &\geq c(d_{GM}((e,2),(\gamma,2)) - 2) - C \\ &= cT_\sigma^+ - (2c + C)\end{aligned}$$

with the constants c, C in Theorem 5.6. Therefore, we have

$$\kappa_{T_\sigma^+}(\sigma) \leq C_Q e^{2c+C} e^{-cT_\sigma^+}.$$

Similarly, we have

$$\begin{aligned}\psi(\mu_\theta(\gamma)) &\leq c' d_{GM}(e,\gamma) + C \\ &\leq c'(d_{GM}((e,2),(\gamma,2)) + 2) + C \\ &= c'T_\sigma^+ + (2c' + C)\end{aligned}$$

where c' is given in Theorem 5.6. Therefore, we have

$$\kappa_{T_\sigma^+}(\sigma) \geq \frac{1}{C_Q} e^{-(2c'+C)} e^{-c'T_\sigma^+}.$$

This finishes the proof. $\square$

We now estimate the contraction rate in the thin part.

Lemma 6.7. *There exists a constant $c_1 \geq 1$ with the following property: if $\sigma \in \mathcal{G}_{thin}$ is such that $\varphi_s\sigma \in \mathcal{G}_{thin}$ for all $0 \leq s \leq t$, then*

$$c_1^{-1} e^{-(3c'-2c)t} \leq \kappa_t(\sigma) \leq c_1 e^{-ct}$$

where $c \leq c'$ are given by Theorem 5.6.

Proof. We fix $\sigma \in \mathcal{G}_{thin}$ such that $\varphi_s\sigma \in \mathcal{G}_{thin}$ for all $0 \leq s \leq t$. Then there exists $P \in \mathcal{P}^\Gamma$ so that $\varphi_s\sigma \in \mathcal{G}_P$ for all $0 \leq s \leq t$. There are three cases to consider:

Case 1. Suppose that $\sigma([0,\infty)) \subset \mathcal{G}_P$. Then $\sigma = \varphi_s\sigma_0$ for some $\sigma_0 \in \partial^+\mathcal{G}_P$ and $s > 0$. In this case, by the definition of the norm, we have

$$\|\cdot\|_{\varphi_t\sigma} = \|\cdot\|_{\varphi_{t+s}\sigma_0} = e^{-c(t+s)}\|\cdot\|_{\sigma_0} = e^{-ct}\|\cdot\|_\sigma.$$

This implies $\kappa_t(\sigma) = e^{-ct}$.

Case 2. Suppose that $\sigma((-\infty,0]) \subset \mathcal{G}_P$. Then $\sigma = \varphi_s\tilde{\sigma}_0$ for some $\tilde{\sigma}_0 \in \partial^-\mathcal{G}_P$ and $s < 0$. We then have

$$\|\cdot\|_{\varphi_t\sigma} = e^{-c(s+t)}\|\cdot\|_{\tilde{\sigma}_0} = e^{-ct}\|\cdot\|_\sigma,$$

and hence $\kappa_t(\sigma) = e^{-ct}$.

Case 3. Suppose that neither $\sigma([0,\infty)) \subset \mathcal{G}_P$ nor $\sigma((-\infty,0]) \subset \mathcal{G}_P$ holds. In this case, we have $\sigma = \varphi_s\sigma_0$ for some $s > 0$ and $\sigma_0 \in \partial^+\mathcal{G}_P$ such that $T_{\sigma_0}^+ < \infty$. We simply write $T := T_{\sigma_0}^+$ and $\sigma_1 = \varphi_T\sigma_0$. We first consider the following three subcases:

- if $s, s+t \in \left(0, \frac{1}{3}T\right]$, then
$$\|\cdot\|_{\varphi_t\sigma} = \|\cdot\|_{\varphi_{s+t}\sigma_0} = e^{-c(s+t)}\|\cdot\|_{\sigma_0} = e^{-ct}\|\cdot\|_\sigma,$$
and hence $\kappa_t(\sigma) = e^{-ct}$.
- if $s, s+t \in \left[\frac{2}{3}T, T\right)$, then
$$\|\cdot\|_{\varphi_t\sigma} = e^{c(T-(t+s))}\|\cdot\|_{\sigma_1} = e^{-ct}\|\cdot\|_\sigma,$$
and hence $\kappa_t(\sigma) = e^{-ct}$.
- if $s, s+t \in \left[\frac{1}{3}T, \frac{2}{3}T\right]$, then we first observe that
$$\begin{aligned}\|\cdot\|_\sigma &= \|\cdot\|_{\varphi_{T/3}\sigma_0}^{2-\frac{3}{T}s}\|\cdot\|_{\varphi_{2T/3}\sigma_0}^{\frac{3}{T}s-1} \\ &= \left(e^{-c\frac{T}{3}}\|\cdot\|_{\sigma_0}\right)^{2-\frac{3}{T}s}\left(e^{c\frac{T}{3}}\|\cdot\|_{\sigma_1}\right)^{\frac{3}{T}s-1} \\ &= e^{c(2s-T)}\|\cdot\|_{\sigma_0}^{2-\frac{3}{T}s}\|\cdot\|_{\sigma_1}^{\frac{3}{T}s-1}\end{aligned}$$
and similarly that
$$\|\cdot\|_{\varphi_t\sigma} = e^{c(2(s+t)-T)}\|\cdot\|_{\sigma_0}^{2-\frac{3}{T}(s+t)}\|\cdot\|_{\sigma_1}^{\frac{3}{T}(s+t)-1}.$$
Combining the above two computations, we obtain
$$\|\cdot\|_{\varphi_t\sigma} = \|\cdot\|_\sigma e^{2ct}\|\cdot\|_{\sigma_0}^{-\frac{3}{T}t}\|\cdot\|_{\sigma_1}^{\frac{3}{T}t}.$$
Evaluating at v_{σ_0}, the above becomes
$$\kappa_{t+s}(\sigma_0) = \kappa_s(\sigma_0)e^{2ct}\kappa_T(\sigma_0)^{\frac{3}{T}t}.$$
Since $\kappa_{t+s}(\sigma_0) = \kappa_t(\sigma)\kappa_s(\sigma_0)$ by (6.4), it follows from Corollary 6.6 and $0 \le t \le \frac{T}{3}$ that
$$\begin{aligned}\kappa_t(\sigma) &= e^{2ct}\kappa_T(\sigma_0)^{\frac{3}{T}t} \\ &\le e^{2ct}(c_0e^{-cT})^{\frac{3}{T}t} = e^{2ct}c_0^{\frac{3}{T}t}e^{-3ct} \\ &\le \max(1, c_0)e^{-ct}.\end{aligned}$$

Similarly, we also obtain from Corollary 6.6 and $0 \le t \le \frac{T}{3}$ that
$$\begin{aligned}\kappa_t(\sigma) &= e^{2ct}\kappa_T(\sigma_0)^{\frac{3}{T}t} \\ &\ge e^{2ct}(c_0^{-1}e^{-c'T})^{\frac{3}{T}t} = e^{2ct}c_0^{\frac{-3}{T}t}e^{-3c't} \\ &\ge \min(1, c_0^{-1})e^{-(3c'-2c)t}.\end{aligned}$$

We now set $c_1 := \max(1, c_0)$. Note also that $c' \ge c$ and hence $e^{-(3c'-2c)t} \le e^{-ct}$ for all $t \ge 0$. In general, we consider the following three consecutive subintervals
$$[s, s+t] \cap \left(0, \tfrac{1}{3}T\right], \quad [s, s+t] \cap \left[\tfrac{1}{3}T, \tfrac{2}{3}T\right], \quad \text{and} \quad [s, s+t] \cap \left[\tfrac{2}{3}T, T\right],$$

and then apply the each of the above three subcases to each subintervals. Then by (6.4), we get

$$c_1^{-1}e^{-(3c'-2c)t} \leq \kappa_t(\sigma) \leq c_1 e^{-ct}$$

as desired. □

We now combine estimates on the thick and thin parts and prove Theorem 6.4. We give proofs of the lower bound and the upper bound separately:

Proof of the lower bound in Theorem 6.4. Let $\sigma \in \mathcal{G}$ and $t \geq 0$. If $\varphi_s\sigma \in \mathcal{G}_{thin}$ for all $0 \leq s \leq t$, then by Lemma 6.7, we have

$$\kappa_t(\sigma) \geq c_1^{-1}e^{-(3c'-2c)t} \tag{6.9}$$

where constants c_1, c', c are given in Lemma 6.7. Now suppose that $\varphi_s\sigma \in \mathcal{G}_{thick}$ for some $s \in [0,t]$ and set

$$\begin{aligned} s_1 &:= \min\{s \in [0,t] : \varphi_s\sigma \in \mathcal{G}_{thick}\}; \\ s_2 &:= \max\{s \in [0,t] : \varphi_s\sigma \in \mathcal{G}_{thick}\} \end{aligned}$$

which are well-defined. It follows from (6.4) and Lemma 6.7 that

$$\begin{aligned} \kappa_t(\sigma) &= \kappa_{t-s_2}(\varphi_{s_2}\sigma)\kappa_{s_2}(\sigma) \\ &= \kappa_{t-s_2}(\varphi_{s_2}\sigma)\kappa_{s_2-s_1}(\varphi_{s_1}\sigma)\kappa_{s_1}(\sigma) \\ &\geq c_1^{-1}e^{-(3c'-2c)(t-s_2)}\kappa_{s_2-s_1}(\varphi_{s_1}\sigma)c_1^{-1}e^{-(3c'-2c)s_1} \\ &= c_1^{-2}e^{-(3c'-2c)t}e^{(3c'-2c)(s_2-s_1)}\kappa_{s_2-s_1}(\varphi_{s_1}\sigma). \end{aligned} \tag{6.10}$$

To estimate $\kappa_{s_2-s_1}(\varphi_{s_1}\sigma)$, we fix a compact fundamental domain $Q \subset X_{GM} - \bigcup_{P\in\mathcal{P}^\Gamma} H_P$ for the Γ-action. We may assume that $e \in Q$. By the definition of s_1 and s_2, there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $(\varphi_{s_1}\sigma)(0) \in \gamma_1 Q$ and $(\varphi_{s_2}\sigma)(0) \in \gamma_2 Q$. In other words, we have $(\gamma_1^{-1}\varphi_{s_1}\sigma)(0) \in Q$ and $(\gamma_1^{-1}\varphi_{s_2}\sigma)(0) \in \gamma_1^{-1}\gamma_2 Q$. Since $(\gamma_1^{-1}\varphi_{s_1}\sigma)(0) = \gamma_1^{-1}\sigma(s_1)$ and $(\gamma_1^{-1}\varphi_{s_2}\sigma)(0) = \gamma_1^{-1}\sigma(s_2)$, this implies that for some constant $q > 0$ depending on Q, we have $|d_{GM}(e, \gamma_1^{-1}\gamma_2) - (s_2 - s_1)| \leq q$. Setting $\gamma := \gamma_1^{-1}\gamma_2$, this is rephrased as

$$|d_{GM}(e, \gamma) - (s_2 - s_1)| \leq q. \tag{6.11}$$

Moreover, noting that $(\varphi_{s_2}\sigma)(0) = (\varphi_{s_1}\sigma)(s_2 - s_1)$, we have

$$(\gamma_1^{-1}\varphi_{s_1}\sigma)(0), \gamma^{-1}(\gamma_1^{-1}\varphi_{s_1}\sigma)(s_2 - s_1) \in Q.$$

Hence, by Lemma 6.3 and Lemma 6.5, we have

$$\begin{aligned} \kappa_{s_2-s_1}(\varphi_{s_1}\sigma) &= \kappa_{s_2-s_1}(\gamma_1^{-1}\varphi_{s_1}\sigma) \\ &\geq \frac{1}{C_Q}e^{-\psi(\mu_\theta(\gamma))} \end{aligned}$$

with the constant C_Q given by Lemma 6.5. By Theorem 5.6 and (6.11), we deduce

$$\kappa_{s_2-s_1}(\varphi_{s_1}\sigma) \geq \frac{1}{C_Q}e^{-c'd_{GM}(e,\gamma)-C} \geq \frac{e^{-c'q-C}}{C_Q}e^{-c'(s_2-s_1)}.$$

Together with (6.10), we have
(6.12)

$$\begin{aligned}\kappa_t(\sigma) &\geq c_1^{-2} e^{-(3c'-2c)t} e^{(3c'-2c)(s_2-s_1)} \frac{e^{-c'q-C}}{C_Q} e^{-c'(s_2-s_1)} \\ &= \frac{c_1^{-2} e^{-c'q-C}}{C_Q} e^{-(3c'-2c)t} e^{(2c'-2c)(s_2-s_1)} \geq \frac{c_1^{-2} e^{-c'q-C}}{C_Q} e^{-(3c'-2c)t}\end{aligned}$$

where the last inequality is due to $c' \geq c$ and $s_2 \geq s_1$.

Now note that $a' \geq 3c' - 2c$ by Theorem 5.6 and choose $b > 1$ such that $b^{-1} \leq \min\left(c_1^{-1}, \frac{c_1^{-2} e^{-c'q-C}}{C_Q}\right)$. Then it follows from (6.9) and (6.12) that

$$\kappa_t(\sigma) \geq \frac{1}{b} e^{-a't}$$

as desired. □

Proof of the upper bound in Theorem 6.4. Let $\sigma \in \mathcal{G}$ and $t \geq 0$. If $\varphi_s \sigma \in \mathcal{G}_{thin}$ for all $0 \leq s \leq t$, then by Lemma 6.7, we have

$$\kappa_t(\sigma) \leq c_1 e^{-ct} \tag{6.13}$$

where c_1 and c are constants given in Lemma 6.7. We now assume that $\varphi_s \sigma \in \mathcal{G}_{thick}$ for some $s \in [0, t]$. As in the proof of the lower bound, we set

$$\begin{aligned}s_1 &:= \min\{s \in [0,t] : \varphi_s \sigma \in \mathcal{G}_{thick}\}; \\ s_2 &:= \max\{s \in [0,t] : \varphi_s \sigma \in \mathcal{G}_{thick}\}\end{aligned}$$

We then have from (6.4) and Lemma 6.7 that

$$\begin{aligned}\kappa_t(\sigma) &= \kappa_{t-s_2}(\varphi_{s_2}\sigma)\kappa_{s_2-s_1}(\varphi_{s_1}\sigma)\kappa_{s_1}(\sigma) \\ &\leq c_1^2 e^{-ct} e^{c(s_2-s_1)} \kappa_{s_2-s_1}(\varphi_{s_1}\sigma).\end{aligned} \tag{6.14}$$

By the similar argument as in the proof of the lower bound, we have

$$\kappa_{s_2-s_1}(\varphi_{s_1}\sigma) \leq C_Q e^{-\psi(\mu_\theta(\gamma))}$$

where $Q \subset X_{GM} - \bigcup_{P \in \mathcal{P}^\Gamma} H_P$ is a compact fundamental domain for the Γ-action, C_Q is the constant given by Lemma 6.5, and $\gamma \in \Gamma$ is such that $|d_{GM}(e, \gamma) - (s_2 - s_1)| \leq q$ for some constant $q \geq 0$ depending only on Q. By Theorem 5.6, this implies

$$\kappa_{s_2-s_1}(\varphi_{s_1}\sigma) \leq C_Q e^{-c d_{GM}(e,\gamma)+C} \leq C_Q e^{cq+C} e^{-c(s_2-s_1)}$$

with the constant C therein. Plugging this into (6.14), we have

$$\kappa_t(\sigma) \leq c_1^2 C_Q e^{cq+C} e^{-ct}. \tag{6.15}$$

We then choose $b \geq \max\left(c_1, c_1^2 C_Q e^{cq+C}\right)$. By (6.13) and (6.15), we finally obtain

$$\kappa_t(\sigma) \leq b e^{-ct}.$$

Since $a = c$ by Theorem 5.6, this completes the proof. □

Proof of Theorem 6.1. As described above, we define the Γ-equivariant continuous section $u : \mathcal{G} \to \mathcal{G} \times \mathbb{R}_+$ by setting $u(\sigma) = (\sigma, v_\sigma)$, and set $\tilde{\Psi} = \Psi_0 \circ u$ so that we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{G} \times \mathbb{R}_+ \arrow[d] \arrow[dr, "\Psi_0"] & \\ \mathcal{G} \arrow[u, bend left, dashed, "u"] \arrow[r, dashed, "\tilde{\Psi}"'] & \tilde{\Omega}_\psi \end{tikzcd}$$

In other words, $\tilde{\Psi}(\sigma) = (\sigma^+, \sigma^-, \log v_\sigma)$.

We first prove that $\tilde{\Psi}$ is proper, from which the properness of Ψ follows. Suppose not. Then there exists a sequence $\sigma_n \in \mathcal{G}$ such that σ_n escapes every compact subset of $\mathcal{G}$ as $n \to \infty$ while $\tilde{\Psi}(\sigma_n) = (\sigma_n^+, \sigma_n^-, \log v_{\sigma_n})$ converges in $\tilde{\Omega}_\psi$. Since the sequence (σ_n^+, σ_n^-) converges in $\Lambda_\theta^{(2)}$, two sequences σ_n^+ and σ_n^- converge to two distinct points in ∂X_{GM}. This implies that there exist a sequence $t_n \in \mathbb{R}$ and a compact subset $Q \subset \mathcal{G}$ so that $\varphi_{t_n} \sigma_n \in Q$ for all $n \geq 1$. Moreover, since the sequence $\tilde{\Psi}(\sigma_n) = (\sigma_n^+, \sigma_n^-, \log v_{\sigma_n})$ converges in $\tilde{\Omega}_\psi$, the sequence v_{σ_n} converges in $\mathbb{R}_+$. This implies that, after passing to a subsequence,

(6.16) the sequence $\|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n}$ converges to a positive number.

On the other hand, since the sequence σ_n escapes any compact subset of $\mathcal{G}$ as $n \to \infty$, we have either $t_n \to \infty$ or $t_n \to -\infty$ as $n \to \infty$, after passing to a subsequence. Suppose first that $t_n \to \infty$ as $n \to \infty$. It follows from Theorem 6.4 that for all sufficiently large $n \geq 1$,

$$\|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n} = \kappa_{t_n}(\sigma_n) \leq b e^{-a t_n} \to 0 \quad \text{as } n \to \infty.$$

This contradicts (6.16). We now assume that $t_n \to -\infty$ as $n \to \infty$. Then for all sufficiently large $n \geq 1$, we have

$$\|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n} = \frac{1}{\|v_{\varphi_{t_n}\sigma_n}\|_{\sigma_n}} = \frac{1}{\kappa_{-t_n}(\varphi_{t_n}\sigma_n)} \geq b^{-1} e^{-a t_n}$$

by (6.4) and Theorem 6.4. Therefore, $\|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n} \to \infty$ as $n \to \infty$, contradicting (6.16). This proves the properness.

We now prove items (1), (2), and (3). Since the Γ-action on $\mathcal{G}$ and $\tilde{\Omega}_\psi$ commute with flows on $\mathcal{G}$ and $\tilde{\Omega}_\psi$, it suffices to prove the statement for $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$. For $(\sigma, s) \in \mathcal{G} \times \mathbb{R} \to \mathbb{R}$, define a continuous function

$$\tilde{\mathfrak{t}}(\sigma, s) := \log v_{\varphi_s \sigma} - \log v_\sigma.$$

By (6.4), we have

$$v_{\varphi_s\sigma} = \frac{v_\sigma}{\|v_\sigma\|_{\varphi_s\sigma}} = \frac{v_\sigma}{\kappa_s(\sigma)}.$$

Therefore

$$\tilde{\mathfrak{t}}(\sigma, s) = -\log \kappa_s(\sigma), \tag{6.17}$$

is Γ-invariant (Lemma 6.3) and hence induces a continuous map $\mathtt{t} : \Gamma\backslash\mathcal{G} \times \mathbb{R} \to \mathbb{R}$. The cocycle property of $\tilde{\mathtt{t}}$ follows from (6.4). By the definition of $\tilde{\Psi}$, we have

$$\tilde{\Psi}(\varphi_s \sigma) = \phi_{\tilde{\mathtt{t}}(\sigma,s)} \tilde{\Psi}(\sigma),$$

from which (1) follows. This also implies (2), noting that

$$\phi_{-\tilde{\mathtt{t}}(\sigma,s)} \tilde{\Psi}(\varphi_s \sigma) = \tilde{\Psi}(\sigma) = \tilde{\Psi}(\varphi_{-s}\varphi_s \sigma) = \phi_{\tilde{\mathtt{t}}(\varphi_s\sigma,-s)}(\varphi_s \sigma).$$

Moreover, by Theorem 6.4 and (6.17), we have that for all $s \geq 0$,

$$as - \log b \leq \tilde{\mathtt{t}}(\sigma, s) \leq a's + \log b \tag{6.18}$$

where $a, a' > 0$ and $b \geq 1$ are given in Theorem 6.4. This shows (3).

To see the surjectivity of $\tilde{\Psi}$, note first that for each $(\xi, \eta, t_0) \in \tilde{\Omega}_\psi$, there exists $\sigma \in \mathcal{G}$ with $\sigma^+ = \xi$ and $\sigma^- = \eta$ as X_{GM} is a proper geodesic Gromov hyperbolic space. For $s_0 \geq 0$, it follows from (6.18) that

$$\tilde{\mathtt{t}}(\sigma, s_0) \geq as_0 - \log b \quad \text{and} \quad \tilde{\mathtt{t}}(\varphi_{-s_0}\sigma, s_0) \geq as_0 - \log b.$$

Since $\tilde{\mathtt{t}}(\varphi_{-s_0}\sigma, s_0) = -\tilde{\mathtt{t}}(\sigma, -s_0)$ due to the cocycle property (6.4), we have

$$\tilde{\mathtt{t}}(\sigma, s_0) \geq as_0 - \log b \quad \text{and} \quad \tilde{\mathtt{t}}(\sigma, -s_0) \leq -as_0 + \log b.$$

Since $\tilde{\Psi}$ is continuous, this implies that the image of $\tilde{\Psi}$ restricted on $\{\varphi_s\sigma : -s_0 \leq s \leq s_0\}$ contains $\{\phi_t \tilde{\Psi}(\sigma) : -as_0 + \log b \leq t \leq as_0 - \log b\}$. Since $\sigma^+ = \xi$ and $\sigma^- = \eta$, $\tilde{\Psi}(\sigma) = (\xi, \eta, t_1)$ for some $t_1 \in \mathbb{R}$. We then take s_0 large enough so that

$$-as_0 + \log b + t_1 \leq t_0 \leq as_0 - \log b + t_1.$$

Then $(\xi, \eta, t_0) \in \{\phi_t \tilde{\Psi}(\sigma) : -as_0 + \log b \leq t \leq as_0 - \log b\}$, and hence (ξ, η, t_0) belongs to the image of $\tilde{\Psi}$. Therefore, $\tilde{\Psi}$ is surjective. This completes the proof. $\square$

7. Uniformity of fibers of reparameterization

Recall the reparameterization $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$ constructed in section 6. The main goal of this section is to establish a uniform bound on the diameters of the fibers of $\tilde{\Psi}$:

Theorem 7.1 (Theorem 1.4(4)). *The fibers of $\tilde{\Psi}$ have uniformly bounded diameter. That is, there exists $C > 0$ such that for any $\sigma, \sigma' \in \mathcal{G}$,*

$$\tilde{\Psi}(\sigma) = \tilde{\Psi}(\sigma') \Longrightarrow d_{GM}(\sigma(0), \sigma'(0)) < C.$$

We prove this result by analyzing the explicit form of our reparameterization. For $\sigma \in \mathcal{G}$,

$$\tilde{\Psi}(\sigma) = (\sigma^+, \sigma^-, \log v_\sigma)$$

where $v_\sigma \in \mathbb{R}_+$ is the unit vector with respect to the norm $\|\cdot\|_\sigma$, as constructed in section 6. Thus, Theorem 7.1 follows from the next proposition:

Proposition 7.2. *There exists a constant $C_0 > 0$ such that the following holds: for any $\sigma, \sigma' \in \mathcal{G}$ with $\sigma^{\pm} = \sigma'^{\pm}$, there exists $s \in \mathbb{R}$ such that*

$$d_{GM}(\sigma(0), \sigma'(s)) < C_0 \quad \textit{and} \quad |\log v_\sigma - \log v_{\varphi_s \sigma'}| < C_0.$$

Moreover, the shift parameter s satisfies:

- *if $s \geq 0$, then*
$$\frac{(\log v_\sigma - \log v_{\sigma'}) - C_0 - B}{a'} \leq s \leq \frac{(\log v_\sigma - \log v_{\sigma'}) + C_0 + B}{a}.$$
- *if $s < 0$, then*
$$\frac{(\log v_\sigma - \log v_{\sigma'}) - C_0 - B}{a} \leq s \leq \frac{(\log v_\sigma - \log v_{\sigma'}) + C_0 + B}{a'}.$$

Here $0 < a < a'$ and $B > 0$ are the constants appearing in Theorem 6.1.

To prove Proposition 7.2, we require several preparatory lemmas. We begin by recalling the definition of the *Gromov product* on $X_{GM} \cup \partial X_{GM}$. For $x, y, z \in X_{GM}$, define

$$(y|z)_x := \frac{1}{2}(d_{GM}(x,y) + d_{GM}(x,z) - d_{GM}(y,z)).$$

For $y, z \in X_{GM} \cup \partial X_{GM}$, define

$$(y|z)_x := \sup \liminf_{i,j \to \infty} (y_i|z_j)_x$$

where the supremum is taken over all sequences $y_i, z_j \in X_{GM}$ converging to y, z, respectively. By the Gromov hyperbolicity of X_{GM} (Theorem 5.1), the Gromov product $(y|z)_x$ estimates the distance from x to a geodesic $[y, z]$, up to a uniformly bounded additive error.

Lemma 7.3. *Let $\sigma_n \in \mathcal{G}$ be a sequence such that $\{\sigma_n(0) \in X_{GM} : n \geq 1\}$ is uniformly bounded. Then there do not exist sequences $T_n, S_n > 0$ tending to ∞ such that both $\sigma_n(T_n)$ and $\sigma_n(-S_n)$ lie in the same horoball $\overline{H_P}$ for some $P \in \mathcal{P}$.*

Proof. Suppose such sequences exist. Then since $\sigma_n^{\pm}$ belong to the shadows $O_1^{GM}(\sigma_n(0), \sigma_n(T_n))$ and $\sigma_n^- \in O_1^{GM}(\sigma_n(0), \sigma_n(-S_n))$, and $\{\sigma_n(0) : n \geq 1\}$ is bounded, we must have $\lim_{n\to\infty} \sigma_n^{\pm} = \xi_P$. On the other hand, the boundedness of $\sigma_n(0)$ implies that $\{\sigma_n\}$ is relatively compact, yielding a contradiction. $\square$

It is a standard fact in the Gromov hyperbolic geometry (cf. [8, Theorem III.H.1.7]) that there exists a constant $c_0 > 0$ such that any two geodesics with same endpoints have Hausdorff distance at most c_0.

Lemma 7.4. *There exists $T'_{\mathsf{h}} > 0$ such that for each $P \in \mathcal{P}^\Gamma$ and $\sigma \in \partial^+ \mathcal{G}_P$ with $T_\sigma^+ > 3T'_{\mathsf{h}}$, the c_0-neighborhood of the segment $\sigma([T'_{\mathsf{h}}, T_\sigma^+ - T'_{\mathsf{h}}])$ is entirely contained in H_P.*

Proof. Suppose not. Since $\mathcal{P}$ is finite, there exist $P \in \mathcal{P}$ and sequences $\sigma_n \in \partial^+\mathcal{G}_P$ with $T^+_{\sigma_n} > 3n$ and $t_n \in [n, T^+_{\sigma_n} - n]$ such that $\sigma_n(t_n)$ is not contained in the c_0-neighborhood of H_P. Hence there exists $p_n \in P$ such that $d_{GM}(\sigma_n(t_n), (p_n, 2)) < c_0$. Replacing σ_n with $p_n^{-1}\sigma_n$, we may assume that $p_n = e$, so $\sigma_n(t_n)$ lies in a fixed bounded neighborhood of $(e, 2)$. Applying Lemma 7.3 to $\varphi_{t_n}\sigma_n$ with $T_n = T^+_{\sigma_n} - t_n$ and $S_n = t_n$ yields a contradiction. □

Lemma 7.5. *There exists $\tilde{T} > 0$ such that for any $P \in \mathcal{P}^\Gamma$ and $\sigma \in \partial^+\mathcal{G}_P$ with $\sigma^+ = \xi_P$, we have $\sigma(t) \in H_P$ for all $t > \tilde{T}$.*

Proof. Suppose not. As in the proof of Lemma 7.4, for some $P \in \mathcal{P}$, there exist $\sigma_n \in \partial^+\mathcal{G}_P$ with $\sigma_n^+ = \xi_P$ and $t_n > n$ such that $\sigma_n(t_n) = (e, 2)$. Since $\sigma_n^+ = \xi_P$, there exist $T_n > n + t_n$ such that $\sigma_n(T_n) \in H_P$ and $\sigma_n(0) \in \partial H_P$. Applying Lemma 7.3 to $\varphi_{t_n}\sigma_n$ gives a contradiction. □

Let $T'_{\mathsf{h}}, \tilde{T} > 0$ be constants given in Lemma 7.4 and Lemma 7.5 respectively.

Lemma 7.6. *There exists $T_{\mathsf{h}} > T'_{\mathsf{h}} + \tilde{T} + c_0 + 2$ with the following property: let $P \in \mathcal{P}^\Gamma$, $\sigma \in \partial^+\mathcal{G}_P$ with $T^+_\sigma > 5T_{\mathsf{h}}$, and $t \in [2T_{\mathsf{h}}, T^+_\sigma - 2T_{\mathsf{h}}]$. Suppose $\sigma' \in \partial^+\mathcal{G}_P$ satisfies $\sigma'^{\pm} = \sigma^{\pm}$ and $d_{GM}(\sigma'([0, T^+_{\sigma'}]), \sigma(t)) < c_0$. Then*

1. *$d_{GM}(\sigma(0), \sigma'(0)) < T_{\mathsf{h}}$;*
2. *$T^+_\sigma < \infty$ if and only if $T^+_{\sigma'} < \infty$, and in this case,*
$$d_{GM}(\sigma(T^+_\sigma), \sigma'(T^+_{\sigma'})) < T_{\mathsf{h}}.$$

Proof. Suppose that there exist $P \in \mathcal{P}$, $\sigma_n, \sigma'_n \in \partial^+\mathcal{G}_P$ with $T^+_{\sigma_n} > 5n$ and $\sigma_n(0) = (e, 2)$, $\sigma_n^\pm = \sigma_n'^\pm$, and $t_n \in [2n, T^+_{\sigma_n} - 2n]$, $s_n \in [0, T^+_{\sigma'_n}]$ such that

$$d_{GM}(\sigma_n(t_n), \sigma'_n(s_n)) < c_0 \quad \text{and} \quad d_{GM}(\sigma_n(0), \sigma'_n(0)) > n.$$

Since $\sigma_n(t_n) \in H_P$, $\sigma_n(0) = (e, 2)$, and $d_{GM}(\sigma_n(t_n), \sigma_n(0)) = t_n \to \infty$, we have $\sigma_n(t_n) \to \xi_P$ as $n \to \infty$. Write $\sigma'_n(0) = (p_n, 2)$ with $p_n \in P$. We claim that

$$d_{GM}(\sigma_n(t_n), \sigma'_n(0)) \to \infty; \tag{7.1}$$

if not, the sequence $p_n^{-1}\sigma_n(t_n)$ is contained in a fixed compact subset. Since $p_n^{-1}\sigma_n(T^+_{\sigma_n}), p_n^{-1}\sigma_n(0) \in \partial H_P$ and $T^+_{\sigma_n} - t_n, t_n \to +\infty$, this contradicts Lemma 7.3.

Let $s'_n \in \mathbb{R}$ be such that $d_{GM}(\sigma_n(0), \sigma'_n(s'_n)) < c_0$, which exists by the Gromov hyperbolicity.

We now divide the argument into two cases:

Case 1: $s'_n \geq 0$ for infinitely many n. Then the Gromov product $(\sigma'_n(0)|\sigma_n'^+)_{\sigma_n(0)}$ is uniformly bounded, passing to a subsequence. Since $\sigma_n(0) = (e, 2)$, it follows that after passing to a subsequence, $\sigma'_n(0) \to \xi$ and $\sigma_n'^+ \to \xi'$ with $\xi \neq \xi'$. But $\sigma'_n(0) = (p_n, 2)$ with $p_n \in P$, and since $d_{GM}(\sigma_n(0), \sigma'_n(0)) > n$, we conclude that $p_n \to \infty$ in P, hence $\sigma'_n(0) \to \xi_P$. On the other hand, since $\sigma_n'^+ = \sigma_n^+ \in O_1^{GM}((e, 2), \sigma_n(T^+_{\sigma_n}))$ and $\sigma_n(T^+_{\sigma_n}) =$

$(q_n, 2)$ with $q_n \to \infty$ in P, it follows from Lemma 5.9 that $\sigma_n'^{+} \to \xi_P$, This contradicts the distinctness $\xi \neq \xi'$.

Case 2: $s_n' < 0$ for all but finitely many $n \geq 1$. In this case, two geodesic segments $\sigma_n([0, t_n])$ and $\sigma_n'([s_n', s_n])$ have c_0-close endpoints. Hence, by Gromov hyperbolicity, there exists $t_n' \in [0, t_n]$ such that $\sigma_n(t_n')$ is uniformly close to $\sigma_n'(0)$. This implies that the Gromov product $(\sigma_n(0)|\sigma_n(t_n))_{\sigma_n'(0)} = (p_n^{-1}\sigma_n(0)|p_n^{-1}\sigma_n(t_n))_{p_n^{-1}\sigma_n'(0)}$ is uniformly bounded. It follows from $p_n \to \infty$ that $p_n^{-1}\sigma_n(0) = (p_n^{-1}, 2)$ converges to ξ_P, after passing to a subsequence. Since $p_n^{-1}\sigma_n'(0) = (e, 2)$, $p_n^{-1}\sigma_n(t_n)$ must converge to a point distinct from ξ_P. On the other hand, we have $p_n^{-1}\sigma_n(t_n) \in H_P$, and from (7.1), we know it diverges from $(e, 2)$, thus converging to ξ_P again, which is a contradiction.

Now let $T_{\mathsf{h}} > 0$ be the constant obtained from the first part. Let $P \in \mathcal{P}$, $\sigma \in \partial^+\mathcal{G}_P$ with $T_\sigma^+ > 5T_{\mathsf{h}}$, and $t \in [2T_{\mathsf{h}}, T_\sigma^+ - 2T_{\mathsf{h}}]$. Let $\sigma' \in \partial^+\mathcal{G}_P$ satisfy $\sigma'^{\pm} = \sigma^{\pm}$, and suppose that there exists $s \in [0, T_{\sigma'}^+]$ such that $d_{GM}(\sigma'(s), \sigma(t)) < c_0$. If $\sigma^+ = \sigma'^+ \neq \xi_P$, then both T_σ^+ and $T_{\sigma'}^+$ are finite. So it suffices to consider the case where $\sigma^+ = \sigma'^+ = \xi_P$. Since $T_\sigma^+ > 5T_{\mathsf{h}} > \tilde{T}$, Lemma 7.5 implies $T_\sigma^+ = \infty$. By the first part, we have $d_{GM}(\sigma(0), \sigma'(0)) < T_{\mathsf{h}}$, and since $t > 2T_{\mathsf{h}}$, we have $T_{\sigma'}^+ \geq s > t - T_{\mathsf{h}} - c_0 > T_{\mathsf{h}} - c_0 > \tilde{T}$, so Lemma 7.5 again implies $T_{\sigma'}^+ = \infty$. Finally, when $T_\sigma^+ < \infty$, and hence $T_{\sigma'}^+ < \infty$, we can apply the same argument to the time-reversed geodesics of $\varphi_{T_\sigma^+}\sigma$ and $\varphi_{T_{\sigma'}^+}\sigma'$, completing the proof. $\square$

Proof of Proposition 7.2. Fix two geodesics $\sigma, \sigma' \in \mathcal{G}$ with the same endpoints $\sigma^{\pm} = \sigma'^{\pm}$. Since the norm $\|\cdot\|_\sigma$ used to define v_σ depends on the position of $\sigma(0)$, we divide the proof into cases based on the geometry of $\sigma(0)$.

Case 1. Suppose that $\sigma(0)$ lies within $5T_{\mathsf{h}}$-neighborhood of the Cayley graph of Γ in X_{GM}. That is, $d_{GM}(\Gamma, \sigma(0)) < 5T_{\mathsf{h}}$. By the definition of $c_0 > 0$, we can find $s \in \mathbb{R}$ so that

$$d_{GM}(\sigma(0), \sigma'(s)) < c_0.$$

Let $\gamma \in \Gamma$ be such that $d_{GM}(\gamma\sigma(0), e) < 5T_{\mathsf{h}}$. Then both $\gamma\sigma(0)$ and $\gamma\sigma'(s)$ lie in the $(5T_{\mathsf{h}} + c_0)$-neighborhood of the identity. Hence the shifted geodesics $\gamma\sigma$ and $\gamma\varphi_s\sigma' = \varphi_s\gamma\sigma'$ lie in a uniformly compact subset of $\mathcal{G}$. Therefore, there exists a uniform constant $C_1 > 0$ such that

$$|\log v_{\gamma\sigma} - \log v_{\gamma\varphi_s\sigma'}| < C_1.$$

By the equivariance formula for $v_{\gamma\sigma}$ (see (6.5)), we have

$$\log v_{\gamma\sigma} = \log v_\sigma + \psi(\beta^{\theta}_{\sigma^+}(\gamma^{-1}, e))$$
$$\log v_{\gamma\varphi_s\sigma'} = \log v_{\varphi_s\sigma'} + \psi(\beta^{\theta}_{\sigma'^+}(\gamma^{-1}, e)).$$

Since $\sigma^+ = \sigma'^+$, the Busemann maps in both expressions coincide and we conclude

$$|\log v_\sigma - \log v_{\varphi_s\sigma'}| < C_1.$$

Choosing $C_0 > \max(c_0, C_1)$ completes the proof in this case.

Case 2. Suppose that $d_{GM}(\Gamma, \sigma(0)) > 5T_{\mathsf{h}}$, $\sigma(0) \in H_P$, and $\sigma^+ = \xi_P$ for some $P \in \mathcal{P}^\Gamma$. In this case, we can write $\sigma = \varphi_t \sigma_0$ for some $\sigma_0 \in \partial^+ \mathcal{G}_P$ and $t > 0$. By hypothesis, $t > 5T_{\mathsf{h}} > \tilde{T}$, and hence $T^+_{\sigma_0} = \infty$ by Lemma 7.5. Then

$$\|\cdot\|_\sigma = e^{-ct}\|\cdot\|_{\sigma_0} \tag{7.2}$$

where $c > 0$ is the constant defined in (6.2).

By the definition of $c_0 > 0$, there exists $s \in \mathbb{R}$ such that $d_{GM}(\sigma'(s), \sigma(0)) < c_0$. Since $t > 5T_{\mathsf{h}} > T'_{\mathsf{h}}$ and $T^+_{\sigma_0} = \infty$, Lemma 7.4 implies $\sigma'(s) \in H_P$. So we may write $\varphi_s \sigma' = \varphi_{t'} \sigma'_0$ for some $\sigma'_0 \in \partial^+ \mathcal{G}_P$ and $t' > 0$. Applying Lemma 7.6 to σ_0 and σ'_0, we obtain

$$d_{GM}(\sigma_0(0), \sigma'_0(0)) < T_{\mathsf{h}} \quad \text{and} \quad T^+_{\sigma'_0} = \infty. \tag{7.3}$$

This gives

$$\|\cdot\|_{\varphi_s \sigma'} = e^{-ct'}\|\cdot\|_{\sigma'_0}. \tag{7.4}$$

Combining (7.2) and (7.4), we compute:

$$\log v_\sigma = ct + \log v_{\sigma_0}$$
$$\log v_{\varphi_s \sigma'} = ct' + \log v_{\sigma'_0}.$$

Hence it suffices to bound $|t - t'|$ and $|\log v_{\sigma_0} - \log v_{\sigma'_0}|$. First,

$$\begin{aligned} t &= d_{GM}(\sigma_0(0), \sigma(0)) \\ &\le d_{GM}(\sigma_0(0), \sigma'_0(0)) + d_{GM}(\sigma'_0(0), \sigma'(s)) + d_{GM}(\sigma'(s), \sigma(0)) < T_{\mathsf{h}} + t' + c_0. \end{aligned}$$

Similarly, $t' < T_{\mathsf{h}} + t + c_0$, and hence

$$|t - t'| < T_{\mathsf{h}} + c_0.$$

Since $\sigma_0, \sigma'_0 \in \partial^+ \mathcal{G}_P$ and their basepoints $\sigma_0(0)$ and $\sigma'_0(0)$ lie in the 2-neighborhood of the Cayley graph of Γ, with distance less than T_{h} by (7.3), we may apply Case 1 to σ_0, σ'_0 to obtain

$$|\log v_{\sigma_0} - \log v_{\sigma'_0}| < C_2$$

for some uniform constant $C_2 > 0$. Therefore,

$$|\log v_\sigma - \log v_{\varphi_s \sigma'}| < c(T_{\mathsf{h}} + c_0) + C_2.$$

Taking $C_0 > \max(c_0, c(T_{\mathsf{h}} + c_0) + C_2)$ verifies the claim in this case.

Case 3. Suppose $d_{GM}(\Gamma, \sigma(0)) > 5T_{\mathsf{h}}$, $\sigma(0) \in H_P$ and $\sigma^- = \xi_P$ for some $P \in \mathcal{P}^\Gamma$. In this case, we apply Lemma 7.5 to the time reversal of σ, obtaining $\sigma = \varphi_t \tilde{\sigma}_0$ for some $\tilde{\sigma}_0 \in \partial^- \mathcal{G}_P$ with $T^-_{\tilde{\sigma}_0} = -\infty$ and $t < 0$. The norm $\|\cdot\|_\sigma$ is given by

$$\|\cdot\|_\sigma = e^{-ct}\|\cdot\|_{\tilde{\sigma}_0}.$$

This case is symmetric to Case 2 and follows by the same argument, which we omit.

Case 4. Suppose that none of Cases 1-3 applies. Then for some $P \in \mathcal{P}^\Gamma$, $\sigma_0 \in \partial^+ \mathcal{G}_P$ with finite $T := T^+_{\sigma_0} < \infty$, and some $t \in [5T_{\mathsf{h}}, T - 5T_{\mathsf{h}}]$, we have $\sigma = \varphi_t \sigma_0$. In particular, $T > 5T_{\mathsf{h}}$ and $t \in [2T_{\mathsf{h}}, T - 2T_{\mathsf{h}}]$. We may assume that $P \in \mathcal{P}$ and $\sigma_0(0) = (e, 2)$.

By definition of $c_0 > 0$, there exists $s'' \in \mathbb{R}$ such that

$$d_{GM}(\sigma(0), \sigma'(s'')) < c_0. \tag{7.5}$$

By Lemma 7.4, $\sigma'(s'') \in H_P$, and hence

$$\varphi_{s''}\sigma' = \varphi_{t''}\sigma_0' \quad \text{for some } t'' > 0 \text{ and } \sigma_0' \in \partial^+ \mathcal{G}_P. \tag{7.6}$$

By Lemma 7.6, we have $T' := T^+_{\sigma_0'} < \infty$ and

$$d_{GM}(\sigma_0(0), \sigma_0'(0)) < T_{\mathsf{h}} \quad \text{and} \quad d_{GM}(\sigma_0(T), \sigma_0'(T')) < T_{\mathsf{h}}. \tag{7.7}$$

In particular,

$$|T - T'| < 2T_{\mathsf{h}}. \tag{7.8}$$

Since all points $\sigma_0(0)$, $\sigma_0'(0)$, $\sigma_0(T)$, and $\sigma_0'(T')$ lie in the 2-neighborhood of the Cayley graph of Γ, we may apply the argument of Case 1 to σ_0 and σ_0' to obtain a uniform constant $C_3 > 0$ such that

$$|\log v_{\sigma_0} - \log v_{\sigma_0'}| < C_3 \quad \text{and} \quad |\log v_{\varphi_T \sigma_0} - \log v_{\varphi_{T'}\sigma_0'}| < C_3 \tag{7.9}$$

As the norm $\| \cdot \|_\sigma$ is defined according to the time parameter t, we now proceed to subcases depending on how t compares the ends of the segment $[0, T]$.

Case 4-1. Suppose that $0 < t \le T/3$. By (7.5), (7.6), (7.7), and (7.8), we have

$$-(T_{\mathsf{h}}+c_0) < t-(T_{\mathsf{h}}+c_0) \le t'' \le t+(T_{\mathsf{h}}+c_0) \le \frac{T}{3}+(T_{\mathsf{h}}+c_0) < \frac{T'}{3}+(2T_{\mathsf{h}}+c_0).$$

Hence, we can take $t' \in (t'' - (2T_{\mathsf{h}} + c_0), t'' + (T_{\mathsf{h}} + c_0))$ so that

$$0 < t' < \frac{T'}{3}.$$

This implies

$$|t - t'| < 3T_{\mathsf{h}} + 2c_0 \quad \text{and} \tag{7.10}$$

$$d_{GM}(\sigma(0), \sigma_0'(t')) \le d_{GM}(\sigma(0), \sigma'(s'')) + d_{GM}(\sigma_0'(t''), \sigma_0'(t')) < 2(T_{\mathsf{h}} + c_0) \tag{7.11}$$

where the last inequality follows from (7.5) and $|t' - t''| < 2T_{\mathsf{h}} + c_0$. From the construction, we have

$$\| \cdot \|_\sigma = e^{-ct} \| \cdot \|_{\sigma_0} \quad \text{and} \quad \| \cdot \|_{\varphi_{t'}\sigma_0'} = e^{-ct'} \| \cdot \|_{\sigma_0'}.$$

and hence

$$\log v_\sigma = ct + \log v_{\sigma_0} \quad \text{and} \quad \log v_{\varphi_{t'}\sigma_0'} = ct' + \log v_{\sigma_0'}.$$

Hence, using (7.10) and (7.9), we deduce

$$|\log v_\sigma - \log v_{\varphi_{t'}\sigma_0'}| < c(3T_{\mathsf{h}} + 2c_0) + C_3.$$

Since $\varphi_s\sigma' = \varphi_{t'}\sigma'_0$ for some $s \in \mathbb{R}$, we conclude the claim in this case hold with $C_0 > \max(2(T_{\mathsf{h}} + c_0), c(3T_{\mathsf{h}} + 2c_0) + C_3)$.

Case 4-2. Suppose that $2T/3 \leq t < T$. In this case, the norm $\|\cdot\|_\sigma$ is given by

$$\|\cdot\|_\sigma = e^{c(T-t)}\|\cdot\|_{\varphi_T\sigma_0}.$$

This case is symmetric to Case 4-1 and follows by the same argument using $T - t$ in place of t, together with (7.8). We omit the details.

Case 4-3. Suppose that $T/3 < t < 2T/3$. Then from the same bounds (7.5), (7.6), (7.7), and (7.8),

$$\begin{aligned}\frac{T'}{3} - (2T_{\mathsf{h}} + c_0) < \frac{T}{3} - (T_{\mathsf{h}} + c_0) < t - (T_{\mathsf{h}} + c_0) \leq t'' \\ \leq t + (T_{\mathsf{h}} + c_0) \leq \frac{2T}{3} + (T_{\mathsf{h}} + c_0) < \frac{2T'}{3} + (3T_{\mathsf{h}} + c_0).\end{aligned}$$

Hence we can find $t' \in (t'' - (3T_{\mathsf{h}} + c_0), t'' + (2T_{\mathsf{h}} + c_0))$ so that

$$\frac{T'}{3} < t' < \frac{2T'}{3}.$$

This gives

$$|t - t'| < 4T_{\mathsf{h}} + 2c_0 \quad \text{and} \tag{7.12}$$

$$d_{GM}(\sigma(0), \sigma'_0(t')) \leq d_{GM}(\sigma(0), \sigma'(s'')) + d_{GM}(\sigma'_0(t''), \sigma'_0(t')) < 3T_{\mathsf{h}} + 2c_0 \tag{7.13}$$

using again (7.5) and $|t' - t''| < 3T_{\mathsf{h}} + c_0$.

Now using the interpolation formula for the norm, we get

$$\begin{aligned}\|\cdot\|_\sigma &= \|\cdot\|_{\varphi_{T/3}\sigma_0}^{2-\frac{3}{T}t}\|\cdot\|_{\varphi_{2T/3}\sigma_0}^{\frac{3}{T}t-1} = e^{c(2t-T)}\|\cdot\|_{\sigma_0}^{2-\frac{3}{T}t}\|\cdot\|_{\varphi_T\sigma_0}^{\frac{3}{T}t-1} \\ \|\cdot\|_{\varphi_{t'}\sigma'_0} &= \|\cdot\|_{\varphi_{T'/3}\sigma'_0}^{2-\frac{3}{T'}t'}\|\cdot\|_{\varphi_{2T'/3}\sigma'_0}^{\frac{3}{T'}t'-1} = e^{c(2t'-T')}\|\cdot\|_{\sigma'_0}^{2-\frac{3}{T'}t'}\|\cdot\|_{\varphi_{T'}\sigma'_0}^{\frac{3}{T'}t'-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\log v_\sigma &= cT - 2ct + \left(2 - \frac{3}{T}t\right)\log v_{\sigma_0} + \left(\frac{3}{T}t - 1\right)\log v_{\varphi_T\sigma_0} \\ &= cT - 2ct + 2\log v_{\sigma_0} - \log v_{\varphi_T\sigma_0} + \frac{3t}{T}\left(\log v_{\varphi_T\sigma_0} - \log v_{\sigma_0}\right) \\ \log v_{\varphi_{t'}\sigma'_0} &= cT' - 2ct' + 2\log v_{\sigma'_0} - \log v_{\varphi_{T'}\sigma'_0} + \frac{3t'}{T'}\left(\log v_{\varphi_{T'}\sigma'_0} - \log v_{\sigma'_0}\right).\end{aligned}$$

Now using the triangle inequality, (7.8), (7.12), (7.9), and the fact that $t' < 2T'/3$, we estimate

$$\begin{aligned}
&|\log v_\sigma - \log v_{\varphi_{t'}\sigma_0'}| \\
&\le 2cT_{\mathsf{h}} + 2c(4T_{\mathsf{h}} + 2c_0) + 2C_3 + C_3 + \left|\frac{3t}{T} - \frac{3t'}{T'}\right| |\log v_{\varphi_T\sigma_0} - \log v_{\sigma_0}| \\
&\quad + \frac{3t'}{T'} |\log v_{\varphi_{T'}\sigma_0'} - \log v_{\varphi_T\sigma_0}| + \frac{3t'}{T'} |\log v_{\sigma_0'} - \log v_{\sigma_0}| \\
&\le 2c(5T_{\mathsf{h}} + 2c_0) + 3C_3 + \left|\frac{3t}{T} - \frac{3t'}{T'}\right| |\log v_{\varphi_T\sigma_0} - \log v_{\sigma_0}| + 4C_3.
\end{aligned}$$

Now recall that $\sigma_0(0) = (e, 2)$ as noted earlier, and denote $\sigma_0(T) = (\gamma, 2)$ for some $\gamma \in P$. Let $Q \subset X_{GM}$ denote the closed 2-ball centered at e. Then $\sigma_0(0) \in Q$ and $\sigma_0(T) \in \gamma Q$. From (6.3) and (6.4), we have $v_{\varphi_T\sigma_0} = \frac{v_{\sigma_0}}{\kappa_T(\sigma_0)}$. In particular,

$$\log v_{\varphi_T\sigma_0} - \log v_{\sigma_0} = -\log \kappa_T(\sigma_0).$$

By Lemma 6.5, there exists $c_Q > 0$ depending only on Q, such that

$$|\log v_{\varphi_T\sigma_0} - \log v_{\sigma_0} - \psi(\mu_\theta(\gamma))| < c_Q.$$

Therefore,

$$\begin{aligned}
&|\log v_\sigma - \log v_{\varphi_{t'}\sigma_0'}| \\
&\le 2c(5T_{\mathsf{h}} + 2c_0) + 7C_3 + \left|\frac{3t}{T} - \frac{3t'}{T'}\right| c_Q + \left|\frac{3t}{T} - \frac{3t'}{T'}\right| |\psi(\mu_\theta(\gamma))| \\
&\le 2c(5T_{\mathsf{h}} + 2c_0) + 7C_3 + c_Q + \left|\frac{3t}{T} - \frac{3t'}{T'}\right| |\psi(\mu_\theta(\gamma))|
\end{aligned}$$

where the last inequality is from $\frac{T}{3} < t < \frac{2T}{3}$ and $\frac{T'}{3} < t' < \frac{2T'}{3}$. Estimate the final term:

$$\begin{aligned}
\left|\frac{3t}{T} - \frac{3t'}{T'}\right| |\psi(\mu_\theta(\gamma))| &\le \frac{3|t-t'|}{T} |\psi(\mu_\theta(\gamma))| + 3t' \left|\frac{1}{T} - \frac{1}{T'}\right| |\psi(\mu_\theta(\gamma))| \\
&\le \frac{3|t-t'|}{T} |\psi(\mu_\theta(\gamma))| + 3t' \frac{|T-T'|}{T'T} |\psi(\mu_\theta(\gamma))| \\
&\le (16T_{\mathsf{h}} + 6c_0) \left|\frac{\psi(\mu_\theta(\gamma))}{T}\right|,
\end{aligned}$$

using (7.12), (7.8), and $t' < 2T'/3$. It follows from $T = d_{GM}(\sigma_0(0), \sigma_0(T)) = d_{GM}((e, 2), (\gamma, 2))$ that

$$|T - d_{GM}(e, \gamma)| \le 2.$$

Then, by Theorem 5.6, there exist uniform constants $c_1, c_2 > 1$ such that

$$c_1^{-1} T - c_2 \le \psi(\mu_\theta(\gamma)) \le c_1 T + c_2.$$

Since $T > T_{\mathsf{h}}$, we conclude:

$$\left|\frac{\psi(\mu_\theta(\gamma))}{T}\right| \le c_1 + \frac{c_2}{T_{\mathsf{h}}}.$$

Combining all altogether,

$$|\log v_\sigma - \log v_{\varphi_{t'}\sigma'_0}| \le 2c(5T_{\mathsf{h}} + 2c_0) + 7C_3 + c_Q + (16T_{\mathsf{h}} + 6c_0)(c_1 + c_2/T_{\mathsf{h}}).$$

Since $\varphi_s\sigma' = \varphi_{t'}\sigma'_0$ for some $s \in \mathbb{R}$, and using (7.13), the claim follows by setting

$$C_0 > 2c(5T_{\mathsf{h}} + 2c_0) + 7C_3 + c_Q + (16T_{\mathsf{h}} + 6c_0)(c_1 + c_2/T_{\mathsf{h}}) > 3T_{\mathsf{h}} + 2c_0.$$

This completes the proof of the first part of Proposition 7.2.

We now prove the second assertion. Let $C_0 > 0$ be the constant from the first part and let $\sigma, \sigma' \in \mathcal{G}$ be such that $\sigma^\pm = \sigma'^\pm$. Then for some $s \in \mathbb{R}$, we have

$$d_{GM}(\sigma(0), \sigma'(s)) < C_0 \quad \text{and} \quad |\log v_\sigma - \log v_{\varphi_s\sigma'}| < C_0.$$

Therefore,

$$\log v_\sigma - \log v_{\sigma'} - C_0 < \log v_{\varphi_s\sigma'} - \log v_{\sigma'} < \log v_\sigma - \log v_{\sigma'} + C_0.$$

Now, from Theorem 6.1, we have

$$\tilde{\Psi}(\varphi_s\sigma') = \phi_t\tilde{\Psi}(\sigma')$$

for some t with $as - B \le t \le a's + B$ if $s \ge 0$ and $a's - B \le t \le as + B$ if $s < 0$, where $0 < a < a'$ and $B > 0$ are constants in the theorem. Since

$$\log v_{\varphi_s\sigma'} = t + \log v_{\sigma'},$$

we deduce the bounds on s as follows

- if $s \ge 0$,
$$\frac{\log v_{\varphi_s\sigma'} - \log v_{\sigma'} - B}{a'} \le s \le \frac{\log v_{\varphi_s\sigma'} - \log v_{\sigma'} + B}{a}.$$
Therefore,
$$\frac{(\log v_\sigma - \log v_{\sigma'}) - C_0 - B}{a'} \le s \le \frac{(\log v_\sigma - \log v_{\sigma'}) + C_0 + B}{a}.$$
- if $s < 0$,
$$\frac{\log v_{\varphi_s\sigma'} - \log v_{\sigma'} - B}{a} \le s \le \frac{\log v_{\varphi_s\sigma'} - \log v_{\sigma'} + B}{a'}.$$
Therefore,
$$\frac{(\log v_\sigma - \log v_{\sigma'}) - C_0 - B}{a} \le s \le \frac{(\log v_\sigma - \log v_{\sigma'}) + C_0 + B}{a'}.$$

This completes the proof. $\square$

Proof of Theorem 7.1. Let $\sigma, \sigma' \in \mathcal{G}$ be such that

$$\tilde{\Psi}(\sigma) = \tilde{\Psi}(\sigma').$$

This implies that $\sigma^{\pm} = \sigma'^{\pm}$ and $\log v_{\sigma} - \log v_{\sigma'} = 0$. By Proposition 7.2, there exist uniform constants $a, B, C_0 > 0$ so that

$$d_{GM}(\sigma(0), \sigma'(s)) < C_0 \text{ for some } s \in \left[-\frac{C_0 + B}{a}, \frac{C_0 + B}{a} \right].$$

Therefore,

$$d_{GM}(\sigma(0), \sigma'(0)) \leq d_{GM}(\sigma(0), \sigma'(s)) + d_{GM}(\sigma'(s), \sigma'(0)) < C_0 + \frac{C_0 + B}{a}.$$

This finishes the proof. $\square$

Disjointness of $\tilde{\Psi}$-images of horoballs. We deduce from Theorem 7.1 that $\tilde{\Psi}$-images of deep horoballs are disjoint. This implies that the reparameterization $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_{\psi}$ and $\Psi : \Gamma \backslash \mathcal{G} \to \Omega_{\psi}$ respectively give genuine decompositions of $\tilde{\Omega}_{\psi}$ and Ω_{ψ} into the non-cuspidal part and disjoint cuspidal components.

To be precise, for each $n \geq 2$, we define the *depth-n horoballs*, similar to the definition of open horoballs H_P, as follows: for $P \in \mathcal{P}$, let $H'_P(n) \subset X_{GM}$ be the subgraph induced by the vertices $\{(g, k) : g \in P, k \geq n\}$ and $\hat{H}_P(n) \subset X_{GM}$ be the subgraph induced by the vertices $\{(g, n) : g \in P\}$. We then set

$$H_P(n) := H'_P - \hat{H}_P.$$

For $\gamma \in \Gamma$, we set

$$H_{\gamma P \gamma^{-1}}(n) := \gamma H_P(n).$$

This results in the collection of depth-n open horoballs $\{H_P(n) : P \in \mathcal{P}^{\Gamma}\}$. Note that $H_P = H_P(2)$ for $P \in \mathcal{P}^{\Gamma}$. For $P \in \mathcal{P}^{\Gamma}$, we consider the set

$$\mathcal{G}_P(n) := \{\sigma \in \mathcal{G} : \sigma(0) \in H_P(n)\}$$

which consists of bi-infinite geodesics based at $H_P(n)$. We now obtain the following disjointness:

Corollary 7.7. *There exists $n_0 \geq 2$ such that for $P, P' \in \mathcal{P}^{\Gamma}$,*

$$P \neq P' \implies \tilde{\Psi}(\mathcal{G}_P(n_0)) \cap \tilde{\Psi}(\mathcal{G}_{P'}(n_0)) = \emptyset.$$

Proof. Let $C > 0$ be the constant given by Theorem 7.1. We fix $n_0 > \frac{C}{2} + 1$ and show that the desired disjointness holds. Suppose on the contrary that for some distinct $P, P' \in \mathcal{P}^{\Gamma}$, there exist $\sigma \in \mathcal{G}_P(n_0)$ and $\mathcal{G}_{P'}(n_0)$ such that $\tilde{\Psi}(\sigma) = \tilde{\Psi}(\sigma')$. Since $\sigma(0) \in H_P(n_0)$, the distance from $\sigma(0)$ to the Cayley graph of Γ is at least $n_0 - 1$. Similarly, the distance from $\sigma'(0)$ to the Cayley graph of Γ is at least $n_0 - 1$. Since two basepoints $\sigma(0)$ and $\sigma'(0)$ are contained in distinct horoballs, a geodesic segment between them must pass through the Cayley graph. Therefore, we have

$$d_{GM}(\sigma(0), \sigma'(0)) \geq 2n_0 - 2 > C.$$

On the other hand, since $\tilde{\Psi}(\sigma) = \tilde{\Psi}(\sigma')$, we have $d_{GM}(\sigma(0), \sigma'(0)) < C$ by Theorem 7.1, which is a contradiction. This shows the claim. □

Remark 7.8. By the above corollary, the reparameterization given in Corollary 6.2 gives us a thick-thin decomposition of Ω_ψ where the thin part is the disjoint union of Ψ-images of bi-infinite geodesics based at the horoballs in $\Gamma \backslash X_{GM}$ corresponding to elements of $\mathcal{P}$.

8. Exponential expansion on unstable foliations

Let $\Gamma < G$ be a θ-Anosov subgroup relative to $\mathcal{P}$. Fix a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$. Recall the space $\tilde{\Omega}_\psi = \Lambda_\theta^{(2)} \times \mathbb{R}$ equipped with the Γ-action given by

$$\gamma(\xi, \eta, s) = (\gamma\xi, \gamma\eta, s + \psi(\beta_\xi^\theta(\gamma^{-1}, e)))$$

for $\gamma \in \Gamma$ and $(\xi, \eta, s) \in \Lambda_\theta^{(2)} \times \mathbb{R}$, and $\Omega_\psi = \Gamma \backslash \tilde{\Omega}_\psi$ as defined in section 3. Recall from (4.1) and (4.2) the unstable and stable foliations $W^\pm$ on Ω_ψ and their lifts $\tilde{W}^\pm$ on $\tilde{\Omega}_\psi$. The goal of this section is to establish the following exponential expansion (resp. contraction) property of the flow $\{\phi_t\}$ on unstable (resp. stable) foliations.

Theorem 8.1. *We have the following:*

(1) *There exist a Γ-invariant non-negative symmetric function $d^+ : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ and constants $\alpha, \alpha' > 0$ and $b \geq 1$ such that for $z \in \tilde{\Omega}_\psi$, the restriction of d^+ defines a semi-metric[3] on $\tilde{W}^+(z)$ and for any $w_1, w_2 \in \tilde{W}^+(z)$ and $t \geq 0$,*

$$\frac{1}{b} e^{\alpha t} d^+(w_1, w_2) \leq d^+(\phi_t w_1, \phi_t w_2) \leq b e^{\alpha' t} d^+(w_1, w_2).$$

(2) *Similarly, there exists a Γ-invariant non-negative symmetric function $d^- : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ such that for $z \in \tilde{\Omega}_\psi$, the restriction of d^- defines a semi-metric on $\tilde{W}^-(z)$ and for any $w_1, w_2 \in \tilde{W}^-(z)$ and $t \geq 0$,*

$$\frac{1}{b} e^{-\alpha' t} d^-(w_1, w_2) \leq d^-(\phi_t w_1, \phi_t w_1) \leq b e^{-\alpha t} d^-(w_1, w_2).$$

(3) *For any small enough $\varepsilon > 0$, there exists a non-negative symmetric function $d_\varepsilon^+ : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ such that for $z \in \tilde{\Omega}_\psi$, the restriction of d_ε^+ defines a metric on $\tilde{W}^+(z)$. Moreover, for any compact subset $Q \subset \tilde{\Omega}_\psi$, there exists a constant $c_Q \geq 1$ such that for any $w_1, w_2 \in Q$,*

$$\frac{1}{c_Q} d^+(w_1, w_2)^\varepsilon \leq d_\varepsilon^+(w_1, w_2) \leq c_Q d^+(w_1, w_2)^\varepsilon.$$

[3] A semi-metric on $\mathcal{X}$ is a non-negative symmetric function $\mathcal{X} \times \mathcal{X} \to \mathbb{R}$ that vanishes precisely on the diagonal.

Remark 8.2. Even though Theorem 8.1 states the exponential expansion and contraction for $t \geq 0$, replacing w_1 and w_2 with $\phi_{-t}w_1$ and $\phi_{-t}w_2$ implies the corresponding estimates for negative-time flow.

The proof of Theorem 8.1 is based on our coarse reparameterization (Theorem 6.1) and the coarse geometry of the Groves-Manning cusp space as a Gromov hyperbolic space.

Groves-Manning cusp space as a Gromov hyperbolic space. Let X_{GM} be the associated Groves-Manning cusp space of $(\Gamma, \mathcal{P})$, which is a proper geodesic Gromov hyperbolic space ([15, Theorem 3.25], Theorem 5.1). We refer to [8, Chapter III.H] for general facts about Gromov hyperbolic spaces.

Recall that $\mathcal{G}$ is the space of all parameterized bi-infinite geodesics in X_{GM}. We define $d^{\pm} : \mathcal{G} \times \mathcal{G} \to [0, \infty)$ as follows: for $\sigma_1, \sigma_2 \in \mathcal{G}$,

$$\begin{aligned} d^{+}(\sigma_1, \sigma_2) &:= \limsup_{t \to \infty} e^{d_{GM}(\sigma_1(t), \sigma_2(t)) - 2t}; \\ d^{-}(\sigma_1, \sigma_2) &:= \limsup_{t \to \infty} e^{d_{GM}(\sigma_1(-t), \sigma_2(-t)) - 2t}. \end{aligned} \tag{8.1}$$

Their well-definedness follows once we explain another formula for $d^{\pm}$ using Gromov products and Busemann functions on X_{GM}. We recall that for $x, p, q \in X_{GM}$, the Gromov product of p, q with respect to x is

$$(p|q)_x := \frac{1}{2}(d_{GM}(x,p) + d_{GM}(x,q) - d_{GM}(p,q)) \geq 0,$$

and this extends to ∂X_{GM} as follows: for $\xi, \eta \in \partial X_{GM}$, we set

$$(\xi|\eta)_x := \sup \liminf_{i,j \to \infty} (p_i|q_j)_x$$

where the supremum is taken over all sequences $p_i, q_j \in X_{GM}$ such that $p_i \to \xi$ and $q_j \to \eta$ as $i, j \to \infty$. Since X_{GM} is Gromov hyperbolic, there exists a uniform constant $\delta > 0$ such that for any $x \in X_{GM}$, $\xi, \eta \in \partial X_{GM}$, and sequences $p_i, q_j \in X_{GM}$ with $\xi = \lim_{i \to \infty} p_i$ and $\eta = \lim_{j \to \infty} q_j$, we have

$$(\xi|\eta)_x - \frac{\delta}{2} \leq \liminf_{i,j \to \infty} (p_i|q_j)_x \leq (\xi|\eta)_x. \tag{8.2}$$

For $\sigma \in \mathcal{G}$ and $p, q \in X_{GM}$, the following *Busemann function* is well-defined:

$$\beta_{\sigma^+}(p,q) := \lim_{t \to \infty} d_{GM}(p, \sigma(t)) - d_{GM}(q, \sigma(t)).$$

We note that the Busemann function is defined for each geodesic $\sigma \in \mathcal{G}$, not for a point in ∂X_{GM}. The notation $+$ in $\beta_{\sigma^+}(p,q)$ is to indicate that the limit is taken along $t \to \infty$. Indeed, this makes the above limit well-defined since the function $f_p : \mathbb{R} \to \mathbb{R}$ defined as

$$f_p(t) = d_{GM}(p, \sigma(t)) - d_{GM}(\sigma(0), \sigma(t))$$

is non-increasing and bounded from above by $d_{GM}(p, \sigma(0))$, and we have $d_{GM}(p, \sigma(t)) - d_{GM}(q, \sigma(t)) = f_p(t) - f_q(t)$.

We have for any $x \in X_{GM}$ that

$$d^+(\sigma_1, \sigma_2) = e^{\beta_{\sigma_1^+}(x,\sigma_1(0))+\beta_{\sigma_2^+}(x,\sigma_2(0))} \limsup_{t\to\infty} e^{-2(\sigma_1(t)|\sigma_2(t))_x}. \tag{8.3}$$

Since $(\sigma_1(t)|\sigma_2(t))_x \geq 0$ for all t, it follows that $d^+(\sigma_1, \sigma_2) < \infty$. Since

$$d^-(\sigma_1, \sigma_2) = d^+(I\sigma_1, I\sigma_2), \tag{8.4}$$

d^- is well-defined as well. The definition of $d^\pm$ is motivated by the Hamenstädt distance in a negatively curved compact manifold [16].

Since Γ acts on X_{GM} by isometries, both d^+ and d^- are Γ-invariant. The geodesic flow on $\mathcal{G}$ exponentially expand and contract d^+ and d^- respectively:

Lemma 8.3. *Let $\sigma_1, \sigma_2 \in \mathcal{G}$ and $s_1, s_2 \in \mathbb{R}$. Then we have*

$$e^{-\delta}e^{s_1+s_2}d^+(\sigma_1, \sigma_2) \leq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta}e^{s_1+s_2}d^+(\sigma_1, \sigma_2);$$
$$e^{-\delta}e^{-(s_1+s_2)}d^-(\sigma_1, \sigma_2) \leq d^-(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta}e^{-(s_1+s_2)}d^-(\sigma_1, \sigma_2).$$

Proof. Fix $x \in X_{GM}$. By (8.3) and (8.2), we have

$$\begin{aligned} d^+(\sigma_1, \sigma_2) &\geq e^{\beta_{\sigma_1^+}(x,\sigma_1(0))+\beta_{\sigma_2^+}(x,\sigma_2(0))} e^{-2(\sigma_1^+|\sigma_2^+)_x}; \\ d^+(\sigma_1, \sigma_2) &\leq e^{\delta} e^{\beta_{\sigma_1^+}(x,\sigma_1(0))+\beta_{\sigma_2^+}(x,\sigma_2(0))} e^{-2(\sigma_1^+|\sigma_2^+)_x}. \end{aligned} \tag{8.5}$$

By the definition of β, we have

$$\begin{aligned} \beta_{\sigma_1^+}(x, (\varphi_{s_1}\sigma_1)(0)) &= \beta_{\sigma_1^+}(x, \sigma_1(0)) + \beta_{\sigma_1^+}(\sigma_1(0), \sigma_1(s_1)) \\ &= \beta_{\sigma_1^+}(x, \sigma_1(0)) + s_1, \end{aligned} \tag{8.6}$$

and similarly

$$\beta_{\sigma_2^+}(x, (\varphi_{s_2}\sigma_2)(0)) = \beta_{\sigma_2^+}(x, \sigma_2(0)) + s_2. \tag{8.7}$$

Since $\varphi_{s_1}\sigma_1^+ = \sigma_1^+$ and $\varphi_{s_2}\sigma_2^+ = \sigma_2^+$, it follows from (8.5), (8.6), and (8.7) that

$$e^{-\delta}e^{s_1+s_2}d^+(\sigma_1, \sigma_2) \leq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta}e^{s_1+s_2}d^+(\sigma_1, \sigma_2).$$

The exponential contraction of d^- follows from the exponential expansion of d^+ shown above and (8.4). $\square$

We fix a basepoint $x \in X_{GM}$. It is a standard fact about Gromov hyperbolic spaces that for $\varepsilon > 0$ small enough, there exists $0 < c_\varepsilon < 1$ and a metric d_ε on ∂X_{GM} such that

$$c_\varepsilon e^{-2\varepsilon(\xi|\eta)_x} \leq d_\varepsilon(\xi, \eta) \leq e^{-2\varepsilon(\xi|\eta)_x} \tag{8.8}$$

for all $\xi, \eta \in \partial X_{GM}$, with the convention that $e^{-\infty} = 0$ [8, Proposition 3.21]. We fix one such $\varepsilon > 0$ and a metric d_ε as above.

Lemma 8.4. *For any compact subset $Q \subset \mathcal{G}$, there exists a constant $b_Q \geq 1$ such that for any $\sigma_1, \sigma_2 \in Q$, we have*

$$\frac{1}{b_Q} d^+(\sigma_1, \sigma_2)^\varepsilon \leq d_\varepsilon(\sigma_1^+, \sigma_2^+) \leq b_Q d^+(\sigma_1, \sigma_2)^\varepsilon.$$

Proof. First note that for any $\sigma \in \mathcal{G}$,

$$|\beta_{\sigma^+}(x, \sigma(0))| \leq d_{GM}(x, \sigma(0)).$$

Given a compact subset $Q \subset \mathcal{G}$, we set

$$b' := \sup_{\sigma \in Q} d_{GM}(x, \sigma(0)) < \infty.$$

Then it follows from (8.8) and (8.5) that

$$\begin{aligned} d_\varepsilon(\sigma_1^+, \sigma_2^+) &\leq e^{-\varepsilon\left(\beta_{\sigma_1^+}(x, \sigma_1(0)) + \beta_{\sigma_2^+}(x, \sigma_2(0))\right)} d^+(\sigma_1, \sigma_2)^\varepsilon \\ &\leq e^{2\varepsilon b'} d^+(\sigma_1, \sigma_2)^\varepsilon. \end{aligned}$$

Similarly, we also have

$$d_\varepsilon(\sigma_1^+, \sigma_2^+) \geq c_\varepsilon e^{-\varepsilon(\delta + 2b')} d^+(\sigma_1, \sigma_2)^\varepsilon$$

where $0 < c_\varepsilon < 1$ is given in (8.8). Setting $b_Q := e^{\varepsilon(\delta+2b')}/c_\varepsilon$ completes the proof. $\square$

Reparameterization revisited. Recall the reparameterization $\Psi : \Gamma\backslash\mathcal{G} \to \Omega_\psi$ in Theorem 6.1, which is induced from the Γ-equivariant map $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$. Since $\tilde{\Psi}$ is proper and surjective, for $w_1, w_2 \in \tilde{\Omega}_\psi$, we define

$$\begin{aligned} d^+(w_1, w_2) &:= \sup_{\sigma_1 \in \tilde{\Psi}^{-1}(w_1),\ \sigma_2 \in \tilde{\Psi}^{-1}(w_2)} d^+(\sigma_1, \sigma_2); \\ d^-(w_1, w_2) &:= \sup_{\sigma_1 \in \tilde{\Psi}^{-1}(w_1),\ \sigma_2 \in \tilde{\Psi}^{-1}(w_2)} d^-(\sigma_1, \sigma_2). \end{aligned} \tag{8.9}$$

Since $\tilde{\Psi}$ is Γ-equivariant, if $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$, then $\gamma\sigma_1 \in \tilde{\Psi}^{-1}(\gamma w_1)$ and $\gamma\sigma_2 \in \tilde{\Psi}^{-1}(\gamma w_2)$ for all $\gamma \in \Gamma$. Since $d^\pm(\gamma\sigma_1, \gamma\sigma_2) = d^\pm(\sigma_1, \sigma_2)$ as well, we have

$$d^\pm(\gamma w_1, \gamma w_2) = d^\pm(w_1, w_2) \quad \text{for all } \gamma \in \Gamma. \tag{8.10}$$

We also have the following expansion and contraction of d^+ and d^- via the flow $\{\phi_t\}$ respectively:

Lemma 8.5. *There exist $\alpha, \alpha' > 0$ and $b \geq 1$ such that for any $w_1, w_2 \in \tilde{\Omega}_\psi$ and $t \geq 0$, we have*

$$\begin{aligned} \frac{1}{b} e^{\alpha t} d^+(w_1, w_2) \leq d^+(\phi_t w_1, \phi_t w_2) \leq b e^{\alpha' t} d^+(w_1, w_2); \\ \frac{1}{b} e^{-\alpha' t} d^-(w_1, w_2) \leq d^-(\phi_t w_1, \phi_t w_2) \leq b e^{-\alpha t} d^-(w_1, w_2). \end{aligned} \tag{8.11}$$

Proof. Let $w_1, w_2 \in \tilde{\Omega}_\psi$ and $t \geq 0$. Let $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$. By Theorem 6.1, there exist $s_1, s_2 \in \mathbb{R}$ such that

$$\varphi_{s_1}\sigma_1 \in \tilde{\Psi}^{-1}(\phi_t w_1) \quad \text{and} \quad \varphi_{s_2}\sigma_2 \in \tilde{\Psi}^{-1}(\phi_t w_2),$$

and moreover, for constants $a, a', B > 0$ in Theorem 6.1, we have:

(1) if $s_1 \geq 0$, then

$$as_1 - B \leq t \leq a's_1 + B$$

(resp. if $s_2 \geq 0$, then $as_2 - B \leq t \leq a's_2 + B$).

(2) if $s_1 \leq 0$, then

$$a's_1 - B \leq t \leq as_1 + B$$

(resp. if $s_2 \leq 0$, then $a's_2 - B \leq t \leq as_2 + B$).

By Lemma 8.3, we have

$$e^{-\delta}e^{s_1+s_2}d^+(\sigma_1,\sigma_2) \leq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta}e^{s_1+s_2}d^+(\sigma_1,\sigma_2). \tag{8.12}$$

Suppose first that $s_1, s_2 \geq 0$. Then by (1) above, we deduce from (8.12) that

$$d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta}e^{\frac{2B}{a}}e^{\frac{2t}{a}}d^+(\sigma_1,\sigma_2) \leq e^{\delta}e^{\frac{2B}{a}}e^{\frac{2t}{a}}d^+(w_1,w_2).$$

Since $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$ are arbitrary, $\varphi_{s_1}\sigma_1$ and $\varphi_{s_2}\sigma_2$ are arbitrary elements of $\tilde{\Psi}^{-1}(\phi_t w_1)$ and $\tilde{\Psi}^{-1}(\phi_t w_2)$ respectively. Hence we have

$$d^+(\phi_t w_1, \phi_t w_2) \leq e^{\delta}e^{\frac{2B}{a}}e^{\frac{2t}{a}}d^+(w_1,w_2). \tag{8.13}$$

Similarly, we deduce from (1) and (8.12) that

$$d^+(\phi_t w_1, \phi_t w_2) \geq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \geq e^{-\delta}e^{-\frac{2B}{a'}}e^{\frac{2t}{a'}}d^+(\sigma_1,\sigma_2).$$

Since $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$ are arbitrary, we have

$$d^+(\phi_t w_1, \phi_t w_2) \geq e^{-\delta}e^{-\frac{2B}{a'}}e^{\frac{2t}{a'}}d^+(w_1,w_2). \tag{8.14}$$

Now consider the case when at least one of s_1 and s_2 is negative. Then by (2), we must have $0 \leq t \leq B$, and hence we deduce from (1) and (2) that $s_1, s_2 \in [-B/a, 2B/a]$. It then follows from (8.12) that

$$d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta}e^{\frac{4B}{a}}d^+(\sigma_1,\sigma_2) \leq e^{\delta}e^{\frac{4B}{a}}d^+(w_1,w_2)$$

and that

$$d^+(\phi_t w_1, \phi_t w_2) \geq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \geq e^{-\delta}e^{-\frac{2B}{a}}d^+(\sigma_1,\sigma_2).$$

Again, since $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$ are arbitrary, these imply

$$e^{-\delta}e^{-\frac{2B}{a}}d^+(w_1,w_2) \leq d^+(\phi_t w_1, \phi_t w_2) \leq e^{\delta}e^{\frac{4B}{a}}d^+(w_1,w_2).$$

Since $0 \leq t \leq B$, we in particular have

$$e^{-\delta}e^{-\frac{2B}{a}-\frac{2B}{a'}}e^{\frac{2t}{a'}}d^+(w_1,w_2) \leq d^+(\phi_t w_1, \phi_t w_2) \leq e^{\delta}e^{\frac{4B}{a}}e^{\frac{2t}{a}}d^+(w_1,w_2). \tag{8.15}$$

Combining (8.13), (8.14), and (8.15), the inequalities for d^+ in (8.11) follows. The inequalities for d^- in (8.11) can be shown by a similar argument. □

For $w_1, w_2 \in \tilde{\Omega}_\psi$, we also define

$$d_\varepsilon^+(w_1, w_2) := d_\varepsilon(\sigma_1^+, \sigma_2^+) \tag{8.16}$$

where $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$. Since every elements of $\tilde{\Psi}^{-1}(w)$ has the common forward endpoint for each $w \in \tilde{\Omega}_\psi$, this is well-defined.

Lemma 8.6. *For any compact subset $Q \subset \tilde{\Omega}_\psi$, there exists a constant $c_Q \geq 1$ such that for any $w_1, w_2 \in Q$, we have*

$$\frac{1}{c_Q} d^+(w_1, w_2)^\varepsilon \leq d_\varepsilon^+(w_1, w_2) \leq c_Q d^+(w_1, w_2)^\varepsilon.$$

Proof. Let $Q \subset \tilde{\Omega}_\psi$ be a compact subset. Since $\tilde{\Psi}$ is proper, it follows from Lemma 8.4 that there exists a uniform constant $c_Q \geq 1$ such that if $w_1, w_2 \in Q$ and $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$, then

$$\frac{1}{c_Q} d^+(\sigma_1, \sigma_2)^\varepsilon \leq d_\varepsilon^+(w_1, w_2) \leq c_Q d^+(\sigma_1, \sigma_2)^\varepsilon \leq c_Q d^+(w_1, w_2)^\varepsilon.$$

Since $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$ are arbitrary, the claim follows. $\square$

Proof of Theorem 8.1. Let $d^\pm : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ be functions defined in (8.9). From the definition, $d^\pm$ are non-negative and symmetric. Moreover, they are Γ-invariant by (8.10).

Let $z \in \tilde{\Omega}_\psi$. We show that the restriction on d^+ defines a semi-metric on $\tilde{W}^+(z)$; the corresponding statement for d^- can be shown by the same argument. It suffices to show that for $w_1, w_2 \in \tilde{W}^+(z)$, $d^+(w_1, w_2) = 0$ if and only if $w_1 = w_2$. Suppose first that $w_1 = w_2$. Then for any $\sigma_1, \sigma_2 \in \tilde{\Psi}^{-1}(w_1) = \tilde{\Psi}^{-1}(w_2)$, we have $\sigma_1^+ = \sigma_2^+$. This implies $(\sigma_1|\sigma_2)_x = \infty$. Hence, by (8.5), we have $d^+(\sigma_1, \sigma_2) = 0$. Since $\sigma_1, \sigma_2 \in \tilde{\Psi}^{-1}(w_1) = \tilde{\Psi}^{-1}(w_2)$ are arbitrary, we have $d^+(w_1, w_2) = 0$. Conversely, suppose that $d^+(w_1, w_2) = 0$. Let $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$. We then have $d^+(\sigma_1, \sigma_2) = 0$, and hence $(\sigma_1^+|\sigma_2^+)_x = \infty$ by (8.5), from which we deduce $\sigma_1^+ = \sigma_2^+$. Since $\tilde{\Psi}(\sigma_1) = w_1$ and $\tilde{\Psi}(\sigma_2) = w_2$, it follows from $w_1, w_2 \in \tilde{W}^+(z)$ and Lemma 4.5 that $w_1 = w_2$, showing the claim.

The inequalities in (1) and (2) follow from Lemma 8.5, finishing the proofs of (1) and (2).

We now show (3). For small enough $\varepsilon > 0$, we consider the function $d_\varepsilon^+ : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ defined in (8.16), that is, for $w_1, w_2 \in \tilde{\Omega}_\psi$,

$$d_\varepsilon^+(w_1, w_2) = d_\varepsilon(\sigma_1^+, \sigma_2^+)$$

where $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$, and d_ε is the visual metric on ∂X_{GM} given in (8.8). Since d_ε is a metric, d_ε^+ is symmetric and satisfies the triangle inequality. Let $z \in \tilde{\Omega}_\psi$ and $w_1, w_2 \in \tilde{W}^+(z)$. As discussed above, for $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$, we have $w_1 = w_2 \Leftrightarrow \sigma_1^+ = \sigma_2^+$ since $w_1, w_2 \in \tilde{W}^+(z)$. Hence $d_\varepsilon^+(w_1, w_2) = 0$ if and only if $w_1 = w_2$, and therefore the restriction of d_ε^+ defines a metric on $\tilde{W}^+(z)$. The inequality stated in (3) is proved in Lemma 8.6. This completes the proof. $\square$

9. Finiteness of Bowen-Margulis-Sullivan measures

Let $\Gamma < G$ be a θ-Anosov subgroup relative to $\mathcal{P}$ and X_{GM} the associated Groves-Manning cusp space. Let $\psi \in \mathfrak{a}_\theta^*$ be a (Γ, θ)-proper linear form tangent to the θ-growth indicator ψ_Γ^θ. By [11], there exists a unique (Γ, ψ)-Patterson-Sullivan measure ν_ψ on Λ_θ and a unique $(\Gamma, \psi \circ \mathrm{i})$-Patterson-Sullivan measure $\nu_{\psi\circ\mathrm{i}}$ on $\Lambda_{\mathrm{i}(\theta)}$. Let m_ψ be the Bowen-Margulis-Sullivan measure on Ω_ψ associated with the pair (ν, ν_{i}) defined in (3.2).

The relatively Anosov subgroups are regarded as the higher-rank generalization of geometrically finite subgroups. Indeed, same as geometrically finite subgroups, relatively Anosov subgroups have finite Bowen-Margulis-Sullivan measures:

Theorem 9.1. *We have*

$$|m_\psi| := m_\psi(\Omega_\psi) < \infty.$$

We prove this finiteness of the Bowen-Margulis-Sullivan measure as a consequence of our reparameterization theorem (Theorem 6.1).

Thick-thin decomposition of Ω_ψ. Let $\Psi : \Gamma \backslash \mathcal{G} \to \Omega_\psi$ be the reparameterization given in Theorem 6.1. Via Ψ, the decomposition $\mathcal{G} = \mathcal{G}_{thick} \cup \mathcal{G}_{thin}$ gives the thick-thin decomposition

$$\Omega_\psi = \Psi(\Gamma \backslash \mathcal{G}_{thick}) \cup \Psi(\Gamma \backslash \mathcal{G}_{thin})$$

into the compact thick part $\Psi(\Gamma \backslash \mathcal{G}_{thick})$ and the thin part $\Psi(\Gamma \backslash \mathcal{G}_{thin})$.

The followings are extra ingredients in the proof:

Lemma 9.2 (Shadow lemma)**.** [18, Lemma 7.2] *For all large enough $R > 0$, there exists $c_0 = c_0(\psi, R) \geq 1$ such that for all $\gamma \in \Gamma$,*

$$c_0^{-1} e^{-\psi(\mu_\theta(\gamma))} \leq \nu_\psi(O_R^\theta(o, \gamma o)) \leq c_0 e^{-\psi(\mu_\theta(\gamma))}.$$

We denote by $0 \leq \delta_\psi(\Gamma) \leq \infty$ the abscissa of convergence of the Poincaré series $s \mapsto \sum_{\gamma \in \Gamma} e^{-s\psi(\mu_\theta(\gamma))}$; this is well-defined by the (Γ, θ)-properness hypothesis on ψ. Indeed, the (Γ, θ)-properness implies $\delta_\psi(\Gamma) < \infty$ as shown in [11, Theorem 1.3]. Since ψ is tangent to ψ_Γ^θ, we furthermore have

$$\delta_\psi(\Gamma) = 1$$

[18, Theorem 4.5]. On the other hand, we have the following:

Theorem 9.3 (Canary-Zhang-Zimmer, [11, Lemma 8.2, Corollary 7.2])**.** *If $\psi \in \mathfrak{a}_\theta^*$ is (Γ, θ)-proper and tangent to ψ_Γ^θ, then the Patterson-Sullivan measure ν_ψ is atomless and for each $P \in \mathcal{P}$, we have*

$$\delta_\psi(P) < 1.$$

Proof of Theorem 9.1. As before, we identify Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ with ∂X_{GM} through the boundary maps. Recall the norm $\|\cdot\|_\sigma$ on $\mathbb{R}_+$ for each $\sigma \in \mathcal{G}$ and the Γ-equivariant surjective proper map $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$, $\sigma \mapsto (\sigma^+, \sigma^-, \log v_\sigma)$, defined in the proof of Theorem 6.1 where $v_\sigma \in \mathbb{R}_+$ is the unique vector such that $\|v_\sigma\|_\sigma = 1$. We then have

$$\tilde{\Omega}_\psi = \tilde{\Psi}(\mathcal{G}_{thick}) \cup \tilde{\Psi}(\mathcal{G}_{thin}).$$

We will use this specific decomposition to show the finiteness of m_ψ. Since Γ acts cocompactly on $\tilde{\Psi}(\mathcal{G}_{thick})$, it suffices to show that the measure of thin part $m_\psi(\Gamma \backslash \tilde{\Psi}(\mathcal{G}_{thin}))$ is finite. Moreover, since $\mathcal{G}_{thin} = \Gamma \cdot \bigcup_{P \in \mathcal{P}} \mathcal{G}_P$ and $\mathcal{P}$ is a finite collection, it suffices to show $m_\psi(P \backslash \tilde{\Psi}(\mathcal{G}_P)) < \infty$ for each $P \in \mathcal{P}$.

Let us fix $P \in \mathcal{P}$ and denote by $\xi_P \in \partial X_{GM}$ the parabolic limit point fixed by P. Since ξ_P is bounded parabolic, we have a compact fundamental domain for the P-action on $\partial X_{GM} - \{\xi_P\}$, which we denote by D. Since ν_ψ and $\nu_{\psi \circ \mathrm{i}}$ are atomless by Theorem 9.3, we have

$$m_\psi(P \backslash \tilde{\Psi}(\mathcal{G}_P)) = \sum_{\gamma \in P} \int_{(\gamma D \times D \times \mathbb{R}) \cap \tilde{\Psi}(\mathcal{G}_P)} e^{\psi(\langle \xi, \eta \rangle)} d\nu_\psi(\xi) d\nu_{\psi \circ \mathrm{i}}(\eta) dt. \tag{9.1}$$

We first estimate the integration with respect to dt. We claim that there exists $C > 0$ such that for any $\gamma \in P$ and $\sigma \in \mathcal{G}_P$ such that $\sigma^- \in D$ and $\sigma^+ \in \gamma D$, we have

$$-C \le \log v_\sigma \le C + \psi(\mu_\theta(\gamma)). \tag{9.2}$$

Let us fix $\gamma \in P$ and let $\sigma \in \mathcal{G}_P$ be such that $\sigma^+ \in \gamma D$ and $\sigma^- \in D$. Recalling that H_P denotes the open horoball in X_{GM} associated to P, this implies that the following two constants are well-defined:

$$\begin{aligned} s_0 &:= \min\{s < 0 : \sigma(s) \in \partial H_P\} \\ s_1 &:= \max\{s > 0 : \sigma(s) \in \partial H_P\}. \end{aligned}$$

In other words, s_0 is the first time that σ enters into ∂H_P and s_1 is the last time that σ exits ∂H_P. We then have from (6.4) and Theorem 6.4 that

$$\begin{aligned} v_{\varphi_{s_0}\sigma} &= \|v_{\varphi_{s_0}\sigma}\|_\sigma v_\sigma = \kappa_{-s_0}(\varphi_{s_0}\sigma) v_\sigma \\ &\le b e^{a s_0} v_\sigma \le b v_\sigma; \\ v_{\varphi_{s_1}\sigma} &= \frac{1}{\|v_\sigma\|_{\varphi_{s_1}\sigma}} v_\sigma = \frac{1}{\kappa_{s_1}(\sigma)} v_\sigma \\ &\ge b^{-1} e^{a s_1} v_\sigma \ge b^{-1} v_\sigma. \end{aligned}$$

Therefore, we have

$$-\log b + \log v_{\varphi_{s_0}\sigma} \le \log v_\sigma \le \log b + \log v_{\varphi_{s_1}\sigma}. \tag{9.3}$$

Now fix $x \in \partial H_P$. Then there exists $R > 0$ with the following property: for any $\sigma_0 \in \mathcal{G}_P$ such that $\sigma_0^- \in D$, the entering point of σ_0 into ∂H_P, i.e. $\sigma_0(s) \in \partial H_P$ with minimal s, must be contained in the R-ball $B_{GM}(x, R)$. Indeed, if not, then there exists a sequence $\sigma_n \in \mathcal{G}_P$ such that $\sigma_n^- \in D$

and the entering point of σ_n into ∂H_P is not contained in $B_{GM}(x, n)$ for all $n \geq 1$. However, since $\sigma_n \in \mathcal{G}_P$ and $\sigma_n^- \in D$ for all $n \geq 1$, two sequences σ_n^+ and σ_n^- converge to two distinct points in ∂X_{GM} as $n \to \infty$, after passing to a subsequence. Hence the images of the bi-infinite geodesics σ_n intersect a single ball centered at x, which contradicts the choice of the sequence σ_n.

Hence we have $(\varphi_{s_0}\sigma)(0) = \sigma(s_0) \in B_{GM}(x, R)$. Since $I(\gamma^{-1}\sigma) \in \mathcal{G}_P$ also satisfies that $I(\gamma^{-1}\sigma)^- = \gamma^{-1}\sigma^+ \in D$ and its entering point into ∂H_P is given by $I(\gamma^{-1}\sigma)(-s_1) = \gamma^{-1}\sigma(s_1)$, we also have $\gamma^{-1}\sigma(s_1) \in B_{GM}(x, R)$. In other words, we have $(\gamma^{-1}\varphi_{s_0}\sigma)(s_1 - s_0) \in B_{GM}(x, R)$. Hence we can apply Lemma 6.5 to $\varphi_{s_0}\sigma$ by setting $Q = \overline{B_{GM}(x, R)}$ and obtain

$$\frac{1}{C_Q} e^{-\psi(\mu_\theta(\gamma))} \leq \kappa_{s_1 - s_0}(\varphi_{s_0}\sigma) \leq C_Q e^{-\psi(\mu_\theta(\gamma))}. \tag{9.4}$$

Since

$$v_{\varphi_{s_1}\sigma} = \frac{1}{\|v_{\varphi_{s_0}\sigma}\|_{\varphi_{s_1}\sigma}} v_{\varphi_{s_0}\sigma} = \frac{1}{\kappa_{s_1 - s_0}(\varphi_{s_0}\sigma)} v_{\varphi_{s_0}\sigma}$$

by (6.4), it follows from (9.4) that

$$\log v_{\varphi_{s_1}\sigma} \leq \log C_Q + \log v_{\varphi_{s_0}\sigma} + \psi(\mu_\theta(\gamma)).$$

Hence we deduce from (9.3) that

$$-\log b + \log v_{\varphi_{s_0}\sigma} \leq \log v_\sigma \leq \log(bC_Q) + \log v_{\varphi_{s_0}\sigma} + \psi(\mu_\theta(\gamma)).$$

Since $(\varphi_{s_0}\sigma)(0) \in B_{GM}(x, R)$ where x is fixed and R is determined by x and P, the constant $\log v_{\varphi_{s_0}\sigma}$ is also uniformly bounded. Therefore, the claim (9.2) follows.

By the claim (9.2), we deduce from (9.1) that

$$m_\psi(P \backslash \tilde{\Psi}(\mathcal{G}_P)) \leq \sum_{\gamma \in P} (2C + \psi(\mu_\theta(\gamma))) \int_{(\gamma D \times D) \cap \{(\sigma^+, \sigma^-) : \sigma \in \mathcal{G}_P\}} e^{\psi(\langle \xi, \eta \rangle)} d\nu_\psi(\xi) d\nu_{\psi \circ \mathrm{i}}(\eta).$$

As we already observed, for $x \in \partial H_P$ and $R > 0$ above, we have that if $\sigma \in \mathcal{G}_P$ is such that $\sigma^- \in D$ and $\sigma^+ \in \gamma D$, then the image of the bi-infinite geodesic σ must intersect $B_{GM}(x, R)$ and $B_{GM}(\gamma x, R)$. Hence it follows from Lemma 5.10 that

$$\psi(\langle \sigma^+, \sigma^- \rangle) \quad \text{is uniformly bounded.} \tag{9.5}$$

Moreover, we also have that $\sigma^+ \in O_{R'}^{GM}(x, \gamma x)$ for some $R' > 0$ depending on x and R. By Proposition 5.7, we then have for some uniform $r > 0$ that

$$\sigma^+ \in O_r^\theta(o, \gamma o). \tag{9.6}$$

By (9.5) and (9.6), we now have

$$m_\psi(P \backslash \tilde{\Psi}(\mathcal{G}_P)) \ll^4 \sum_{\gamma \in P} (2C + \psi(\mu_\theta(\gamma))) \nu_\psi(O_r^\theta(o, \gamma o)).$$

[4] The notation $f \ll g$ means that there is a constant $c > 0$ such that $f \leq cg$

Applying the shadow lemma (Lemma 9.2), we finally obtain

$$m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) \ll \sum_{\gamma\in P}(2C+\psi(\mu_\theta(\gamma)))e^{-\psi(\mu_\theta(\gamma))}.$$

Let $0<\varepsilon<1$. Since ψ is (Γ,θ)-proper, $\liminf_{\gamma\in P}\psi(\mu_\theta(\gamma))=\infty$, and hence $\psi(\mu_\theta(\gamma))\ll e^{\varepsilon\psi(\mu_\theta(\gamma))}$. Hence

$$m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) \ll \sum_{\gamma\in P}(2C+\psi(\mu_\theta(\gamma)))e^{-\psi(\mu_\theta(\gamma))} \ll \sum_{\gamma\in P} e^{-(1-\varepsilon)\psi(\mu_\theta(\gamma))}.$$

By Theorem 9.3, for $\varepsilon>0$ sufficiently small, we have

$$m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) \ll \sum_{\gamma\in P} e^{-(1-\varepsilon)\psi(\mu_\theta(\gamma))}<\infty.$$

This completes the proof of Theorem 9.1. $\square$

10. Unique measure of maximal entropy

Let Γ be a relatively θ-Anosov subgroup and $\psi\in\mathfrak{a}_\theta^*$ a (Γ,θ)-proper form tangent to ψ_Γ^θ. Let m_ψ be the Bowen-Margulis-Sullivan measure on Ω_ψ. This section is devoted to the proof of the following: by Theorem 9.1, m_ψ is of finite measure.

Theorem 10.1. *Let m be a probability $\{\phi_t\}$-invariant measure on Ω_ψ. Then the metric entropy $h_m(\{\phi_t\})$ is at most $\delta_\psi=1$, and $h_m(\{\phi_t\})=1$ if and only if $m=m_\psi/|m_\psi|$, the normalized probability measure of m_ψ.*

We recall some basic notions about entropy; we refer to ([17], [14]) for details.

Measurable partitions and entropy. Let $(\mathcal{X},\mathcal{M},m)$ be a probability space, where $\mathcal{M}$ is a σ-algebra and m is a probability measure. By a partition ζ of $\mathcal{X}$, we mean a collection of disjoint non-empty measurable subsets of $\mathcal{X}$ whose union is $\mathcal{X}$. For a partition ζ of $\mathcal{X}$ and $x\in\mathcal{X}$, we denote by $\zeta(x)$ the element of ζ containing x, called the *atom* at x. Let $\mathcal{M}_\zeta\subset\mathcal{M}$ be the sub σ-algebra generated by the atoms of ζ. A partition ζ of $\mathcal{X}$ is called *m-measurable* if it admits a separation by countably many elements in $\mathcal{M}_\zeta$. More precisely, ζ is m-measurable if there exist a m-conull subset $\mathcal{Y}\subset\mathcal{X}$ and a sequence $\{Y_i\in\mathcal{M}_\zeta : i\in\mathbb{N}\}$ such that for any distinct atoms z,z' of ζ, there exists $i\in\mathbb{N}$ such that either $z\cap\mathcal{Y}\subset Y_i$ and $z'\cap\mathcal{Y}\subset\mathcal{X}-Y_i$, or $z\cap\mathcal{Y}\subset\mathcal{X}-Y_i$ and $z'\cap\mathcal{Y}\subset Y_i$.

For an m-measurable partition ζ and m-a.e. $x\in\mathcal{X}$, we denote by $m_{\zeta(x)}$ the *conditional measure* on the atom $\zeta(x)$ so that the following holds [14, Theorem 5.9]: for any measurable $Y\subset\mathcal{X}$, we have

- $x\mapsto m_{\zeta(x)}(Y\cap\zeta(x))$ is measurable;
- $m(Y)=\int_{\mathcal{X}} m_{\zeta(x)}(Y\cap\zeta(x))\ dm(x)$.

For two m-measurable partitions ζ, ζ', we say that ζ is *finer* than ζ' and write $\zeta \succ \zeta'$ if for m-a.e. $x \in \mathcal{X}$, $\zeta(x) \subset \zeta'(x)$. For a sequence of m-measurable partitions ζ_i, we denote by $\bigvee_i \zeta_i$ the smallest m-measurable partition finer than all ζ_i.

Given an m-measurable partition ζ and an m-measurable map $\varphi : \mathcal{X} \to \mathcal{X}$, the pull-back $\varphi^{-1}\zeta$ is an m-measurable partition with atoms $(\varphi^{-1}\zeta)(x) = \varphi^{-1}(\zeta(\varphi(x)))$. We say that ζ is *φ-decreasing* if $\varphi^{-1}\zeta \succ \zeta$ and *φ-generating* if $\bigvee_{i\in\mathbb{N}} \varphi^{-i}\zeta$ is m-equivalent to the partition consisting of points.

Let $\varphi : \mathcal{X} \to \mathcal{X}$ be an m-measure-preserving transformation. For a countable partition ζ, the *entropy of ζ relative to m* is

$$H_m(\zeta) := \int_{\mathcal{X}} -\log m(\zeta(x))\ dm(x)$$

with the convention that $\infty \cdot 0 = 0$. The average entropy of ζ is defined as

$$H_m(\varphi, \zeta) := \lim_{n\to\infty} \frac{1}{n} H_m\left(\bigvee_{i=0}^{n-1} \varphi^{-i}\zeta\right).$$

The *metric entropy* of φ with respect to m is defined as

$$h_m(\varphi) := \sup H_m(\varphi, \zeta)$$

where the supremum is taken over all countable partitions ζ with $H_m(\zeta) < \infty$. For a flow $\{\phi_t\}_{t\in\mathbb{R}}$ on $\mathcal{X}$, we have $h_m(\phi_t) = |t| h_m(\phi_1)$ for all $t \neq 0$. The metric entropy of the flow $\{\phi_t\}$ with respect to m is defined as

$$h_m(\{\phi_t\}) := h_m(\phi_1).$$

For a φ-decreasing m-measurable partition ζ, we also define

$$h_m(\varphi, \zeta) := \int_{\mathcal{X}} -\log m_{\zeta(x)}((\varphi^{-1}\zeta)(x))\ dm(x).$$

Partition realizing the entropy. Recall the foliations $\tilde{W}^{\pm}$ of $\tilde{\Omega}_\psi$ and $W^{\pm}$ of Ω_ψ from (4.1) and (4.2). Let m be a probability measure on Ω_ψ and $\tilde{m}$ the Γ-invariant lift of m to $\tilde{\Omega}_\psi$. A Γ-invariant partition $\tilde{\zeta}$ of $\tilde{\Omega}_\psi$ is called *$\tilde{m}$-measurable* if the induced partition ζ on Ω_ψ is m-measurable. We say that an $\tilde{m}$-measurable partition $\tilde{\zeta}$ is *subordinated to* $\tilde{W}^+$ if for $\tilde{m}$-a.e. $\tilde{x} \in \tilde{\Omega}_\psi$, there exist precompact open neighborhoods $\tilde{\mathcal{U}}_1$ and $\tilde{\mathcal{U}}_2$ of $\tilde{x}$ in $\tilde{W}^+(\tilde{x})$ such that

$$\tilde{\mathcal{U}}_1 \subset \tilde{\zeta}(\tilde{x}) \subset \tilde{\mathcal{U}}_2$$

Proposition 10.2. *Let $\tau > 0$. Let m be a probability measure on Ω_ψ which is invariant and ergodic under ϕ_τ and $\tilde{m}$ its lift to $\tilde{\Omega}_\psi$. Then there exists a Γ-invariant $\tilde{m}$-measurable partition $\tilde{\zeta}$ of $\tilde{\Omega}_\psi$ subordinated to $\tilde{W}^+$ such that its projection ζ is an m-measurable ϕ_τ-decreasing and generating partition of Ω_ψ which satisfies*

$$h_m(\phi_\tau) = h_m(\phi_\tau, \zeta) < \infty.$$

The most delicate part of the proof of this proposition lies in the construction of the partition which is subordinated to the unstable foliation $\tilde{W}^+$. The exponential expansion property of the flow $\{\phi_t\}$ on Ω_ψ (Theorem 8.1) was obtained precisely for this purpose. Other parts of Proposition 10.2 can be obtained by similar argument in [24].

Proof of Proposition 10.2. Let $d^\pm$ and d_ε^+ be functions on $\tilde{\Omega}_\psi \times \tilde{\Omega}_\psi$ given in Theorem 8.1 for some fixed $\varepsilon > 0$. Fix $u \in \tilde{\Omega}_\psi$. For $r > 0$, we set

$$\tilde{C}(u,r) = \left\{ v \in \tilde{\Omega}_\psi : \begin{array}{c} \exists s \in (-r,r), w \in \tilde{W}^-(\phi_s u) \text{ with } d^-(\phi_s u, w) < r \\ \text{s.t. } v \in \tilde{W}^+(w) \text{ and } d_\varepsilon^+(w,v) < r \end{array} \right\}.$$

Fix $\rho > 0$ small enough so that the projection $\tilde{\Omega}_\psi \to \Omega_\psi$ is injective on $\tilde{C}(u, 4\rho)$. For $0 < r < 4\rho$, we denote by $C(u,r)$ the image of $\tilde{C}(u,r)$ under the projection $\tilde{\Omega}_\psi \to \Omega_\psi$.

We define a function $\ell : \Omega_\psi \to \mathbb{R}$ as follows: for each $x \in C(u,\rho)$, let $\tilde{x} \in \tilde{C}(u,\rho)$ be the unique lift of x. It follows from the description of $\tilde{W}^\pm$ in Lemma 4.5 that there exist unique $s \in (-\rho,\rho)$ and $\tilde{y} \in \tilde{W}^-(\phi_s u)$ such that $\tilde{x} \in \tilde{W}^+(\tilde{y})$, $d^-(\phi_s u, \tilde{y}) < \rho$ and $d_\varepsilon^+(\tilde{y},\tilde{x}) < \rho$. We set

$$\ell(x) := \max(s, d^-(\phi_s u, \tilde{y}), d_\varepsilon^+(\tilde{y},\tilde{x})).$$

For $x \in \Omega_\psi - C(u,\rho)$, we then set $\ell(x) := \rho$.

For each $0 < r < \rho$, let $\tilde{\zeta}_r'$ be the partition of $\tilde{\Omega}_\psi$ with atoms $\gamma\tilde{C}(u,r) \cap \tilde{W}^+(\tilde{x})$ for $\tilde{x} \in \tilde{\Omega}_\psi$, $\gamma \in \Gamma$ and $\tilde{\Omega}_\psi - \Gamma\tilde{C}(u,r)$. We then define

$$\tilde{\zeta}_r := \bigvee_{i=0}^{\infty} \phi_\tau^i \tilde{\zeta}_r'.$$

Let ζ_r' and ζ_r be the partitions obtained by projecting $\tilde{\zeta}_r'$ and $\tilde{\zeta}_r$ to Ω_ψ respectively. Then $\zeta_r = \bigvee_{i=0}^\infty \phi_\tau^i \zeta_r'$ since the Γ-action commutes with the flow $\{\phi_t\}$. It is clear that ζ_r is ϕ_τ-decreasing. In view of the construction of $\tilde{\zeta}$ which uses atoms $\gamma\tilde{C}(u,r) \cap W^+(\tilde{x})$, we can verity that ζ_r is m-measurable by a same argument as in [24, Proposition 1]. Denote by $\tilde{m}$ is the lift of m to $\tilde{\Omega}_\psi$. Let d be the metric on Ω_ψ considered in Proposition 4.6. By the ergodicity of m, we have that for m-a.e. $x \in \Omega_\psi$, $\phi_\tau^k x \in C(u,r)$ for infinitely many $k \in \mathbb{N}$, and hence $\zeta_r'(\phi_\tau^k x)$ is contained in a uniformly bounded set $C(u,r) \cap W^+(\phi_\tau^k x)$ with respect to d. Since $(\phi_\tau^{-k}\zeta_r)(x) \subset \phi_\tau^{-k}(\zeta_r'(\phi_\tau^k x))$, it follows from Proposition 4.6 that ζ_r is ϕ_τ-generating. Similarly, for $\tilde{m}$-a.e. $\tilde{x} \in \tilde{\Omega}_\psi$, we have $\phi_\tau^{-k}\tilde{x} \in \gamma\tilde{C}(u,r)$ for some $k \in \mathbb{N}$ and $\gamma \in \Gamma$. Hence we have $\tilde{\zeta}_r(\tilde{x}) \subset \phi_\tau^k(\tilde{\zeta}_r'(\phi_\tau^{-k}\tilde{x})) \subset \phi_\tau^k \gamma\tilde{C}(u,r) \cap \tilde{W}^+(\tilde{x})$, and therefore $\tilde{\zeta}_r(\tilde{x})$ is a precompact subset of $\tilde{W}^+(\tilde{x})$.

We now show the most delicate part of the proof that we can take $r > 0$ so that $\tilde{\zeta}_r(\tilde{x})$ contains an open neighborhood of $\tilde{x}$ in $W^+(\tilde{x})$ for $\tilde{m}$-a.e. $\tilde{x} \in \tilde{\Omega}_\psi$. We use Theorem 8.1 in a crucial way.

Consider the push-forward $\ell_* m$ of the measure m by ℓ, which is a probability measure on $[0,\rho] \subset \mathbb{R}$. For any $\varepsilon_0 \in (0,1)$, we have that

$$\mathrm{Leb}\left(\left\{r \in (0,\rho) : \sum_{k=0}^{\infty} (\ell_* m)([r-\varepsilon_0^k, r+\varepsilon_0^k]) < \infty\right\}\right) = \rho$$

by [20, Proposition 3.2]. Since m is ϕ_τ-invariant, this is same to say that

$$\mathrm{Leb}\left(\left\{r \in (0,\rho) : \sum_{k=0}^{\infty} m(\{x : |\ell(\phi_\tau^{-k}x) - r| < \varepsilon_0^k\}) < \infty\right\}\right) = \rho.$$

We fix a constant $e^{-\varepsilon\alpha\tau} < \varepsilon_0 < 1$ where $\alpha > 0$ is a constant given in Theorem 8.1. We can therefore choose $0 < r < \rho/2$ so that $m(\partial C(u,r)) = 0$ and that

$$\sum_{k=0}^{\infty} m(\{x : |\ell(\phi_\tau^{-k}x) - r| < \varepsilon_0^k\}) < \infty.$$

Let Ω'_ψ be the set of all $x \in \Omega_\psi - \bigcup_{k=0}^{\infty} \phi^k \partial C(u,r)$ satisfying that for some $N_0 = N_0(x) > 0$, we have

$$\ell(\phi_\tau^{-k}x) < r - \varepsilon_0^k \quad \text{or} \quad \ell(\phi_\tau^{-k}x) > r + \varepsilon_0^k \tag{10.1}$$

for all $k \geq N_0$. Since $m(\partial C(u,r)) = 0$, it follows from the classical Borel-Cantelli lemma that $m(\Omega'_\psi) = 1$. Let $x \in \Omega'_\psi$ be an arbitrary point and corresponding $N_0 = N_0(x)$. We fix a lift $\tilde{x} \in \tilde{\Omega}_\psi$ of x.

For $\tilde{y} \in \tilde{\Omega}_\psi$, we write y for its projection to Ω_ψ. Fix a compact subset $Q \subset \tilde{\Omega}_\psi$ containing

$$\bigcup_{v_0 \in \tilde{C}(u,\rho)} \{v \in \tilde{W}^+(v_0) : d^+(v, v_0) \leq b\}$$

where $b \geq 1$ is the constant given in Theorem 8.1.

We set

$$r_1 := \min\left(\frac{1}{2}, \frac{1}{b(2c)^{1/\varepsilon}}\right) > 0$$

where $c = c_Q \geq 1$ is as given in Theorem 8.1(3). Let

$$\tilde{\mathcal{U}} = \{\tilde{y} \in \tilde{W}^+(\tilde{x}) : d^+(\tilde{x}, \tilde{y}) < r_1\};$$

this is a precompact neighborhood of $\tilde{x}$ in $\tilde{W}^+(\tilde{x})$. Let $\mathcal{U}$ be the image of $\tilde{\mathcal{U}}$ in Ω_ψ. We claim that for each $k \geq N_0$, either

$$\phi_\tau^{-k}(\tilde{\mathcal{U}}) \subset \gamma^{-1}\tilde{C}(u,r) \text{ for some } \gamma \in \Gamma \quad \text{or} \quad \phi_\tau^{-k}(\tilde{\mathcal{U}}) \cap \Gamma\tilde{C}(u,r) = \emptyset. \tag{10.2}$$

Fix $k \geq N_0$. Recall that x satisfies either $\ell(\phi_\tau^{-k}x) < r - \varepsilon_0^k$ or $\ell(\phi_\tau^{-k}x) > r + \varepsilon_0^k$. Consider the first case. This implies that there exists $\gamma \in \Gamma$ such that $\gamma\phi_\tau^{-k}\tilde{x} \in \tilde{C}(u, r - \varepsilon_0^k)$. We then have

$$d^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y}) = d^+(\phi_\tau^{-k}\tilde{x}, \phi_\tau^{-k}\tilde{y}) \leq b e^{-\alpha\tau k} d^+(\tilde{x}, \tilde{y}).$$

by (8.10) and Theorem 8.1(1). In particular, we have $\gamma\phi_\tau^{-k}\tilde{y} \in Q$ and hence

$$d_\varepsilon^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y}) \le c d^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y})^\varepsilon \le cb^\varepsilon e^{-\varepsilon\alpha\tau k} d^+(\tilde{x}, \tilde{y})^\varepsilon \tag{10.3}$$

by Theorem 8.1(3). Let $\tilde{y} \in \tilde{\mathcal{U}}$, and hence $d^+(\tilde{x}, \tilde{y}) < r_1$. Since $e^{-\varepsilon\alpha\tau} < \varepsilon_0$, we then have

$$d_\varepsilon^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y}) < \varepsilon_0^k$$

by (10.3), and therefore $\gamma\phi_\tau^{-k}\tilde{y} \in \tilde{C}(u, r)$. Hence

$$\phi_\tau^{-k}(\tilde{\mathcal{U}}) \subset \gamma^{-1}\tilde{C}(u, r),$$

proving (10.2) in this case.

Now consider the case when $\ell(\phi_\tau^{-k}x) > r + \varepsilon_0^k$. In this case, we claim that $\phi_\tau^{-k}(\tilde{\mathcal{U}}) \cap \Gamma\tilde{C}(u, r) = \emptyset$. Suppose not. Then there exists $\gamma \in \Gamma$ and some $\tilde{y} \in \tilde{W}^+(\tilde{x})$ such that $d^+(\tilde{x}, \tilde{y}) < r_1$ and $\gamma\phi_\tau^{-k}\tilde{y} \in \tilde{C}(u, r)$. By the same argument as above, $\gamma\phi_\tau^{-k}\tilde{x} \in Q$ and hence

$$d_\varepsilon^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y}) \le cb^\varepsilon e^{-\varepsilon\alpha\tau k} d^+(\tilde{x}, \tilde{y})^\varepsilon.$$

Since $d^+(\tilde{x}, \tilde{y}) < r_1$, we have $\gamma\phi_\tau^{-k}\tilde{x} \in \tilde{C}(u, r + \varepsilon_0^k)$. This is a contradiction since $\ell(\phi_\tau^{-1}x) > r + \varepsilon_0^k$, proving the claim.

The claim (10.2) implies that $\phi_\tau^{-k}(\tilde{\mathcal{U}})$ lies in a single atom of $\tilde{\zeta}_r'$ for each $k \ge N_0$.

Since $\phi_\tau^{-k}\tilde{x} \notin \partial\gamma^{-1}\tilde{C}(u, r)$ for all $k \in \mathbb{N}$ and $\gamma \in \Gamma$, we can find a small neighborhood $\tilde{\mathcal{U}}' \subset \tilde{\mathcal{U}}$ of $\tilde{x}$ in $\tilde{W}^+(\tilde{x})$ such that $\phi_\tau^{-k}(\tilde{\mathcal{U}}')$ is entirely contained in some $\gamma^{-1}\tilde{C}(u, r)$, $\gamma \in \Gamma$ or disjoint from $\Gamma\overline{C(u, r)}$ for each $0 \le k \le N_0$. Therefore $\phi_\tau^{-k}(\tilde{\mathcal{U}}')$ is contained in a single atom of $\tilde{\zeta}_r'$ for all $k \in \mathbb{N}$. This proves that the atom of $\tilde{\zeta}_r$ containing $\tilde{x}$ also contains $\tilde{\mathcal{U}}'$. Since $x \in \Omega_\psi'$ is arbitrary, $\tilde{\zeta}_r$ is subordinated to $\tilde{W}^+$.

The rest of the argument is a similar entropy computation as in the deduction of [24, Proposition 4] from [24, Proposition 1]. $\square$

Proof of Theorem 10.1. The deduction of Theorem 10.1 from Proposition 10.2 can be done similarly to [24].

First, note that $\delta_\psi = 1$ since ψ is tangent to ψ_Γ^θ ([11, Theorem 10.1], [18, Theorem 4.5]). For $g \in G$ such that $[g] \in \tilde{\Omega}_\psi$, we consider the measure $\mu_{\tilde{W}^+([g])}$ on $\tilde{W}^+([g])$ given by

$$d\mu_{\tilde{W}^+([g])}([gn]) = e^{\psi(\beta^\theta_{(gn)^+}(e, gn))} d\nu((gn)^+)$$

for $n \in N_\theta^+$. It follows from the definition that for all $a \in A_\theta$, we have

$$\frac{da_*\mu_{\tilde{W}^+([g])}}{d\mu_{\tilde{W}^+([ga])}}(x) = e^{-\psi(\log a)}. \tag{10.4}$$

We write m^{pr} for the normalized probability measure $m_\psi/|m_\psi|$. Denote by $\tilde{m}^{pr}$ its lift to $\tilde{\Omega}_\psi$. The following can be obtained by directly checking the condition for conditional measures:

Lemma 10.3. *Let $\tilde{\zeta}$ be an $\tilde{m}^{pr}$-measurable partition of $\tilde{\Omega}_\psi$ subordinated to $\tilde{W}^+$. Then the family of conditional measures of $\tilde{m}^{pr}$ with respect to $\tilde{\zeta}$ is given by*

$$d\tilde{m}^{pr}_{\tilde{\zeta}(\tilde{x})}(w) := \frac{\mathbb{1}_{\tilde{\zeta}(\tilde{x})}(w)}{\mu_{\tilde{W}^+(\tilde{x})}(\tilde{\zeta}(\tilde{x}))} d\mu_{\tilde{W}^+(\tilde{x})}(w) \quad \text{for } \tilde{x} \in \tilde{\Omega}_\psi.$$

By Theorem 9.1, m_ψ is finite, and hence it follows from Theorem 4.2 that m^{pr} is $\{\phi_t\}$-ergodic. It is a general fact that m^{pr} is ergodic for the transformation ϕ_t for uncountably many t [24, Lemma 7]. Fix $\tau > 0$ so that m^{pr} is ϕ_τ-ergodic. Now let m be a probability $\{\phi_t\}$-invariant measure on Ω_ψ. Considering the ergodic decomposition of m, we may assume that m is ϕ_τ-ergodic without loss of generality [14, (3.5a)].

We now consider the partition $\tilde{\zeta}$ given by Proposition 10.2 for the measure m, its lift $\tilde{m}$, and the transformation ϕ_τ. Since $\tilde{\zeta}$ is subordinated to $\tilde{W}^+$, the measure

$$d\tilde{m}^{pr}_{\tilde{\zeta}(\tilde{x})}(w) := \frac{\mathbb{1}_{\tilde{\zeta}(\tilde{x})}(w)}{\mu_{\tilde{W}^+(\tilde{x})}(\tilde{\zeta}(\tilde{x}))} d\mu_{\tilde{W}^+(\tilde{x})}(w)$$

and the function

$$\tilde{G}(\tilde{x}) := -\log \mu_{\tilde{W}^+(\tilde{x})}(\tilde{\zeta}(\tilde{x}))$$

are well-defined for $\tilde{m}$-a.e. $\tilde{x} \in \tilde{\Omega}_\psi$. Note that since $\tilde{\zeta}$ is a partition for the measure $\tilde{m}$, it may not be $\tilde{m}^{pr}$-measurable and hence Lemma 10.3 does not apply to $\tilde{\zeta}$. It follows from (10.4) that for $\tilde{m}$-a.e. $\tilde{x} \in \tilde{\Omega}_\psi$, we have

$$-\log \tilde{m}^{pr}_{\tilde{\zeta}(\tilde{x})}((\phi_\tau^{-1}\tilde{\zeta})(\tilde{x})) = \tau + (\tilde{G} \circ \phi_\tau)(\tilde{x}) - \tilde{G}(\tilde{x}). \tag{10.5}$$

This implies

$$\tilde{G} \circ \phi_\tau - \tilde{G} \geq -\tau$$

$\tilde{m}$-a.e. Since $\tilde{G}$ is Γ-invariant, it induces the function $G : \Omega_\psi \to \mathbb{R}$. By [24, Lemme 8], we have $\int G \circ \phi_\tau - G \; dm = 0$ and therefore

$$\int -\log m^{pr}_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x)) \; dm(x) = \tau. \tag{10.6}$$

where $m^{pr}_{\zeta(x)}$ is the measure on $\zeta(x)$ induced by $\tilde{m}^{pr}_{\tilde{\zeta}(\tilde{x})}$.

We can now show $h_{m^{pr}}(\{\phi_t\}) = 1$. Indeed, if we consider the special case that $m = m^{pr}$, then the partition ζ becomes an m^{pr}-measurable partition given by Proposition 10.2. Hence by Lemma 10.3, the measure $m^{pr}_{\zeta(x)}$ forms the family of conditional measure for m^{pr}. Therefore the above identity (10.6) yields

$$h_{m^{pr}}(\phi_\tau) = h_{m^{pr}}(\phi_\tau, \zeta) = \int -\log m^{pr}_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x)) \; dm(x) = \tau.$$

Hence

$$h_{m^{pr}}(\{\phi_t\}) = h_{m^{pr}}(\phi_\tau)/\tau = 1.$$

It remains to show that for a general m, $h_m(\{\phi_t\}) \le 1$ and that $h_m(\{\phi_t\}) = 1$ implies $m = m^{pr}$. We define the following function: for m-a.e. $x \in \Omega_\psi$,

$$F(x) := \frac{m^{pr}_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x))}{m_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x))} \quad \text{if} \quad m_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x)) > 0,$$

and $F(x) := 0$ otherwise. By [24, Fait 9], both functions F and $\log F$ are m-integrable and $\int F\, dm \le 1$. Since

$$\int \log F\, dm = -\tau + h_m(\phi_\tau, \zeta) = -\tau + h_m(\phi_\tau) = -\tau + \tau h_m(\{\phi_t\})$$

by (10.6) and the choice of ζ, we apply Jensen's inequality and obtain

$$-\tau + \tau h_m(\{\phi_t\}) \le \log\left(\int F\, dm\right) \le 0.$$

This proves

$$h_m(\{\phi_t\}) \le 1.$$

Now suppose that $h_m(\{\phi_t\}) = 1$. This implies that the equality holds in Jensen's inequality, that is, $\log\left(\int F\, dm\right) = 0$, which means that $F = 1$ m-a.e. It follows that the two conditional measures $m^{pr}_{\zeta(x)}$ and $m_{\zeta(x)}$ coincide on the σ-algebra generated by $(\phi_\tau^{-1}\zeta)(x)$ for m-a.e. x. Since this holds after replacing ϕ_τ with ϕ_τ^k for any $k \in \mathbb{N}$ and the partition ζ is ϕ_τ-generating, we have

$$m^{pr}_{\zeta(x)} = m_{\zeta(x)} \quad \text{for } m\text{-a.e. } x \in \Omega_\psi.$$

Then the equality between measures $m = m^{pr}$ follows from the Hopf argument. Indeed, let $f : \Omega_\psi \to \mathbb{R}$ be a compactly supported continuous function. By the Birkhoff ergodic theorem, the set

$$\mathcal{Z} := \left\{x \in \Omega_\psi : \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\phi_s x)ds = m^{pr}(f)\right\}$$

has a full m^{pr}-measure. Then $\mathcal{Z}$ is invariant under the flow $\{\phi_t\}$ and moreover, since f is uniformly continuous, $x \in \mathcal{Z}$ implies $W^-(x) \subset \mathcal{Z}$ by Proposition 4.6. By the quasi-product structure of the BMS measure m^{pr}, this implies that for all $x \in \Omega_\psi$, $\mathcal{Z} \cap W^+(x)$ has full $\mu_{W^+(x)}$-measure. Hence $\mathcal{Z} \cap \zeta(x)$ has full $m^{pr}_{\zeta(x)}$-measure for m-a.e. $x \in \Omega_\psi$ by the definition of $m^{pr}_{\zeta(x)}$. Hence $\mathcal{Z} \cap \zeta(x)$ has full $m_{\zeta(x)}$-measure for m-a.e. $x \in \Omega_\psi$. Since $m_{\zeta(x)}$ is a conditional measure for m, this implies $m(\mathcal{Z}) = 1$, and therefore $m(f) = m^{pr}(f)$ by applying the Birkhoff ergodic theorem again to m. This finishes the proof. $\square$

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Department of Mathematics, Yale University, New Haven, CT 06511
Email address: `dongryul.kim@yale.edu`

Department of Mathematics, Yale University, New Haven, CT 06511
Email address: `hee.oh@yale.edu`