

A GLOBAL SHADOW LEMMA FOR RELATIVELY MORSE GROUPS IN HIGHER RANK

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ABSTRACT. Patterson-Sullivan measures encode the distribution of orbits of discrete group actions near the boundary. In this paper, we prove a global shadow lemma for Patterson-Sullivan measures associated to relatively Morse subgroups of higher-rank semisimple Lie groups. The estimate is uniform for shadows centered at arbitrary points in a Gromov model, including points deep in the cuspidal part. This extends the global shadow lemma of Stratmann-Velani for geometrically finite real hyperbolic groups. As applications, we obtain uniform local estimates for Patterson-Sullivan measures, and we give sufficient conditions under which these measures agree, up to scale, with the Hausdorff measure defined by the associated visual quasi-metric.

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1. INTRODUCTION

One of the basic themes in geometric group theory and dynamics is that the large-scale geometry of a discrete group action is reflected in the way its orbits accumulate at infinity. Patterson-Sullivan theory makes this principle quantitative: starting from the exponential growth of an orbit, it produces natural measures on the boundary, and these measures reveal the asymptotic distribution of the orbit through shadow lemmas, counting estimates, and equidistribution phenomena. In real hyperbolic geometry, this theory has

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been a fundamental tool in the study of Kleinian groups and negatively curved manifolds. This paper develops a higher-rank analogue for relatively Morse subgroups of semisimple Lie groups, where the visual boundary is replaced by an appropriate flag manifold. The main goal is to prove a global shadow lemma for the corresponding Patterson-Sullivan measures. We begin with the classical real hyperbolic case, both as motivation and as a guide to the higher-rank relative setting considered later.

Let $\Gamma < \text{Isom}^+(\mathbb{H}^n)$ be a non-elementary discrete subgroup, let $o \in \mathbb{H}^n$, and let $\Lambda \subset \partial\mathbb{H}^n$ be the limit set of Γ . Patterson [20] and Sullivan [24] constructed a Borel probability measure ν supported on Λ whose transformation rule is governed by the critical exponent $\delta_\Gamma > 0$:

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\delta_\Gamma \beta_\xi(o, \gamma o)} \quad \text{for all } \gamma \in \Gamma \text{ and } \nu\text{-a.e. } \xi \in \Lambda.$$

Here β_ξ denotes the Busemann function. The measure ν is now referred to as the Patterson-Sullivan measure of Γ .

A fundamental feature of this measure is Sullivan's shadow lemma. For $x, y \in \mathbb{H}^n$ and $R > 0$, let $O_R(x, y) \subset \partial\mathbb{H}^n$ denote the shadow of the ball $B(y, R)$ seen from x , namely the set of endpoints ξ such that the geodesic ray $[x, \xi] \subset \mathbb{H}^n$ intersects the R -neighborhood of y .

Theorem 1.1 (Shadow Lemma [24, Proposition 3]). *For all sufficiently large $R > 0$, there exists $C > 1$ such that*

$$C^{-1} e^{-\delta_\Gamma d(o, \gamma o)} \leq \nu(O_R(o, \gamma o)) \leq C e^{-\delta_\Gamma d(o, \gamma o)} \quad \text{for all } \gamma \in \Gamma.$$

Thus the Patterson-Sullivan measure of a shadow is comparable to the exponential of minus the orbit distance: $\nu(O_R(o, \gamma o)) \asymp e^{-\delta_\Gamma d(o, \gamma o)}$, with implied constants independent of γ . This estimate is the basic prototype for the global shadow lemma proved in this paper.

Geometrically finite groups. For geometrically finite groups, Sullivan's shadow lemma admits a global version due to Stratmann-Velani [23]. Let $\Gamma < \text{Isom}^+(\mathbb{H}^n)$ be geometrically finite with limit set $\Lambda \subset \partial\mathbb{H}^n$, and let hull $\Lambda \subset \mathbb{H}^n$ be its convex hull. Choose a Γ -invariant family \mathcal{B} of pairwise disjoint cusp horoballs in hull Λ whose complement has compact quotient. Let \mathcal{P} be a set of representatives of the corresponding maximal parabolic subgroups, and write $\delta_{\mathcal{P}}$ for the critical exponent of $P \in \mathcal{P}$. It is known that $\delta_{\mathcal{P}} < \delta_\Gamma$ ([24], [25]).

Theorem 1.2 (Global Shadow Lemma [23, Theorem 2]). *For all sufficiently large $R > 0$, the following holds. Let $\xi \in \Lambda$ and $x \in [o, \xi]$. If x lies in a horoball $B \in \mathcal{B}$ whose stabilizer is conjugate to $P \in \mathcal{P}$, then*

$$\nu(O_R(o, x)) \asymp e^{-\delta_\Gamma d(o, x)} e^{(2\delta_{\mathcal{P}} - \delta_\Gamma)d(\Gamma o, x)},$$

with implied constants independent of ξ , x , and B .

In hull Λ but away from the cusp horoballs, the quantity $d(\Gamma o, x)$ is uniformly bounded, and the estimate reduces to the usual shadow lemma $\nu(O_R(o, x)) \asymp e^{-\delta_\Gamma d(o, x)}$.

Relatively Anosov and relatively Morse groups. Relatively Anosov subgroups provide a higher-rank analogue of geometrically finite Kleinian groups. Let G be a connected semisimple real algebraic group with Cartan decomposition $G = KA^+K$ where K is a maximal compact subgroup. Let $X := G/K$ denote the associated Riemannian symmetric space. Let $\mathfrak{a}^+ := \log A^+$, let Π be the corresponding set of simple roots, and fix a non-empty subset $\theta \subset \Pi$. We denote by P_θ the associated standard parabolic subgroup and by

$$\mathcal{F}_\theta := G/P_\theta$$

the corresponding flag variety. We also set

$$\mathfrak{a}_\theta := \bigcap_{\alpha \in \Pi - \theta} \ker \alpha$$

and regard \mathfrak{a}_θ^* as a subspace of \mathfrak{a}^* via the canonical projection $p_\theta : \mathfrak{a} \rightarrow \mathfrak{a}_\theta$. After replacing θ by $\theta \cup i(\theta)$, if necessary, we assume that

$$\theta = i(\theta),$$

where i is the opposition involution on \mathfrak{a} .

Let $\Gamma < G$ be a discrete subgroup which is hyperbolic relative to a finite collection \mathcal{P} of finitely generated infinite subgroups. We write $\partial(\Gamma, \mathcal{P})$ for its Bowditch boundary. A Gromov model for (Γ, \mathcal{P}) is a proper geodesic Gromov hyperbolic space Y on which Γ acts properly discontinuously, together with a Γ -invariant collection \mathcal{B} of disjoint horoballs whose stabilizers are conjugates of subgroups in \mathcal{P} , and on whose complement Γ acts cocompactly. We identify ∂Y with $\partial(\Gamma, \mathcal{P})$, and we assume that it contains at least three points, i.e., (Γ, \mathcal{P}) is non-elementary.

We say that Γ is θ -Anosov relative to \mathcal{P} if it is θ -regular and admits a transverse Γ -equivariant boundary map

$$\zeta : \partial Y \rightarrow \mathcal{F}_\theta.$$

See section 4 for its precise definition. The image of this map is the θ -limit set, denoted by Λ_θ .

For $\psi \in \mathfrak{a}_\theta^*$, a (Γ, ψ) -Patterson-Sullivan measure is a Borel probability measure ν on Λ_θ satisfying

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi^\theta(e, \gamma))} \quad \text{for all } \gamma \in \Gamma \text{ and } \nu\text{-a.e. } \xi \in \Lambda_\theta,$$

where β^θ is the \mathfrak{a}_θ -valued Busemann map; see (2.3). This notion of higher-rank Patterson-Sullivan measure was introduced by Quint [21].

Let $\mu : G \rightarrow \mathfrak{a}^+$ be the Cartan projection. Let $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ denote the asymptotic cone of the Cartan projection $\mu(\Gamma)$ of Γ , called the limit cone of Γ . The shadow lemma for Patterson-Sullivan measures was proved for

shadows in the higher-rank symmetric space $X = G/K$ in [18, Lemma 7.8] and [17, Lemma 7.2]. For relatively θ -Anosov subgroups, the compatibility between shadows in the Gromov model Y and shadows in X was established in [16, Proposition 5.7].¹ Combining these results gives the following orbit-shadow estimate.

Theorem 1.3 (Shadow Lemma). *Let $\Gamma < G$ be θ -Anosov relative to \mathcal{P} , and fix $o_Y \in Y$. Let ν be a (Γ, ψ) -Patterson-Sullivan measure on Λ_θ for some $\psi \in \mathfrak{a}_\theta^*$. Then, for all sufficiently large $R > 0$,*

$$\nu(\zeta(O_R(o_Y, \gamma o_Y))) \asymp e^{-\psi(\mu(\gamma))}$$

for all $\gamma \in \Gamma$, with implied constants independent of γ .

The relatively Morse condition strengthens the relative Anosov condition by requiring the relative geometry of Y to be realized inside the symmetric space. We say that Γ is θ -Morse relative to \mathcal{P} if there exists a Γ -equivariant quasi-isometric embedding

$$f : Y \rightarrow X$$

such that, writing $\mathcal{L}_f \subset \mathfrak{a}^+$ for the asymptotic cone of the Cartan projections

$$\{\mu(f(x)^{-1}f(y)) : x, y \in Y\},$$

we have

$$\mathcal{L}_f \cap \ker \alpha = \{0\} \quad \text{for every } \alpha \in \theta.$$

Thus being relative Anosov gives a boundary map into the flag variety, whereas the relatively Morse condition gives a coarse geometric model in the symmetric space whose Cartan projections stay uniformly away from the walls indexed by θ . By the higher-rank Morse lemma of Kapovich-Leeb-Porti [14, Theorem 1.4], the map f extends continuously to a transverse Γ -equivariant homeomorphism

$$f : \partial Y \rightarrow \Lambda_\theta.$$

Fix a basepoint $o_Y \in Y$. For $\psi \in \mathfrak{a}_\theta^*$ positive on $\mathcal{L}_f - \{0\}$, define

$$\mathbf{d}_\psi(x, y) := \psi(\mu(f(x)^{-1}f(y))) \quad \text{for } x, y \in Y,$$

and, for subsets $E, F \subset Y$, set

$$\mathbf{d}_\psi(E, F) := \inf_{x \in E, y \in F} \mathbf{d}_\psi(x, y).$$

Theorem 1.3 estimates shadows centered at orbit points. The main result of this paper is a global version for relatively Morse groups, where the center of the shadow may be an arbitrary point along a geodesic ray in the Gromov model.

¹The reference [16] treats a specific Gromov model, but the same proof applies to the Gromov models considered here.

For a subgroup $H < \Gamma$, denote by $\delta_\psi(H)$ the critical exponent of the Poincaré series

$$s \mapsto \sum_{\gamma \in H} e^{-s\psi(\mu(\gamma))}.$$

In the setting below, the strict inequality $\delta_\psi(\mathbf{P}) < \delta_\psi(\Gamma)$ was proved by Canary-Zhang-Zimmer [8].

The following theorem is the higher-rank relatively Morse analogue of the global shadow lemma of Stratmann–Velani [23].

Theorem 1.4 (Global Shadow Lemma). *Let $\Gamma < G$ be θ -Morse relative to \mathcal{P} , and let $\psi \in \mathfrak{a}_\theta^*$ be such that $\psi > 0$ on $\mathcal{L}_f - \{0\}$ and $\delta_\psi(\Gamma) = 1$. Let ν be a (Γ, ψ) -Patterson-Sullivan measure on Λ_θ . Then there exists $C_0 > 0$ such that, for all sufficiently large $R > 0$, the following holds.*

Let $\xi \in \partial Y$ and $x \in [o_Y, \xi]$. Suppose that $x \in B$ for some horoball $B \in \mathcal{B}$ whose stabilizer is conjugate to $\mathbf{P} \in \mathcal{P}$. Then

$$\begin{aligned} \nu(f(O_R(o_Y, x))) &\asymp e^{-d_\psi(o_Y, x)} e^{d_\psi(\Gamma o_Y, x)} e^{2(\delta_{\bar{\psi}}(\mathbf{P}) - 1)d_{\bar{\psi}}(\Gamma o_Y, x)} \\ &\cdot (C_0 + d_{\bar{\psi}}(\Gamma o_Y, x))^{a_{\bar{\psi}}(\mathbf{P})}, \end{aligned}$$

where $\bar{\psi} = \frac{\psi + \psi \circ i}{2}$ and $a_{\bar{\psi}}(\mathbf{P})$ is a non-negative integer depending on $\bar{\psi}$ and \mathbf{P} . The implied constants are independent of ξ , x , and B .

We also show that when \mathbf{P} is virtually cyclic or when G has rank one, then $a_{\bar{\psi}}(\mathbf{P})$ is zero (Theorem 5.1). It would be interesting to know whether $a_{\bar{\psi}}(\mathbf{P})$ is always zero in the general setting of this theorem.

If x lies in the thick part $Y - \bigcup_{B \in \mathcal{B}} B$, then Theorem 1.4 recovers the usual orbit-shadow estimate. The content of the theorem is therefore the precise correction term that appears when x penetrates a cusp.

Remark 1.5. Bray-Tiozzo [6] proved a global shadow lemma for relatively hyperbolic groups using Patterson-Sullivan measures associated to the Busemann functions of a Gromov model. Our setting is different: although shadows are taken in the Gromov model, the Busemann maps, Patterson-Sullivan measures, and critical exponents come from the ambient higher-rank Lie group.

Example 1.6. Here are two standard examples of relatively Π -Morse groups.

- (1) Let $G = \prod_{i=1}^k \mathrm{SO}^\circ(n_i, 1)$, $n_i \geq 2$. It follows from the work of Bowditch [3] and Yaman [27] that a discrete subgroup $\Gamma < G$ is relatively Π -Anosov if and only if there exists a geometrically finite subgroup $\Gamma_1 < \mathrm{SO}^\circ(n_1, 1)$ and geometrically finite type-preserving representations $\rho_i : \Gamma_1 \rightarrow \mathrm{SO}^\circ(n_i, 1)$, $2 \leq i \leq k$, so that the diagonal embedding $(\mathrm{id} \times \rho_2 \times \cdots \times \rho_k)(\Gamma_1)$ is a finite-index subgroup of Γ . The work of Tukia [26] (also see [9]) then implies that Γ is relatively Π -Morse. Moreover, there exists a Morse embedding f such that $\mathcal{L}_f = \mathcal{L}_\Gamma$.

- (2) Let $G = \mathrm{SL}(n, \mathbb{R})$, $n \geq 2$. We consider a relatively θ -Anosov $\Gamma < G$ with peripheral subgroups \mathcal{P} , such that (Γ, \mathcal{P}) is isomorphic to a geometrically finite Fuchsian group. In this case, Zhu-Zimmer showed that Γ is relatively θ -Morse ([29, Corollary 1.14], [28, Proposition 11.3]). They explicitly constructed a Morse embedding f in [28, Sections 6, 10, 11], based on the notion of cusp representations introduced by Canary-Zhang-Zimmer [7, Proposition 3.6], and their construction gives that $\mathcal{L}_f = \mathcal{L}_\Gamma$.

Local properties of Patterson-Sullivan measures. The global shadow lemma also gives local information on Patterson-Sullivan measures. Assume, in addition, that $\psi = \psi \circ i$. For $\xi, \eta \in \Lambda_\theta$, set

$$d_\psi(\xi, \eta) := e^{-\psi(\mathcal{G}^\theta(\xi, \eta))}$$

where \mathcal{G}^θ is the \mathfrak{a}_θ -valued Gromov product. This is a higher-rank visual quasi-metric on Λ_θ ; denote by $B_\psi(\xi, r)$ the corresponding balls. As applications, we obtain the following uniform local estimates for Patterson-Sullivan measures.

Corollary 1.7 (Uniform local estimates). *Let Γ , ψ , ν be as in Theorem 1.4. Suppose further that $\psi = \psi \circ i$. For every $\kappa \geq 1$, there exists $L_\kappa > 1$ and $C_\kappa > 1$ such that*

$$C_\kappa^{-1} \nu(B_\psi(\xi, L_\kappa r)) \leq \nu(B_\psi(\xi, r)) \leq \kappa^{-1} \nu(B_\psi(\xi, L_\kappa r))$$

for all $\xi \in \Lambda_\theta$ and all $0 < r \leq L_\kappa^{-1}$.

This corollary follows immediately from Theorems 8.1 and 8.2; it records the resulting two-sided comparison at the scale L_κ , while Theorem 8.1 gives the lower bound for every fixed scale $L > 1$.

Finally, we compare ν with the Hausdorff measure defined by the quasi-metric d_ψ in Theorem 9.1. The resulting criterion is governed by the parabolic critical exponents $\delta_\psi(\mathbf{P})$, which measure the growth of peripheral subgroups, compared to the ambient exponent $\delta_\psi(\Gamma)$.

2. PRELIMINARIES

Let G be a connected semisimple real algebraic group. Let P be a minimal parabolic subgroup with a fixed Langlands decomposition $P = MAN$ where A is a maximal real split torus of G , M is the maximal compact subgroup of P commuting with A and N is the unipotent radical of P . Let \mathfrak{g} and \mathfrak{a} denote, respectively, the Lie algebras of G and A . Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ so that $\log N$ consists of positive root subspaces and set $A^+ = \exp \mathfrak{a}^+$. We fix a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. We denote by $\mu : G \rightarrow \mathfrak{a}^+$ the Cartan projection defined by the condition $g \in K \exp \mu(g)K$ for $g \in G$. Let $X = G/K$ be the associated Riemannian symmetric space and $o = [K] \in X$. Fix a K -invariant norm $\|\cdot\|$ on \mathfrak{g} . This induces the left G -invariant Riemannian metric d on X .

Lemma 2.1. [1, Lemma 4.6] *For any compact subset $Q \subset G$, there exists $C = C(Q) > 0$ such that for all $g \in G$,*

$$\sup_{q_1, q_2 \in Q} \|\mu(q_1 g q_2) - \mu(g)\| \leq C.$$

Let Π denote the set of simple roots determined by \mathfrak{a}^+ . We fix a non-empty subset $\theta \subset \Pi$. Let P_θ denote the corresponding standard parabolic subgroup with the convention that $P_\Pi = P$, and set $\mathcal{F}_\theta := G/P_\theta$. We also set

$$\mathfrak{a}_\theta := \bigcap_{\alpha \in \Pi - \theta} \ker \alpha \quad \text{and} \quad \mathfrak{a}_\theta^+ := \mathfrak{a}_\theta \cap \mathfrak{a}^+.$$

Let

$$(2.1) \quad p_\theta : \mathfrak{a} \rightarrow \mathfrak{a}_\theta$$

denote the projection invariant under all Weyl elements fixing \mathfrak{a}_θ pointwise. We write $\mu_\theta := p_\theta \circ \mu : G \rightarrow \mathfrak{a}_\theta^+$. We identify $\mathfrak{a}_\theta^* = \text{Hom}(\mathfrak{a}_\theta, \mathbb{R})$ with the subspace of \mathfrak{a}^* consisting of linear forms invariant under p_θ .

Abusing notation, for $p, q \in X$, we set

$$\mu(p) := \mu(g) \quad \text{and} \quad \mu(p^{-1}q) := \mu(g^{-1}h)$$

for $g, h \in G$ such that $go = p$ and $ho = q$. This definition is independent of the choice of g and h , and we similarly define $\mu_\theta(p)$ and $\mu_\theta(p^{-1}q)$.

Let $w_0 \in K$ represent the longest Weyl element. The opposition involution is

$$i := -\text{Ad}_{w_0} : \mathfrak{a} \rightarrow \mathfrak{a},$$

and it induces an involution of Π , again denoted by i .

Limit set. The subgroup K acts transitively on \mathcal{F}_θ , and hence $\mathcal{F}_\theta \simeq K/M_\theta$ where $M_\theta := P_\theta \cap K$. Set $\xi_\theta := [M_\theta] \in \mathcal{F}_\theta$.

Definition 2.2. For a sequence $g_i \in G$ and $\xi \in \mathcal{F}_\theta$, we write $\lim_{i \rightarrow \infty} g_i = \xi$ and say g_i converges to ξ if

- for each $\alpha \in \theta$, $\alpha(\mu(g_i)) \rightarrow \infty$ as $g_i \rightarrow \infty$;
- $\lim_{i \rightarrow \infty} \kappa_i \xi_\theta = \xi$ in \mathcal{F}_θ for some $\kappa_i \in K$ such that $g_i \in \kappa_i A^+ K$.

The θ -limit set of a discrete subgroup Γ can be defined as follows:

$$(2.2) \quad \Lambda_\theta = \Lambda_\theta(\Gamma) := \{\lim \gamma_i \in \mathcal{F}_\theta : \gamma_i \in \Gamma\}$$

where $\lim \gamma_i$ is defined as in Definition 2.2. If Γ is Zariski dense, this is the unique Γ -minimal subset of \mathcal{F}_θ [1].

Busemann map. The \mathfrak{a} -valued Busemann map $\beta : \mathcal{F}_\Pi \times G \times G \rightarrow \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi)$$

where $\sigma(g^{-1}, \xi) \in \mathfrak{a}$ is the unique element such that $g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N$ for any $k \in K$ with $\xi = kP$. For $(\xi, g, h) \in \mathcal{F}_\theta \times G \times G$, we define

$$(2.3) \quad \beta_\xi^\theta(g, h) := p_\theta(\beta_{\xi_0}(g, h))$$

for any $\xi_0 \in \mathcal{F}_\Pi$ projecting to ξ . This is well-defined independent of the choice of ξ_0 [21, Lemma 6.1]. Moreover, since product map $K \times A \times N \rightarrow G$ is a diffeomorphism, Busemann maps are continuous.

Gromov product. Two points $\xi \in \mathcal{F}_\theta$ and $\eta \in \mathcal{F}_{i(\theta)}$ are said to be *transverse* or be *in general position* if

$$(2.4) \quad \xi = gP_\theta \quad \text{and} \quad \eta = gw_0P_{i(\theta)} \quad \text{for some } g \in G.$$

We set

$$(2.5) \quad \mathcal{F}_\theta^{(2)} = \{(\xi, \eta) \in \mathcal{F}_\theta \times \mathcal{F}_{i(\theta)} : \xi, \eta \text{ are in general position}\}$$

which is the unique open G -orbit in $\mathcal{F}_\theta \times \mathcal{F}_{i(\theta)}$ under the diagonal G -action.

For $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$, we define the \mathfrak{a}_θ -valued Gromov product as

$$(2.6) \quad \mathcal{G}^\theta(\xi, \eta) = \frac{1}{2} \left(\beta_\xi^\theta(e, g) + i \beta_\eta^{i(\theta)}(e, g) \right)$$

where $g \in G$ satisfies $(gP_\theta, gw_0P_{i(\theta)}) = (\xi, \eta)$. This is independent of the choice of g [17, Lemma 9.13].

Patterson-Sullivan measures. For $\psi \in \mathfrak{a}_\theta^*$, a (Γ, ψ) -conformal measure is a Borel probability measure on \mathcal{F}_θ such that

$$(2.7) \quad \frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi^\theta(e, \gamma))} \quad \text{for all } \gamma \in \Gamma \text{ and } \xi \in \mathcal{F}_\theta$$

where $\gamma_*\nu(D) = \nu(\gamma^{-1}D)$ for any Borel subset $D \subset \mathcal{F}_\theta$ and β_ξ^θ denotes the \mathfrak{a}_θ -valued Busemann map defined in (2.3). A (Γ, ψ) -conformal measure supported on Λ_θ is called a (Γ, ψ) -Patterson-Sullivan measure.

3. RELATIVELY HYPERBOLIC GROUPS AND GROMOV MODELS

In this section, we recall relatively hyperbolic groups, Gromov hyperbolic spaces, and Gromov models.

Relatively hyperbolic groups. Let Γ be a countable group acting by homeomorphisms on a compact metrizable space \mathcal{X} . The action is called a *convergence group action* if, for every sequence of distinct elements $\gamma_n \in \Gamma$, there exist a subsequence γ_{n_k} and points $a, b \in \mathcal{X}$ such that $\gamma_{n_k}(x)$ converges to a for all $x \in \mathcal{X} - \{b\}$, uniformly on compact subsets.

An infinite-order element $\gamma \in \Gamma$ is called *loxodromic* if it fixes exactly two points of \mathcal{X} , and *parabolic* if it fixes exactly one point. An infinite subgroup $P < \Gamma$ is called *parabolic* if it fixes a point of \mathcal{X} and every infinite-order element of P is parabolic.

A point $\xi \in \mathcal{X}$ is called a *conical limit point* if there exist a sequence of distinct elements $\gamma_n \in \Gamma$ and distinct points $a, b \in \mathcal{X}$ such that

$$\gamma_n^{-1}\xi \rightarrow a \quad \text{and} \quad \gamma_n^{-1}\eta \rightarrow b \quad \text{for all } \eta \in \mathcal{X} - \{\xi\}.$$

A point $\xi \in \mathcal{X}$ is called a *parabolic limit point* if it is fixed by a parabolic subgroup of Γ . Such a point is called *bounded parabolic* if

$$\text{Stab}_\Gamma(\xi) \backslash (\mathcal{X} - \{\xi\})$$

is compact. The action of Γ on \mathcal{X} is called a *geometrically finite convergence group action* if every point of \mathcal{X} is either conical or bounded parabolic. A typical example is the action of a geometrically finite Kleinian group on its limit set.

Let Γ be a finitely generated group and let \mathcal{P} be a finite collection of finitely generated infinite subgroups of Γ . We say that Γ is *hyperbolic relative to \mathcal{P}* , or that (Γ, \mathcal{P}) is *relatively hyperbolic*, if Γ admits a geometrically finite convergence group action on a compact perfect metrizable space \mathcal{X} whose maximal parabolic subgroups are precisely

$$\mathcal{P}^\Gamma := \{\gamma P \gamma^{-1} : P \in \mathcal{P}, \gamma \in \Gamma\}.$$

Bowditch [5] showed that, if Γ is hyperbolic relative to \mathcal{P} , then the space \mathcal{X} satisfying the above condition is unique up to Γ -equivariant homeomorphism. This space is called the *Bowditch boundary* and is denoted by $\partial(\Gamma, \mathcal{P})$. Since $\partial(\Gamma, \mathcal{P})$ is assumed to be perfect, we have $\#\partial(\Gamma, \mathcal{P}) \geq 3$; equivalently, (Γ, \mathcal{P}) is non-elementary.

Gromov hyperbolic spaces. A proper geodesic metric space (Y, d) is called *Gromov hyperbolic* if there exists $\delta > 0$ such that every geodesic triangle in Y is δ -thin; that is, each side is contained in the δ -neighborhood of the union of the other two sides. The Gromov boundary ∂Y is the set of equivalence classes of geodesic rays, where two rays are equivalent if they have finite Hausdorff distance. We write

$$\overline{Y} := Y \cup \partial Y$$

for the corresponding compactification.

For $C_1, C_2 \geq 1$ and an interval $I \subset \mathbb{R}$, a map $\sigma : I \rightarrow Y$ is called a (C_1, C_2) -quasi-geodesic if

$$C_1^{-1}|t - s| - C_2 \leq d(\sigma(t), \sigma(s)) \leq C_1|t - s| + C_2 \quad \text{for all } t, s \in I.$$

We also call the image $\sigma(I)$ a (C_1, C_2) -quasi-geodesic. We shall use the following standard stability property.

Lemma 3.1. *For any $C_1, C_2 \geq 1$, there exists $R > 0$ such that any two (C_1, C_2) -quasi-geodesics in Y with the same endpoints in \bar{Y} have Hausdorff distance at most R .*

We use the following notation.

Definition 3.2. Let $y, y_1, y_2 \in Y$, let $z_1, z_2 \in \bar{Y}$, and let $R > 0$.

(1) Let

$$B(y, R) = \{x \in Y : d(y, x) < R\}$$

denote the ball of radius R centered at y .

(2) The notation $[y_1, y_2]$ denotes a choice of geodesic in Y connecting y_1 to y_2 . Such a geodesic need not be unique.

(3) The shadow $O_R(y_1, y_2) \subset \partial Y$ is

$$O_R(y_1, y_2) = \{\xi \in \partial Y : \text{some geodesic ray } [y_1, \xi] \text{ intersects } B(y_2, R)\}.$$

(4) We denote by $\pi_{z_1, z_2}(y)$ the set of nearest-point projections of y to all geodesics between z_1 and z_2 .

(5) We denote by $\pi_{[z_1, z_2]}(y)$ the set of nearest-point projections of y to the chosen geodesic $[z_1, z_2]$.

It is well known that, for any $x \in \pi_{z_1, z_2}(y)$, the concatenation $[y, x] \cup [x, z_1]$ is a $(1, O(\delta))$ -quasi-geodesic, where $O(\delta)$ denotes a constant depending only on δ .

Given $o_Y \in Y$ and two distinct points $\xi, \eta \in Y \cup \partial Y$, define their Gromov product with respect to o_Y by

$$\langle \xi, \eta \rangle_{o_Y} := \sup \liminf_{i, j \rightarrow \infty} \frac{1}{2} (d(o_Y, x_i) + d(o_Y, y_j) - d(x_i, y_j)),$$

where the supremum is taken over all sequences $x_i, y_j \in Y$ such that $x_i \rightarrow \xi$ and $y_j \rightarrow \eta$. This quantity measures the distance from o_Y to $\pi_{\xi, \eta}(o_Y)$ up to a uniform additive error depending only on δ . We shall use the standard inequality

$$(3.1) \quad \langle \xi, \eta \rangle_{o_Y} \geq \min(\langle \xi, \zeta \rangle_{o_Y}, \langle \zeta, \eta \rangle_{o_Y}) - O(\delta)$$

for all $\xi, \eta, \zeta \in Y \cup \partial Y$.

The following standard comparison between shadows and Gromov products will be used repeatedly.

Lemma 3.3. *Let $o_Y \in Y$, $\xi \in \partial Y$, and $\xi_t \in [o_Y, \xi]$ be such that $d(o_Y, \xi_t) = t$ for $t \geq 0$. For any $R > 0$:*

(1) *if $\eta \in O_R(o_Y, \xi_t)$, then*

$$\langle \xi, \eta \rangle_{o_Y} \geq t - R - O(\delta);$$

(2) *if $\langle \xi, \eta \rangle_{o_Y} \geq t - R + O(\delta)$, then*

$$\eta \in O_R(o_Y, \xi_t).$$

Proof. We first prove (1). Let $\eta \in O_R(o_Y, \xi_t)$ and let $y \in \pi_{[\xi, \eta]}(o_Y)$. It suffices to consider the case $\langle \xi, \eta \rangle_{o_Y} \leq t$. In this case, ξ_t is $O(\delta)$ -close to $[\xi, \eta]$. Choose $x \in [\xi, \eta]$ with $d(\xi_t, x) < O(\delta)$. Since $\eta \in O_R(o_Y, \xi_t)$, Lemma 3.1 implies that ξ_t is $R + O(\delta)$ -close to $[o_Y, y] \cup [y, \eta]$. Hence x is also $R + O(\delta)$ -close to $[o_Y, y] \cup [y, \eta]$.

If x is $R + O(\delta)$ -close to $[y, \eta]$, then $d(x, y) < R + O(\delta)$, since both x and $[y, \eta]$ lie on the geodesic $[\xi, \eta]$. Therefore

$$d(o_Y, y) \geq d(o_Y, x) - R - O(\delta) \geq d(o_Y, \xi_t) - R - O(\delta).$$

Since $d(o_Y, \xi_t) = t$ and $d(o_Y, y) \leq \langle \xi, \eta \rangle_{o_Y} + O(\delta)$, this proves the desired estimate in this case.

If x is $R + O(\delta)$ -close to $[o_Y, y]$, then choose $w \in \pi_{[o_Y, y]}(x)$. Then we have $d(x, w) < R + O(\delta)$. On the other hand, the concatenation $[w, y] \cup [y, x]$ is a $(1, O(\delta))$ -quasi-geodesic. Hence y is also $R + O(\delta)$ -close to x , and the same argument as above proves (1).

For (2), suppose that

$$\langle \xi, \eta \rangle_{o_Y} \geq t - R + O(\delta).$$

Then $\xi_{t-R+O(\delta)}$ is $O(\delta)$ -close to $[o_Y, \eta]$. Hence ξ_t is R -close to $[o_Y, \eta]$, and therefore $\eta \in O_R(o_Y, \xi_t)$. \square

Gromov models. For the rest of this section, let (Γ, \mathcal{P}) be a relatively hyperbolic group. Following [13], we recall the basic properties of Gromov models. Such a model is a proper geodesic Gromov hyperbolic space on which Γ acts in a way analogous to the action of a geometrically finite Kleinian group on the real hyperbolic space.

Let (Y, d) be a proper geodesic Gromov hyperbolic space. For any $\eta \in \partial Y$, a horofunction based at η is obtained as follows: if $y_n \in Y$ is a sequence converging to η , then, after passing to a subsequence, there exist $t_n \rightarrow \infty$ and a function $h : Y \rightarrow \mathbb{R}$ such that

$$h(x) = \lim_{n \rightarrow \infty} d(x, y_n) - t_n,$$

uniformly on compact subsets. There exists a constant $C > 0$, depending only on Y , such that every such function satisfies

$$(3.2) \quad |(h(\ell_\eta(t_2)) - h(\ell_\eta(t_1))) - (t_1 - t_2)| \leq C \quad \text{for all } t_1, t_2 \geq 0,$$

for every unit-speed geodesic ray $\ell_\eta : [0, \infty) \rightarrow Y$ asymptotic to η . By a horofunction at η , we mean any function satisfying (3.2) for every such ray.

A subset $H \subset Y$ is called a horoball at η if there exists a horofunction h at η such that

$$\{h \leq 0\} \subset H \subset \{h \leq 10C\}.$$

Then $\overline{H} \cap \partial Y = \{\eta\}$, and horoballs are uniformly quasi-convex.

Definition 3.4. A proper geodesic Gromov hyperbolic space (Y, d) is called a *Gromov model* for (Γ, \mathcal{P}) if:

- (1) Γ acts properly discontinuously on Y by isometries;

- (2) Y is taut, meaning that there exists $R > 0$ such that every point of Y lies within distance R of a bi-infinite geodesic;
- (3) there exists a Γ -invariant collection $\mathcal{B} = \{B_i\}$ of disjoint open horoballs such that the stabilizer of each B_i in Γ is of the form $\gamma\mathcal{P}\gamma^{-1}$ for some $\gamma \in \Gamma$ and $\mathcal{P} \in \mathcal{P}$;
- (4) the action of Γ on $Y - \bigcup_i B_i$ is cocompact.

By the uniqueness of the Bowditch boundary, the Gromov boundary of a Gromov model Y for (Γ, \mathcal{P}) is Γ -equivariantly homeomorphic to $\partial(\Gamma, \mathcal{P})$. By [4, Proposition 2.12], the action of Γ on ∂Y is a convergence group action.

For each $B \in \mathcal{B}$, write

$$\{p\} = \overline{B} \cap \partial Y.$$

We call p the basepoint of B and also write $B = B_p$. Given $\mathcal{P} \in \mathcal{P}^\Gamma$, we denote its basepoint by $\xi_{\mathcal{P}} \in \partial Y$.

These basepoints are precisely the parabolic limit points of (Γ, \mathcal{P}) in ∂Y . We shall use the following standard properties of parabolic subgroups, whose proofs are included for completeness.

Lemma 3.5. *Let $o_Y \in Y$, let $\mathcal{P} \in \mathcal{P}^\Gamma$, and set $\xi = \xi_{\mathcal{P}}$. For every $R > 0$, there exists a compact set $Q \subset \partial Y - \{\xi\}$ such that*

$$O_R(o_Y, go_Y) \subset gQ \quad \text{for all but finitely many } g \in \mathcal{P}.$$

Proof. Fix $R > 0$ and suppose the claim fails. Let $Q_n \subset \partial Y - \{\xi\}$ be an increasing sequence of compact sets with

$$\bigcup_n Q_n = \partial Y - \{\xi\}.$$

Then there exist sequences $g_n \in \mathcal{P}$ and $\eta_n \in O_R(o_Y, g_n o_Y)$ such that $\eta_n \notin g_n Q_n$. We may assume that the sequence g_n is infinite. Since $g_n^{-1} \eta_n \notin Q_n$ for every n , $g_n^{-1} \eta_n \rightarrow \xi$. On the other hand,

$$g_n^{-1} \eta_n \in O_R(g_n^{-1} o_Y, o_Y) \quad \text{and} \quad g_n^{-1} o_Y \rightarrow \xi,$$

which is impossible. This proves the lemma. \square

Lemma 3.6. *Let $o_Y \in Y$, let $\mathcal{P} \in \mathcal{P}^\Gamma$, and set $\xi = \xi_{\mathcal{P}}$. For every compact set $Q \subset \partial Y - \{\xi\}$, there exists $R > 0$ such that*

$$gQ \subset O_R(o_Y, go_Y) \quad \text{for all } g \in \mathcal{P}.$$

Proof. Let $Q \subset \partial Y - \{\xi\}$ be compact and suppose the conclusion fails. Then, for every $n \geq 1$, there exists $g_n \in \mathcal{P}$ such that

$$g_n Q \not\subset O_n(o_Y, g_n o_Y).$$

Thus g_n is an infinite sequence and

$$Q \not\subset O_n(g_n^{-1} o_Y, o_Y) \quad \text{for every } n \geq 1.$$

Since $g_n^{-1} o_Y \rightarrow \xi$, this forces $\xi \in Q$, a contradiction. \square

Finally, a point $\xi \in \partial Y$ is conical if and only if, for any $o_Y \in Y$, there exist $R > 0$ and an infinite sequence $\gamma_n \in \Gamma$ such that

$$\xi \in O_R(o_Y, \gamma_n o_Y) \quad \text{for all } n.$$

In this case, we say that $\gamma_n o_Y$ converges conically to ξ . It is easy to see that this notion of conicality is equivalent to the conicality defined in terms of convergence action.

4. RELATIVELY MORSE GROUPS

Let $\Gamma < G$ be a discrete subgroup which is hyperbolic relative to a finite collection \mathcal{P} of finitely generated infinite subgroups of Γ . We fix a non-empty subset $\theta \subset \Pi$.

Definition 4.1. We say that Γ is θ -Morse relative to \mathcal{P} if there exist a Gromov model Y for (Γ, \mathcal{P}) and a Γ -equivariant quasi-isometric embedding

$$f : Y \rightarrow X$$

such that, if $\mathcal{L}_f \subset \mathfrak{a}^+$ denotes the asymptotic cone of

$$\{\mu(f(x)^{-1}f(y)) : x, y \in Y\},$$

then

$$\mathcal{L}_f \cap \ker \alpha = \{0\} \quad \text{for every } \alpha \in \theta.$$

Such a map f is called a Morse embedding of Y .

For a Morse embedding $f : Y \rightarrow X$, we call \mathcal{L}_f the Morse limit cone of f . We also set

$$\mathcal{L}_{\theta, f} := p_\theta(\mathcal{L}_f)$$

and call it the Morse θ -limit cone. Since $\mathcal{L}_f = i(\mathcal{L}_f)$, being θ -Morse relative to \mathcal{P} is equivalent to being $\theta \cup i(\theta)$ -Morse relative to \mathcal{P} . Thus, without loss of generality, we assume

$$\theta = i(\theta)$$

throughout the rest of this section.

A subgroup $\Gamma < G$ which is hyperbolic relative to \mathcal{P} is called θ -Anosov relative to \mathcal{P} if:

- Γ is θ -regular, meaning that

$$\min_{\alpha \in \theta} \alpha(\mu(\gamma_n)) \rightarrow \infty$$

for every infinite sequence $\gamma_n \in \Gamma$;

- there exists a transverse Γ -equivariant embedding

$$\zeta : \partial Y \rightarrow \mathcal{F}_\theta,$$

i.e., it sends distinct points of ∂Y to points in general position.

If Γ is θ -Morse relative to \mathcal{P} , then Γ is θ -Anosov relative to \mathcal{P} . Indeed, θ -regularity follows from the fact that f is a quasi-isometric embedding and

$$\mathcal{L}_f \cap \bigcup_{\alpha \in \theta} \ker \alpha = \{0\}.$$

Moreover, by [14, Theorem 1.4], the Morse embedding $f : Y \rightarrow X$ extends continuously to a transverse Γ -equivariant embedding

$$f : \partial Y \rightarrow \mathcal{F}_\theta.$$

Its image is the θ -limit set Λ_θ .

Remark 4.2. A map $f : Y \rightarrow X$ satisfying

$$\mathcal{L}_f \cap \bigcup_{\alpha \in \theta} \ker \alpha = \{0\}$$

is called uniformly θ -regular in [13]. Although this condition is a priori different from the original definition of a Morse embedding in [13], the Morse lemma [14] implies that the two notions are equivalent for quasi-isometric embeddings.

Geometric properties of relatively Morse groups. For the rest of this section, let $\Gamma < G$ be θ -Morse relative to \mathcal{P} , with Morse embedding $f : Y \rightarrow X$ from a Gromov model (Y, d) and continuous extension $f : \partial Y \rightarrow \mathcal{F}_\theta$. We fix a basepoint $o_Y \in Y$ and we may assume that $f(o_Y) = o$.

The following proposition is a key ingredient in the proof of the global shadow lemma. Its second assertion follows from Lemma 4.4 below.

Proposition 4.3. *Let $P \in \mathcal{P}$ and let $p = \xi_P$. For every compact subset $Q \subset \partial Y - \{p\}$, there exists a constant $c = c(Q) > 0$ such that*

$$\sup_{\xi \in Q, \gamma \in P} \left\| \mathcal{G}^\theta(f(p), \gamma f(\xi)) - \frac{1}{2} \mu_\theta(\gamma) \right\| \leq c.$$

In particular, $\mu_\theta(\gamma) - i \mu_\theta(\gamma)$ is uniformly bounded for all $\gamma \in P$.

We first collect several lemmas. The following was proved in [18, Lemma 6.6] when Γ is Borel Anosov, using the Morse property of the orbit map of an Anosov group into X [14]. The same argument applies in our setting, replacing the Cayley graph of a hyperbolic group by the Gromov hyperbolic space Y .

Lemma 4.4. *There exists $C > 0$ such that, for any distinct $\xi, \eta \in \partial Y$,*

$$\sup_{z \in \pi_{\xi, \eta}(o_Y)} \left\| \mathcal{G}^\theta(f(\xi), f(\eta)) - \frac{1}{2} (\mu_\theta(f(z)) + i \mu_\theta(f(z))) \right\| \leq C.$$

The following was proved in [11, Corollary 4.12]. Although the original statement applies to the values of a linear form on the Cartan projections, the same proof gives the following vector-valued version.

Lemma 4.5. *There exists $D_0 > 0$ such that, for any $x, z \in Y$ and any $y \in [x, z]$,*

$$\|\mu_\theta(f(x)^{-1}f(z)) - (\mu_\theta(f(x)^{-1}f(y)) + \mu_\theta(f(y)^{-1}f(z)))\| < D_0.$$

We also need the following elementary observation.

Lemma 4.6. *Let $P \in \mathcal{P}$ and let $p = \xi_P$. For every $R > 0$, there exists a finite subset $P(R) \subset P$ such that, for every $\gamma \in P - P(R)$, every geodesic $[\xi, p]$ with $\xi \in \partial Y - \{p\}$, and every $u \in B(o_Y, R) \cap [\xi, p]$, we have*

$$\pi_{\gamma[\xi, p]}(u) \subset \gamma[u, p]$$

where we choose $[u, p] \subset [\xi, p]$.

Proof. Suppose not. Then there exist $R > 0$, an infinite sequence $\gamma_n \in P$, and sequences

$$\xi_n \in \partial Y - \{p\}, \quad u_n \in B(o_Y, R) \cap [\xi_n, p], \quad \text{and} \quad y_n \in \pi_{\gamma_n[\xi_n, p]}(u_n),$$

such that

$$y_n \notin \gamma_n[u_n, p]$$

where we choose $[u_n, p] \subset [\xi_n, p]$, for all $n \geq 1$.

For each n , choose $[y_n, p] \subset \gamma_n[\xi_n, p]$. By the assumption, $\gamma_n u_n \in [y_n, p]$. By Gromov hyperbolicity, the concatenation

$$[u_n, y_n] \cup [y_n, p]$$

is a $(1, O(\delta))$ -quasi-geodesic. Hence

$$p \in O_{O(\delta)}(u_n, \gamma_n u_n).$$

Since $d(o_Y, u_n) < R$, it follows that for all n ,

$$p \in O_{R'}(o_Y, \gamma_n o_Y)$$

for some $R' > 0$ depending only on R and δ . Thus $\gamma_n o_Y$ converges conically to p , contradicting the fact that p is a parabolic limit point. \square

We will use the following coarse midpoint estimate in the Gromov model:

Lemma 4.7. [6, Lemma 3.7] *Let $P \in \mathcal{P}$ and let $p = \xi_P$. Let $R > 0$, and let $Q \subset \partial Y - \{p\}$ be compact. Then there exists $D > 0$ such that, for every $x \in B(o_Y, R)$ and every $\gamma \in P$,*

$$\sup_{\xi \in Q, y \in \pi_{\gamma[\xi, p]}(x)} \left| d(x, y) - \frac{1}{2}d(x, \gamma x) \right| \leq D.$$

As an intermediate step in the proof of Proposition 4.3, we prove the following. We write \approx for equality up to a uniform additive error.

Lemma 4.8. *Let $P \in \mathcal{P}$ and let $p = \xi_P$. Let $R > 0$, and let $Q \subset \partial Y - \{p\}$ be compact. Then there exists a constant $C = C(R, Q) > 0$ such that, for every $\gamma \in P$, every geodesic $[\xi, p]$ with $\xi \in Q$, every $x \in [\xi, p] \cap B(o_Y, R)$, and every $w \in \gamma^{-1}\pi_{\gamma[\xi, p]}(x)$, there exists $y \in [x, \gamma x]$ such that*

$$d(y, w) < C, \quad d(y, \gamma w) < C, \quad \text{and} \quad d(w, \gamma w) < C.$$

Proof. Let $\mathsf{P}(R) \subset \mathsf{P}$ be the finite subset given by Lemma 4.6. It suffices to consider $\gamma \in \mathsf{P} - \mathsf{P}(R)$. Let $\xi \in Q$, and choose a geodesic $[\xi, p]$. Let $x \in [\xi, p] \cap B(o_Y, R)$, and let $w \in [\xi, p]$ be such that $\gamma w \in \pi_{\gamma[\xi, p]}(x)$. By Lemma 4.7,

$$(4.1) \quad d(x, \gamma w) \approx \frac{1}{2}d(x, \gamma x),$$

with additive error depending only on R and Q .

Consider the geodesic $[\gamma x, p] \subset \gamma[\xi, p]$ and the geodesic triangle

$$[x, p] \cup [x, \gamma x] \cup [\gamma x, p].$$

By Lemma 4.6, we have $\gamma w \in [\gamma x, p]$. Since $w \in [\xi, p]$, we also have $w \in [x, p]$. Since $\gamma w \in \pi_{\gamma[\xi, p]}(x)$, Gromov hyperbolicity implies that there exist $y \in [x, \gamma x]$ and $z \in [x, p]$ such that the three points γw , y , and z are uniformly close to one another. Together with (4.1), this gives

$$\frac{1}{2}d(x, \gamma x) \approx d(x, \gamma w) \approx d(x, y) \approx d(x, z).$$

Hence

$$(4.2) \quad d(x, \gamma x) \approx d(x, y) + d(x, z).$$

On the other hand,

$$\begin{aligned} d(x, \gamma x) &= d(x, y) + d(y, \gamma x) \\ &\approx d(x, y) + d(\gamma w, \gamma x) \\ &= d(x, y) + d(w, x). \end{aligned}$$

Comparing this with (4.2), we obtain

$$d(x, w) \approx d(x, z).$$

Since $z, w \in [x, p]$, this implies that the points z and w are uniformly close. Since γw , y , and z are uniformly close, it follows that γw , y , and w are uniformly close as well. This proves the lemma. \square

Proof of Proposition 4.3. There exists $R = R(Q) > 0$ such that every geodesic $[\xi, p]$ with $\xi \in Q$ intersects $B(o_Y, R)$. Let $\gamma \in \mathsf{P}$.

Let $\xi \in Q$, and choose a geodesic $[\xi, p]$. Choose $x \in [\xi, p] \cap B(o_Y, R)$, and let $w \in [\xi, p]$ be such that

$$\gamma w \in \pi_{\gamma[\xi, p]}(x).$$

Since $x \in B(o_Y, R)$, the sets $\pi_{\gamma[\xi, p]}(x)$ and $\pi_{\gamma[\xi, p]}(o_Y)$ have uniformly bounded Hausdorff distance. Hence, by Lemmas 2.1 and 4.4,

$$\mathcal{G}^\theta(f(p), \gamma f(\xi)) \approx \frac{1}{2}(\mu_\theta(f(\gamma w)) + i\mu_\theta(f(\gamma w))).$$

By Lemma 4.8, there exists $y \in [x, \gamma x]$ such that y , w , and γw are uniformly close. Since $y \in [x, \gamma x]$, Lemma 4.5 gives

$$\mu_\theta(f(x)^{-1}f(\gamma x)) \approx \mu_\theta(f(x)^{-1}f(y)) + \mu_\theta(f(y)^{-1}f(\gamma x)).$$

Since f is a quasi-isometric embedding, the points $f(y)$, $f(w)$, and $f(\gamma w)$ are uniformly close. Since $x \in B(o_Y, R)$, the point $f(x)$ remains in a uniformly bounded subset of X . Therefore, by Lemma 2.1,

$$\begin{aligned} \mu_\theta(\gamma) &\approx \mu_\theta(f(x)^{-1}f(\gamma x)) \\ &\approx \mu_\theta(f(x)^{-1}f(\gamma w)) + \mu_\theta(f(w)^{-1}f(x)) \\ &\approx \mu_\theta(f(x)^{-1}f(\gamma w)) + \mathbf{i}\mu_\theta(f(x)^{-1}f(w)) \\ &\approx \mu_\theta(f(\gamma w)) + \mathbf{i}\mu_\theta(f(\gamma w)) \\ &\approx 2\mathcal{G}^\theta(f(p), \gamma f(\xi)). \end{aligned}$$

This finishes the proof of the main claim.

For the second assertion, note that by Lemma 4.4,

$$\mathcal{G}^\theta(f(p), \gamma f(\xi)) \approx \frac{1}{2}(\mu_\theta(f(z_\gamma)) + \mathbf{i}\mu_\theta(f(z_\gamma)))$$

for some $z_\gamma \in \pi_{p, \gamma\xi}(o_Y)$. Hence

$$\mu_\theta(\gamma) \approx \mu_\theta(f(z_\gamma)) + \mathbf{i}\mu_\theta(f(z_\gamma)).$$

Since the right-hand side is \mathbf{i} -invariant, this implies that $\mu_\theta(\gamma) - \mathbf{i}\mu_\theta(\gamma)$ is uniformly bounded. \square

A metric-like function on Y . We identify \mathfrak{a}_θ^* with the subspace of \mathfrak{a}^* obtained by precomposing with the projection $p_\theta : \mathfrak{a} \rightarrow \mathfrak{a}_\theta$. Let $\psi \in \mathfrak{a}_\theta^*$ be such that

$$\psi > 0 \quad \text{on } \mathcal{L}_f - \{0\}.$$

Define

$$\mathbf{d}_\psi(x, y) := \psi(\mu(f(x)^{-1}f(y))) \quad \text{for } x, y \in Y.$$

It was shown in [11] that \mathbf{d}_ψ behaves like a metric on Y ; for example, it satisfies a coarse triangle inequality. The following is a consequence of Lemma 4.5 and the Gromov hyperbolicity of Y . Since $\psi > 0$ on $\mathcal{L}_f - \{0\}$, there exists $c > 0$ such that

$$(4.3) \quad \mathbf{d}_\psi(x, y) > -c \quad \text{for all } x, y \in Y.$$

Proposition 4.9. *There exists $C > 0$ such that, for all $x, y, z \in Y$,*

$$\mathbf{d}_\psi(x, z) \leq \mathbf{d}_\psi(x, y) + \mathbf{d}_\psi(y, z) + C.$$

Proof. Let $w \in \pi_{[x, z]}(y)$ be a nearest-point projection of y to a geodesic $[x, z]$. By Lemma 4.5,

$$\mathbf{d}_\psi(x, z) \leq \mathbf{d}_\psi(x, w) + \mathbf{d}_\psi(w, z) + D_0\|\psi\|,$$

where D_0 is the constant from Lemma 4.5. Since the concatenation $[x, w] \cup [w, y]$ is a uniform quasi-geodesic, Lemmas 3.1 and 4.5 imply that

$$\mathbf{d}_\psi(x, y) \approx \mathbf{d}_\psi(x, w) + \mathbf{d}_\psi(w, y) > \mathbf{d}_\psi(x, w) - c,$$

where c is as in (4.3). Similarly,

$$\mathbf{d}_\psi(y, z) \approx \mathbf{d}_\psi(y, w) + \mathbf{d}_\psi(w, z) > \mathbf{d}_\psi(w, z) - c.$$

Combining these three estimates gives the desired inequality. \square

We also note that the metric-like function d_ψ behaves quasi-isometrically to the Gromov model (Y, d) as follows:

Lemma 4.10. (1) *For any $x, y, z \in Y$, if $y \in [x, z]$, then*

$$d_\psi(x, z) \approx d_\psi(x, y) + d_\psi(y, z).$$

(2) *There exist $a, b > 0$ such that for all $x, y \in Y$,*

$$a d(x, y) - b \leq d_\psi(x, y) \leq b d(x, y) + b.$$

(3) *There exists $A > 0$ such that, for every $\xi \in \partial Y$ and every $s \geq 0$, there exists $x \in [o_Y, \xi]$ such that*

$$|d_\psi(o_Y, x) - s| \leq A.$$

Proof. (1) is an immediate consequence of Lemma 4.5.

Since $\psi > 0$ on $\mathcal{L}_f - \{0\}$ and f is a quasi-isometric embedding, there exist $a, b > 0$ such that

$$(4.4) \quad a d(x, y) - b \leq d_\psi(x, y) \leq b d(x, y) + b$$

for all $x, y \in Y$. This shows (2).

Finally, fix $\xi \in \partial Y$ and denote by $\xi_t \in [o_Y, \xi]$ the point such that $d(o_Y, \xi_t) = t$ for $t \geq 0$. We then set

$$F(t) := d_\psi(o_Y, \xi_t).$$

Note that definition of $F(t)$ involves a choice of $[o_Y, \xi]$, but different choices only make uniformly bounded error which is allowed for our purpose. By Lemma 4.5, if $0 \leq t \leq u$, then

$$F(u) \approx F(t) + d_\psi(\xi_t, \xi_u).$$

Together with (4.4), this gives constants $a_1, b_1 > 0$, independent of ξ , such that for all $0 \leq t \leq u$,

$$(4.5) \quad a_1(u - t) - b_1 \leq F(u) - F(t) \leq b_1(u - t) + b_1.$$

Since $F(0) = 0$ and $F(t) \rightarrow \infty$ as $t \rightarrow \infty$, for any $s \geq 0$ choose the smallest integer $n \geq 0$ with $F(n) \geq s$. If $n = 0$, then $s = 0$. If $n \geq 1$, then $F(n - 1) < s$, and hence by (4.5),

$$0 \leq F(n) - s \leq F(n) - F(n - 1) \leq 2b_1.$$

Thus (3) holds with $A = 2b_1$. \square

For subsets $E, F \subset Y$, set

$$d_\psi(E, F) := \inf_{x \in E, y \in F} d_\psi(x, y).$$

Lemma 4.11. *Let $P \in \mathcal{P}$ and let $p = \xi_P$. For every $x \in [o_Y, p]$, we have*

$$d_\psi(x, \Gamma o_Y) \approx d_\psi(x, o_Y),$$

where the implied constant depends only on the ambient constants and ψ .

Proof. Let $B_p \in \mathcal{B}$ be the horoball based at p . We first claim that

$$\mathbf{d}_\psi(x, \Gamma o_Y) \approx \mathbf{d}_\psi(x, \mathbf{P} o_Y).$$

It suffices to consider the case that x is sufficiently deep in B_p .

Since \mathbf{P} acts cocompactly on ∂B_p , there exists $A > 0$ such that every point of ∂B_p is within distance A of some point of $\mathbf{P} o_Y$. Let $h \in \Gamma$. A geodesic from x to $h o_Y$ leaves B_p through a point $u_h \in \partial B_p$. Choose $\gamma_h \in \mathbf{P}$ such that

$$d(u_h, \gamma_h o_Y) \leq A.$$

By Lemma 2.1,

$$\mathbf{d}_\psi(x, \gamma_h o_Y) \approx \mathbf{d}_\psi(x, u_h).$$

On the other hand, since $u_h \in [x, h o_Y]$, Lemma 4.5 and (4.3) give

$$\mathbf{d}_\psi(x, h o_Y) \approx \mathbf{d}_\psi(x, u_h) + \mathbf{d}_\psi(u_h, h o_Y) \geq \mathbf{d}_\psi(x, u_h) - c$$

where $c > 0$ is given in (4.3). Hence

$$\mathbf{d}_\psi(x, \gamma_h o_Y) \leq \mathbf{d}_\psi(x, h o_Y) + c'$$

for some uniform constant $c' > 0$ independent of h and x . Since this holds for all $h \in \Gamma$ and $\mathbf{P} < \Gamma$, the claim follows.

By the claim, we can choose $\gamma \in \mathbf{P}$ such that $\mathbf{d}_\psi(x, \Gamma o_Y) \approx \mathbf{d}_\psi(x, \gamma o_Y)$. Let $w \in \pi_{[o_Y, p]}(\gamma o_Y)$. By tautness of Y and Lemma 4.6, we may apply Lemma 4.8 to obtain a point $y \in [o_Y, \gamma^{-1} o_Y]$ such that the points w , $\gamma^{-1} w$, and y are uniformly close. Since f is a quasi-isometric embedding, the points $f(w)$, $f(\gamma^{-1} w)$, and $f(y)$ are also uniformly close. Hence

$$\mu_\theta(f(w)^{-1}) \approx \mu_\theta(f(\gamma^{-1} w)^{-1}) = \mu_\theta(f(w)^{-1} f(\gamma o_Y)).$$

We distinguish two cases according to the relative positions of w and x on $[o_Y, p]$. First suppose that $w \in [o_Y, x]$. By Lemma 4.5,

$$\begin{aligned} \mu_\theta(f(x)^{-1}) &\approx \mu_\theta(f(x)^{-1} f(w)) + \mu_\theta(f(w)^{-1}) \\ &\approx \mu_\theta(f(x)^{-1} f(w)) + \mu_\theta(f(w)^{-1} f(\gamma o_Y)) \\ &\approx \mu_\theta(f(x)^{-1} f(\gamma o_Y)), \end{aligned}$$

since the concatenation $[x, w] \cup [w, \gamma o_Y]$ is a $(1, O(\delta))$ -quasi-geodesic. Applying ψ , we get

$$\mathbf{d}_\psi(x, o_Y) \approx \mathbf{d}_\psi(x, \gamma o_Y) \approx \mathbf{d}_\psi(x, \Gamma o_Y),$$

as desired.

Now suppose that $x \in [o_Y, w]$. Again by Lemma 4.5,

$$\begin{aligned} \mu_\theta(f(x)^{-1} f(\gamma o_Y)) &\approx \mu_\theta(f(x)^{-1} f(w)) + \mu_\theta(f(w)^{-1} f(\gamma o_Y)) \\ &\approx \mu_\theta(f(x)^{-1} f(w)) + \mu_\theta(f(w)^{-1}) \\ &\approx \mu_\theta(f(x)^{-1} f(w)) + \mu_\theta(f(w)^{-1} f(x)) + \mu_\theta(f(x)^{-1}). \end{aligned}$$

Applying ψ gives

$$\begin{aligned} \mathbf{d}_\psi(x, \gamma o_Y) &\approx \mathbf{d}_\psi(x, w) + \mathbf{d}_\psi(w, x) + \mathbf{d}_\psi(x, o_Y) \\ &> \mathbf{d}_\psi(x, o_Y) - 2c, \end{aligned}$$

where the last inequality follows from (4.3). Since $\mathbf{d}_\psi(x, \gamma o_Y) \approx \mathbf{d}_\psi(x, \Gamma o_Y)$ and $o_Y \in \Gamma o_Y$, the desired estimate follows in this case as well. \square

\mathfrak{a}_θ -valued Gromov products in shadows. We now translate the elementary comparison between shadows and Gromov products in the Gromov hyperbolic space Y into a comparison involving \mathfrak{a}_θ -valued Gromov products on the flag variety. The main input from the relatively Morse property is that nearest-point projections in Y coarsely control the higher-rank Gromov product $\mathcal{G}^\theta(f(\xi), f(\eta))$ through Cartan projections; see Lemma 4.4. Together with the coarse additivity of Cartan projections along geodesics in Y , Lemma 4.5, this gives the following analogue of Lemma 3.3. For Anosov groups, this comparison was proved in [11]; the same argument applies in the relatively Morse setting.

For $\psi \in \mathfrak{a}_\theta^*$, write

$$\bar{\psi} := \frac{\psi + \psi \circ \mathbf{i}}{2}.$$

Proposition 4.12. *Let $\psi \in \mathfrak{a}_\theta^*$ be such that $\psi > 0$ on $\mathcal{L}_f - \{0\}$.*

- (1) *For every $R > 0$, there exists $c_1 > 0$ such that, for every $\xi \in \partial Y$ and every $z \in [o_Y, \xi]$, if $\eta \in O_R(o_Y, z) - \{\xi\}$, then*

$$\psi(\mathcal{G}^\theta(f(\xi), f(\eta))) \geq \mathbf{d}_{\bar{\psi}}(o_Y, z) - c_1.$$

- (2) *For every $c_2 > 0$, there exists $R > 0$ such that, for every $\xi \in \partial Y$ and every $z \in [o_Y, \xi]$, if $\eta \in \partial Y - \{\xi\}$ satisfies*

$$\psi(\mathcal{G}^\theta(f(\xi), f(\eta))) \geq \mathbf{d}_{\bar{\psi}}(o_Y, z) + c_2,$$

then

$$\eta \in O_R(o_Y, z).$$

Proof. We first prove (1). Fix $R > 0$, $\xi \in \partial Y$, and $z \in [o_Y, \xi]$. Let $\eta \in O_R(o_Y, z)$ and $y \in \pi_{[\xi, \eta]}(o_Y)$. Let $[z, \xi] \subset [o_Y, \xi]$.

We claim that there exists $r = r(R) > 0$ such that

$$(4.6) \quad [z, \xi] \cap B(y, r) \neq \emptyset.$$

Since $\eta \in O_R(o_Y, z)$, Lemma 3.1 implies that the uniform quasi-geodesic

$$[o_Y, y] \cup [y, \eta]$$

intersects $B(z, R + O(\delta))$. Choose $w \in [o_Y, y] \cup [y, \eta]$ such that

$$d(z, w) < R + O(\delta).$$

We consider two cases, according to the position of w relative to y .

If $w \in [o_Y, y]$, then both

$$[w, y] \cup [y, \xi] \quad \text{and} \quad [w, z] \cup [z, \xi]$$

are uniform quasi-geodesics with the same endpoints. By Lemma 3.1, the distance from y to $[w, z] \cup [z, \xi]$ is bounded above by a uniform constant depending R . Since $[w, z]$ has length at most $R + O(\delta)$, the claim follows in this case.

If $w \in [y, \eta]$, choose a geodesic $[w, \xi] \subset [\xi, \eta]$. Then $y \in [w, \xi]$. As in the previous case,

$$[z, w] \cup [w, \xi]$$

is a uniform quasi-geodesic. Hence, by Lemma 3.1, the point $y \in [w, \xi]$ lies in a uniform neighborhood of $[z, \xi]$. Now the claim follows.

By (4.6), there exists $u \in [z, \xi] \subset [o_Y, \xi]$ such that $d(y, u) < r$. Since f is a quasi-isometric embedding, Lemma 2.1 implies that

$$\mu(f(y)) \approx \mu(f(u)),$$

where the implied constant depends on r . By Lemma 4.4,

$$\psi(\mathcal{G}^\theta(f(\xi), f(\eta))) \approx \mathbf{d}_{\bar{\psi}}(o_Y, y) \approx \mathbf{d}_{\bar{\psi}}(o_Y, u).$$

Since $u \in [z, \xi] \subset [o_Y, \xi]$, Lemma 4.5 gives

$$\mathbf{d}_{\bar{\psi}}(o_Y, u) \approx \mathbf{d}_{\bar{\psi}}(o_Y, z) + \mathbf{d}_{\bar{\psi}}(z, u).$$

By (4.3), the term $\mathbf{d}_{\bar{\psi}}(z, u)$ is bounded from below by a uniform constant. Combining these estimates, we obtain a constant $c_1 > 0$ such that

$$\psi(\mathcal{G}^\theta(f(\xi), f(\eta))) \geq \mathbf{d}_{\bar{\psi}}(o_Y, z) - c_1.$$

This proves (1).

We now prove (2). Let $c_2 > 0$, let $\xi \in \partial Y$, and let $z \in [o_Y, \xi]$. Suppose that $\eta \in \partial Y$ satisfies

$$\psi(\mathcal{G}^\theta(f(\xi), f(\eta))) \geq \mathbf{d}_{\bar{\psi}}(o_Y, z) + c_2.$$

We claim that there exists $r_0 = r_0(c_2) > 0$ such that, for every $y \in \pi_{[\xi, \eta]}(o_Y)$, the union

$$[o_Y, y] \cup [y, \eta]$$

intersects $B(z, r_0)$. Since $[o_Y, y] \cup [y, \eta]$ and $[o_Y, \eta]$ have uniformly bounded Hausdorff distance by Lemma 3.1, the claim implies that

$$\eta \in O_R(o_Y, z) \quad \text{for } R = r_0 + O(\delta).$$

Let C be the constant from Lemma 4.4. Then

$$(4.7) \quad \mathbf{d}_{\bar{\psi}}(o_Y, y) \geq \mathbf{d}_{\bar{\psi}}(o_Y, z) + c_2 - \|\psi\|C.$$

By Lemma 3.1, there exists $w \in [o_Y, \xi]$ such that $d(y, w) < O(\delta)$. Since f is a quasi-isometric embedding,

$$(4.8) \quad \mathbf{d}_{\bar{\psi}}(o_Y, w) \approx \mathbf{d}_{\bar{\psi}}(o_Y, y).$$

We distinguish two cases according to the position of w relative to z . If $w \in [z, \xi] \subset [o_Y, \xi]$, then $z \in [o_Y, w]$. Since $d(y, w) < O(\delta)$, the concatenation

$$[o_Y, w] \cup [w, y]$$

is a $(1, O(\delta))$ -quasi-geodesic. Hence, by Lemma 3.1, the geodesic $[o_Y, y]$ intersects a uniform neighborhood of z . Thus $[o_Y, y] \cup [y, \eta]$ intersects a uniform neighborhood of z .

If $w \in [o_Y, z] \subset [o_Y, \xi]$, then Lemma 4.5 gives

$$d_{\bar{\psi}}(o_Y, z) \approx d_{\bar{\psi}}(o_Y, w) + d_{\bar{\psi}}(w, z).$$

Together with (4.7) and (4.8), this implies that $d_{\bar{\psi}}(w, z)$ is uniformly bounded above. Since $\psi > 0$ on $\mathcal{L}_f - \{0\}$, the same is true for $\bar{\psi}$, and therefore $\|\mu(f(w)^{-1}f(z))\|$ is uniformly bounded. Since f is a quasi-isometric embedding, $d(w, z)$ is uniformly bounded. As $d(y, w) < O(\delta)$, it follows that $[o_Y, y] \cup [y, \eta]$ intersects a uniform neighborhood of z .

In both cases the claim follows, and this completes the proof. \square

5. COUNTING IN PARABOLIC SUBGROUPS

Throughout this section, let $\Gamma < G$ be θ -Anosov relative to a finite collection \mathcal{P} . Let $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ denote the limit cone of Γ , that is, the asymptotic cone of $\mu(\Gamma)$.

Fix $\psi \in \mathfrak{a}_\theta^*$ that is positive on $\mathcal{L}_\Gamma - \{0\}$. For a subgroup $H < \Gamma$, let $\delta_\psi(H)$ denote the abscissa of convergence of the Poincaré series

$$s \mapsto \sum_{\gamma \in H} e^{-s\psi(\mu(\gamma))}.$$

We prove the following counting estimate for parabolic subgroups.

Theorem 5.1. *Let $\mathbf{P} \in \mathcal{P}$. Then there exist $a_\psi(\mathbf{P}) \in \mathbb{Z}_{\geq 0}$, $C > 1$, $k \in \mathbb{N}$, and $T_0 > 0$ such that, for all $T > T_0$ and all $n \geq 0$,*

$$\begin{aligned} & C^{-1} e^{\delta_\psi(\mathbf{P})(T+kn)} (1+T+kn)^{a_\psi(\mathbf{P})} \\ & \leq \#\{\gamma \in \mathbf{P} : T+kn \leq \psi(\mu(\gamma)) < T+k(n+1)\} \\ & \leq C e^{\delta_\psi(\mathbf{P})(T+kn)} (1+T+kn)^{a_\psi(\mathbf{P})}. \end{aligned}$$

Moreover, if \mathbf{P} is virtually cyclic or if G has rank one, then $a_\psi(\mathbf{P}) = 0$.

We deduce Theorem 5.1 from the following counting estimate. We write \ll for inequality up to a uniform multiplicative constant, and similarly for \gg . We write \asymp when we have both \ll and \gg .

Proposition 5.2. *Let $\mathbf{P} \in \mathcal{P}$. Then there exist $a_\psi(\mathbf{P}) \in \mathbb{Z}_{\geq 0}$ and $c > 0$ such that, for all sufficiently large T ,*

$$e^{\delta_\psi(\mathbf{P})T} (T-c)^{a_\psi(\mathbf{P})} \ll \#\{\gamma \in \mathbf{P} : \psi(\mu(\gamma)) \leq T\} \ll e^{\delta_\psi(\mathbf{P})T} (T+c)^{a_\psi(\mathbf{P})}.$$

Moreover, if \mathbf{P} is virtually cyclic or if G has rank one, then $a_\psi(\mathbf{P}) = 0$.

The key input is a volume estimate of Benoist-Oh [2]. We first recall the structural description of parabolic subgroups in the relatively Anosov setting. After the reduction in [8, Section 4.4], we may assume that P_θ contains no simple factor of G . Let $\mathbf{P} \in \mathcal{P}$. By [8, Theorem 4.4], there exists a closed subgroup $H < G$ with finitely many connected components such

that P is a cocompact lattice in H . Moreover, if U denotes the unipotent radical of H , then

$$(5.1) \quad H = L \rtimes U \quad \text{and} \quad H^\circ = L^\circ \times U,$$

where $L < H$ is compact and its identity component L° is abelian.

Let $\mathfrak{u} = \text{Lie}(U)$, and let ℓ denote Lebesgue measure on \mathfrak{u} . We shall use the following consequence of [2]:

Theorem 5.3. *Let R_1, \dots, R_m be positive rational functions on \mathfrak{u} which are defined everywhere. Let $c_1, \dots, c_m \in \mathbb{R}$, and suppose that $R := R_1^{c_1} \cdots R_m^{c_m}$ is a proper function. Then*

$$\ell(\{Y \in \mathfrak{u} : R(Y) \leq T\}) \sim c_0 T^r (\log T)^q$$

for some $c_0 > 0$, $r \geq 0$, and $q \in \mathbb{Z}_{\geq 0}$.

Proof. This is immediate from [2, Proposition 7.2] when all exponents c_i are positive rational numbers. The same proof also gives the present form; we briefly indicate the additional points needed for real exponents. Following the notation in [2, Proposition 7.2], let Z be the affine space over \mathbb{R} associated to the real vector space \mathfrak{u} , so that $Z(\mathbb{R}) = \mathfrak{u}$. Let ω be the standard algebraic volume form on Z inducing ℓ . Write $R_i = f_i/g_i$, with f_i and g_i polynomial. Choose a smooth projective real compactification V of Z . After replacing V by a resolution, and keeping the same notation, we may assume that $D := V - Z$, the divisors of the rational functions f_i, g_i , and the divisor of ω , have simple normal crossings. Put $F = R$ and $f = 1/F$.

Let $y_0 \in D(\mathbb{R}) \cap \overline{Z(\mathbb{R})}$. In a neighborhood of y_0 , choose local real analytic coordinates (x_1, \dots, x_N) such that

$$D = \{x_1 \cdots x_r = 0\}.$$

By the above resolution of singularities, and arguing as in the proof of [8, Lemma 6.7], we may write locally

$$f_i = x_1^{a_{i1}} \cdots x_r^{a_{ir}} \widehat{f}_i, \quad \text{and} \quad g_i = x_1^{b_{i1}} \cdots x_r^{b_{ir}} \widehat{g}_i,$$

where $a_{ij}, b_{ij} \in \mathbb{Z}$ and $\widehat{f}_i, \widehat{g}_i$ are nowhere-vanishing real analytic functions. After restricting to one orthant and replacing x_j by $|x_j|$, we may assume $x_j > 0$ for $j = 1, \dots, r$. Then $R_i = \widehat{R}_i \prod_{j=1}^r x_j^{a_{ij} - b_{ij}}$ where $\widehat{R}_i := \widehat{f}_i / \widehat{g}_i$, and hence

$$f = R^{-1} = \widehat{f} \prod_{j=1}^r x_j^{s_j}.$$

for some $s_j \in \mathbb{R}$ and some positive real analytic unit \widehat{f} . Since R is proper on $Z(\mathbb{R})$, $f = R^{-1}$ tends to 0 along every boundary component meeting the closure of $Z(\mathbb{R})$. Hence $s_j > 0$ for every relevant boundary component.

Moreover, locally

$$\omega = a_0(x) \prod_{j=1}^r x_j^{\beta_j} dx_1 \cdots dx_N$$

with $\beta_j \in \mathbb{Z}$ and a_0 a nowhere-vanishing real analytic function.

Thus the local integrals appearing in the proof of [2, Proposition 7.2] are of monomial type. The remainder of their argument applies without change. \square

Proof of Proposition 5.2. By [8, Proposition 2.3], for each $\alpha \in \theta$ there exist $m_\alpha \in \mathbb{N}$, $C_\alpha > 1$, and an everywhere-defined positive rational function $R_\alpha : \mathfrak{u} \rightarrow \mathbb{R}$ such that, for all $Y \in \mathfrak{u}$,

$$C_\alpha^{-1} R_\alpha(Y)^{1/m_\alpha} \leq e^{\omega_\alpha(\mu(\exp Y))} \leq C_\alpha R_\alpha(Y)^{1/m_\alpha},$$

where $\omega_\alpha \in \mathfrak{a}_\theta^*$ is the fundamental weight associated to α .

Write

$$\psi = \sum_{\alpha \in \theta} c_\alpha \omega_\alpha$$

for coefficients $c_\alpha \in \mathbb{R}$, $\alpha \in \theta$. Set

$$R_\psi := \prod_{\alpha \in \theta} R_\alpha^{c_\alpha/m_\alpha} \quad \text{and} \quad C_\psi := \prod_{\alpha \in \theta} C_\alpha^{|c_\alpha|}.$$

Then, for all $Y \in \mathfrak{u}$,

$$C_\psi^{-1} R_\psi(Y)^{-1} \leq e^{-\psi(\mu(\exp Y))} \leq C_\psi R_\psi(Y)^{-1}.$$

By [8, Lemma 7.3], the function R_ψ is proper.

Let

$$P_1 := \pi(P \cap H^\circ),$$

where $\pi : H^\circ \rightarrow U$ is the projection. Then P_1 is a cocompact lattice in U . Since $P \cap H^\circ$ has finite index in P and $\ker \pi$ is compact, we have

$$\#\{g \in P_1 : \psi(\mu(g)) \leq T\} \asymp \#\{\gamma \in P : \psi(\mu(\gamma)) \leq T\}$$

for all sufficiently large T . It therefore suffices to count elements of P_1 .

Let λ_U denote a Haar measure on U . Let $Q_1 \subset U$ be a bounded open set such that the translates gQ_1 , $g \in P_1$, are pairwise disjoint. Then

$$\begin{aligned} \#\{g \in P_1 : \psi(\mu(g)) \leq T\} &\ll \lambda_U \left(\bigcup_{\substack{g \in P_1 \\ \psi(\mu(g)) \leq T}} gQ_1 \right) \\ &\ll \lambda_U (\{u \in U : \psi(\mu(u)) \leq T + c'\}) \\ &\ll \ell (\{Y \in \mathfrak{u} : R_\psi(Y) \leq ce^T\}) \end{aligned}$$

for some constants $c', c > 1$.

Similarly, choose a compact set $Q_2 \subset U$ such that $P_1 Q_2 = U$. Then, after increasing c', c with $c > 1$, if necessary,

$$\begin{aligned} \#\{g \in P_1 : \psi(\mu(g)) \leq T\} &\gg \lambda_U \left(\bigcup_{\substack{g \in P_1 \\ \psi(\mu(g)) \leq T}} gQ_2 \right) \\ &\gg \lambda_U (\{u \in U : \psi(\mu(u)) \leq T - c'\}) \\ &\gg \ell(\{Y \in \mathfrak{u} : R_\psi(Y) \leq c^{-1}e^T\}). \end{aligned}$$

The proposition now follows from Theorem 5.3, together with the fact that the exponential growth rate of

$$\#\{g \in P_1 : \psi(\mu(g)) \leq T\}$$

is $\delta_\psi(\mathbf{P})$. If \mathbf{P} is virtually cyclic, then the corresponding unipotent group is one-dimensional, and the volume asymptotic has no logarithmic factor. Hence $a_\psi(\mathbf{P}) = 0$. See Proposition 5.5 for the claim about the case $\text{rank } G = 1$. \square

Before proving Theorem 5.1, we record the following entropy gap for parabolic subgroups, due to Canary-Zhang-Zimmer.

Theorem 5.4. [8, Lemma 7.4, Theorem 6.1, Theorem 7.1] *For each $\mathbf{P} \in \mathcal{P}$, we have*

$$0 < \delta_\psi(\mathbf{P}) < \delta_\psi(\Gamma).$$

Proof of Theorem 5.1. By Proposition 5.2, there exist $c > 0$ and $T_0 > 0$ such that, for all $T > T_0$,

$$e^{\delta_\psi(\mathbf{P})T-c}(1+T)^{a_\psi(\mathbf{P})} \leq \#\{\gamma \in \mathbf{P} : \psi(\mu(\gamma)) < T\} \leq e^{\delta_\psi(\mathbf{P})T+c}(1+T)^{a_\psi(\mathbf{P})}.$$

For simplicity, write

$$\delta = \delta_\psi(\mathbf{P}) \quad \text{and} \quad a = a_\psi(\mathbf{P}),$$

and define

$$N(T) := \#\{\gamma \in \mathbf{P} : \psi(\mu(\gamma)) < T\}.$$

Choose $k \in \mathbb{N}$ so large that

$$e^{\delta k - c} - e^c > 0,$$

which is possible since $\delta > 0$ by Theorem 5.4. Set $S := T + kn$. Then

$$\begin{aligned} N(S+k) - N(S) &\leq e^{\delta(S+k)+c}(1+S+k)^a - e^{\delta S - c}(1+S)^a \\ &= e^{\delta S}(1+S)^a \left[e^{\delta k + c} \left(1 + \frac{k}{1+S}\right)^a - e^{-c} \right] \\ &\leq C e^{\delta S}(1+S)^a \end{aligned}$$

for a constant $C > 1$ independent of T and n . Similarly,

$$\begin{aligned} N(S+k) - N(S) &\geq e^{\delta(S+k)-c}(1+S+k)^a - e^{\delta S+c}(1+S)^a \\ &= e^{\delta S}(1+S)^a \left[e^{\delta k-c} \left(1 + \frac{k}{1+S}\right)^a - e^c \right] \\ &\geq C^{-1} e^{\delta S}(1+S)^a, \end{aligned}$$

after increasing C , if necessary. Since

$$N(S+k) - N(S) = \#\{\gamma \in \mathbf{P} : S \leq \psi(\mu(\gamma)) < S+k\},$$

and $S = T + kn$, this proves the theorem. \square

Rank-one parabolic subgroups. We record a rank-one refinement of Proposition 5.2, which shows the absence of the polynomial term in Theorem 5.1 for rank-one G . In this subsection, assume that G has real rank one. Let $\Pi = \{\alpha\}$, so that $\theta = \{\alpha\}$ and $\mathfrak{a}_\theta = \mathfrak{a} \simeq \mathbb{R}$. Let $\mathfrak{n} := \text{Lie}(N)$ and write

$$\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha},$$

where \mathfrak{g}_α and $\mathfrak{g}_{2\alpha}$ are the corresponding root spaces, with the convention that $\mathfrak{g}_{2\alpha} = 0$ if 2α is not a root.

Let $\mathbf{P} \in \mathcal{P}$. Let $H = L \times U$ be the subgroup associated to \mathbf{P} as in (5.1). Up to conjugation, we may assume that $U < N$, and use the same notation $\mathfrak{u} := \text{Lie}(U)$ as before. Define

$$V_{\mathbf{P}} := \text{pr}_{\mathfrak{g}_\alpha}(\mathfrak{u}), \quad Z_{\mathbf{P}} := \mathfrak{u} \cap \mathfrak{g}_{2\alpha}, \quad \text{and} \quad Q(\mathbf{P}) := \dim V_{\mathbf{P}} + 2 \dim Z_{\mathbf{P}}.$$

Since \mathfrak{a} is one-dimensional, $\mathcal{L}_\Gamma = \mathfrak{a}^+$ and hence any $\psi \in \mathfrak{a}^*$ positive on $\mathcal{L}_\Gamma - \{0\}$ is a multiplication by a positive real number. The norm $\|\psi\|$ is given by $\psi(H_0)$ where $H_0 \in \mathfrak{a}^+$ is the unique unit vector.

Theorem 5.5. *Suppose that $\text{rank } G = 1$. For any positive $\psi \in \mathfrak{a}^*$ on $\mathfrak{a}^+ - \{0\}$, we have*

$$\#\{\gamma \in \mathbf{P} : \psi(\mu(\gamma)) \leq T\} \asymp e^{\delta_\psi(\mathbf{P})T},$$

and $\delta_\psi(\mathbf{P}) = \frac{\|\alpha\|Q(\mathbf{P})}{2\|\psi\|}$. In particular, in Theorem 5.1, we have

$$a_\psi(\mathbf{P}) = 0 = a_{\bar{\psi}}(\mathbf{P}).$$

Proof. Let $H_0 \in \mathfrak{a}^+$ be the unit vector. Let $b := \alpha(H_0)$ and $\lambda_\psi := \psi(H_0)$. We use the same reduction as in Proposition 5.2. Recall that $\pi : H^\circ \rightarrow U$ is the projection and $\mathbf{P}_1 = \pi(\mathbf{P} \cap H^\circ)$. Then \mathbf{P}_1 is a cocompact lattice in U . Since $\mathbf{P} \cap H^\circ$ has finite index in \mathbf{P} and $\ker \pi$ is compact, Lemma 2.1 implies that counting \mathbf{P} and counting \mathbf{P}_1 give the same estimates, up to multiplicative constants, as before.

Write $u = \exp(X + Z) \in U$ with $X \in \mathfrak{g}_\alpha$ and $Z \in \mathfrak{g}_{2\alpha}$. By [19, (2.5)], following [22, p. 72],

$$\cosh^2\left(\frac{d_{NA}(o, uo)}{2}\right) = \left(1 + \frac{\|X\|^2}{8}\right)^2 + \frac{1}{4}\|Z\|^2$$

where d_{NA} is the distance induced from the norm whose unit vector H_1 satisfies $\alpha(H_1) = 1/2$. It follows that d_{NA} is the $2b$ multiple of the Riemannian distance d induced from the norm for which H_0 is a unit vector.

Therefore

$$d(o, uo) \approx \frac{2}{b} \log \max\{1, \|X\|, \|Z\|^{1/2}\}.$$

Define

$$\|u\|_{\text{cusp}} := \max\{1, \|X\|, \|Z\|^{1/2}\},$$

where the term involving Z is omitted if $\mathfrak{g}_{2\alpha} = 0$. Since $\mu(u) = d(o, uo)H_0$, we get

$$\psi(\mu(u)) = \lambda_\psi d(o, uo) \approx \frac{2\lambda_\psi}{b} \log \|u\|_{\text{cusp}}.$$

Hence

$$\psi(\mu(u)) \leq T \iff \|u\|_{\text{cusp}} \ll e^{bT/(2\lambda_\psi)},$$

up to changing the implicit constants.

It remains to compute the volume growth of these $\|\cdot\|_{\text{cusp}}$ -balls inside U . Choose a linear complement W to $Z_{\mathfrak{P}}$ in \mathfrak{u} . The projection $W \rightarrow V_{\mathfrak{P}}$ is an isomorphism, so every $Y \in \mathfrak{u}$ can be written uniquely as

$$Y = v + \phi(v) + z \quad \text{with } v \in V_{\mathfrak{P}} \text{ and } z \in Z_{\mathfrak{P}},$$

for some linear map $\phi : V_{\mathfrak{P}} \rightarrow \mathfrak{g}_{2\alpha}$. The condition

$$\|\exp Y\|_{\text{cusp}} \leq R$$

is equivalent, up to uniform constants, to

$$\|v\| \ll R \quad \text{and} \quad \|\phi(v) + z\| \ll R^2.$$

Thus the v -variables contribute $R^{\dim V_{\mathfrak{P}}}$ and the z -variables contribute $R^{2 \dim Z_{\mathfrak{P}}}$. Therefore

$$\lambda_U(\{u \in U : \|u\|_{\text{cusp}} \leq R\}) \asymp R^{Q(\mathfrak{P})}.$$

Since \mathfrak{P}_1 is a cocompact lattice in U , a compact fundamental domain comparison gives

$$\#\{g \in \mathfrak{P}_1 : \|g\|_{\text{cusp}} \leq R\} \asymp R^{Q(\mathfrak{P})}.$$

Substituting $R = e^{bT/(2\lambda_\psi)}$ yields

$$\#\{g \in \mathfrak{P}_1 : \psi(\mu(g)) \leq T\} \asymp e^{\frac{bQ(\mathfrak{P})}{2\lambda_\psi} T}.$$

This finishes the proof. Note that the last claim follows since i is trivial in rank one. \square

6. SHADOW ESTIMATES AT PARABOLIC LIMIT POINTS

Let $\Gamma < G$ be θ -Morse relative to \mathcal{P} , with Morse embedding $f : Y \rightarrow X$ of a Gromov model (Y, d) for (Γ, \mathcal{P}) . Let $\psi \in \mathfrak{a}_\theta^*$ be positive on $\mathcal{L}_f - \{0\}$. We normalize ψ so that $\delta_\psi(\Gamma) = 1$. Then the existence of Patterson-Sullivan measure is a consequence of the work of Canary-Zhang-Zimmer.

Theorem 6.1. [8] *There exists a unique (Γ, ψ) -Patterson-Sullivan measure ν on Λ_θ . Moreover, ν is atomless.*

Let ν be a (Γ, ψ) -Patterson-Sullivan measure on Λ_θ given in Theorem 6.1. In this section we prove the global shadow lemma for shadows along geodesic rays towards parabolic limit points and provide estimates on their complements. Since all shadows we consider are taken in $Y \cup \partial Y$, we identify ν with its pullback to ∂Y under the Γ -equivariant homeomorphism $f : \partial Y \rightarrow \Lambda_\theta$.

Suppose $\theta = i(\theta)$ and set

$$\bar{\psi} := \frac{\psi + \psi \circ i}{2}.$$

Since $\psi > 0$ on $\mathcal{L}_f - \{0\}$, so is $\bar{\psi}$, and hence there exists $C_{\bar{\psi}} > 0$ such that

$$(6.1) \quad \bar{\psi}(v) > 1 - C_{\bar{\psi}} \quad \text{for all } v \in \mu(f(Y)^{-1}f(Y)).$$

Shadows at parabolic limit points. We first estimate shadows along geodesic rays ending at parabolic limit points. Recall that $o_Y \in Y$ is chosen so that $f(o_Y) = o$. For $\xi \in \partial Y$ and $t \geq 0$, let $\xi_t \in [o_Y, \xi]$ denote the point with $d(o_Y, \xi_t) = t$, after choosing a geodesic ray $[o_Y, \xi]$.

Theorem 6.2. *Let $P \in \mathcal{P}$ and let $\xi = \xi_P$. For all sufficiently large $R > 0$,*

$$\nu(O_R(o_Y, \xi_t)) \asymp e^{2(\delta_{\bar{\psi}}(P)-1)d_{\bar{\psi}}(o_Y, \xi_t)} (C_{\bar{\psi}} + d_{\bar{\psi}}(o_Y, \xi_t))^{a_{\bar{\psi}}(P)}$$

for all $t \geq 0$, with implied constants independent of t .

For an arbitrary parabolic limit point, we obtain the following translated form.

Theorem 6.3. *There exist constants $c, R_0 > 0$ with the following property. Let $P \in \mathcal{P}$ and let $\xi = \gamma\xi_P$ for some $\gamma \in \Gamma$. Suppose that*

$$d(\xi_{t_0}, \gamma o_Y) < c \quad \text{for some } t_0 \geq 0.$$

Then, for every $C \geq C_{\bar{\psi}}$, $R > R_0$, and $t \geq t_0$,

$$\nu(O_R(o_Y, \xi_t)) \asymp e^{-\psi(\mu(\gamma))} e^{2(\delta_{\bar{\psi}}(P)-1)d_{\bar{\psi}}(\gamma o_Y, \xi_t)} \cdot (C + d_{\bar{\psi}}(\gamma o_Y, \xi_t))^{a_{\bar{\psi}}(P)}.$$

For $g \in G$, define the translated measure ν_g by

$$d\nu_g(\eta) = e^{\psi(\beta_\eta^\theta(e, g))} d\nu(\eta).$$

Then, for $g, h \in G$,

$$d\nu_g(\eta) = e^{\psi(\beta_\eta^\theta(h, g))} d\nu_h(\eta).$$

For $x \in Y$, we write

$$\nu_x := \nu_g$$

where $g \in G$ satisfies $f(x) = go$. This is independent of the choice of g .

The next lemma relates ν to its translate at the center of a shadow.

Lemma 6.4. *For all $\xi \in \partial Y$, $t \geq 0$, and $R > 0$,*

$$\nu(O_R(o_Y, \xi_t)) \asymp e^{-d_\psi(o_Y, \xi_t)} \nu_{\xi_t}(O_R(o_Y, \xi_t)).$$

Lemma 6.4 is an immediate consequence of the following comparison between Busemann maps and Cartan projections inside shadows. The corresponding statement for shadows in the symmetric space $X = G/K$ was proved in [18, Lemma 5.7]; the present version follows from comparing shadows in Y with shadows in X under the Morse embedding f .

Lemma 6.5. *For every $R > 0$, there exists $C > 0$ such that, for all $x, y \in Y$ and $\xi \in O_R(x, y)$,*

$$\left\| \beta_{f(\xi)}^\theta(g, h) - \mu_\theta(g^{-1}h) \right\| < C$$

whenever $g, h \in G$ satisfy $go = f(x)$ and $ho = f(y)$.

Since Γ acts cocompactly on $Y - \bigcup \mathcal{B}$, Lemma 6.4 and the ordinary shadow lemma imply the following thick-part estimate.

Lemma 6.6. *There exists $R_0 > 0$ such that, for every $R > R_0$ and every $x \in Y - \bigcup \mathcal{B}$,*

$$\nu_x(O_R(o_Y, x)) \asymp 1.$$

The main estimate needed for Theorems 6.2 and 6.3 is the following.

Proposition 6.7. *Let $P \in \mathcal{P}$ and let $\xi = \xi_P$. For all sufficiently large $R > 0$, there exists $C > 0$ such that, for all $t \geq 0$,*

$$\begin{aligned} \nu_{\xi_t}(O_R(o_Y, \xi_t)) &\ll e^{d_\psi(o_Y, \xi_t)} \sum_{\substack{g \in P \\ \bar{\psi}(\mu(g)) \geq 2d_{\bar{\psi}}(o_Y, \xi_t) - C}} e^{-\psi(\mu(g))}, \\ \nu_{\xi_t}(O_R(o_Y, \xi_t)) &\gg e^{d_\psi(o_Y, \xi_t)} \sum_{\substack{g \in P \\ \bar{\psi}(\mu(g)) \geq 2d_{\bar{\psi}}(o_Y, \xi_t) + C}} e^{-\psi(\mu(g))}. \end{aligned}$$

Proof. Choose $R > 0$ large enough so that the ordinary shadow lemma holds. Let $Q \subset \partial Y - \{\xi\}$ be compact such that $PQ = \partial Y - \{\xi\}$ and such that the conclusion of Lemma 3.5 holds for the chosen R . Increasing R if necessary, Proposition 4.12 gives constants $c_1, c_2 > 0$ such that

$$\eta \in O_R(o_Y, \xi_t) - \{\xi\} \implies \psi(\mathcal{G}^\theta(f(\xi), f(\eta))) \geq d_{\bar{\psi}}(o_Y, \xi_t) - c_1,$$

$$\psi(\mathcal{G}^\theta(f(\xi), f(\eta))) \geq d_{\bar{\psi}}(o_Y, \xi_t) + c_2 \implies \eta \in O_R(o_Y, \xi_t).$$

By Proposition 4.3, and using that $\mu_\theta(g) - i\mu_\theta(g)$ is uniformly bounded for $g \in P$, there exists $c > 0$ such that

$$(6.2) \quad \bigcup_{\substack{g \in P \\ \frac{1}{2}\bar{\psi}(\mu(g)) \geq d_{\bar{\psi}}(o_Y, \xi_t) + c + c_2}} gQ \subset O_R(o_Y, \xi_t) - \{\xi\} \subset \bigcup_{\substack{g \in P \\ \frac{1}{2}\bar{\psi}(\mu(g)) \geq d_{\bar{\psi}}(o_Y, \xi_t) - c - c_1}} gQ.$$

If $gQ \subset O_R(o_Y, \xi_t)$, then Lemma 6.5 gives

$$\nu_{\xi_t}(gQ) \asymp e^{\mathbf{d}_\psi(o_Y, \xi_t)} \nu(gQ).$$

Similarly, if $gQ \cap O_R(o_Y, \xi_t) \neq \emptyset$, then

$$\nu_{\xi_t}(gQ \cap O_R(o_Y, \xi_t)) \ll e^{\mathbf{d}_\psi(o_Y, \xi_t)} \nu(gQ).$$

Since ν is atomless (Theorem 6.1), using (6.2) and the bounded multiplicity of the \mathbf{P} -translates of Q , we obtain

$$\begin{aligned} \nu_{\xi_t}(O_R(o_Y, \xi_t)) &\ll e^{\mathbf{d}_\psi(o_Y, \xi_t)} \sum_{\substack{g \in \mathbf{P} \\ \frac{1}{2}\bar{\psi}(\mu(g)) \geq \mathbf{d}_{\bar{\psi}}(o_Y, \xi_t) - c - c_1}} \nu(gQ), \\ \nu_{\xi_t}(O_R(o_Y, \xi_t)) &\gg e^{\mathbf{d}_\psi(o_Y, \xi_t)} \sum_{\substack{g \in \mathbf{P} \\ \frac{1}{2}\bar{\psi}(\mu(g)) \geq \mathbf{d}_{\bar{\psi}}(o_Y, \xi_t) + c + c_2}} \nu(gQ). \end{aligned}$$

By Lemmas 3.5 and 3.6, and the ordinary shadow lemma,

$$\nu(gQ) \asymp e^{-\psi(\mu(g))}.$$

The proposition follows. \square

Proof of Theorem 6.2. Let $C > 0$ be the constant from Proposition 6.7. Let k and T_0 be as in Theorem 5.1, applied to $\bar{\psi}$.

First suppose that

$$2\mathbf{d}_{\bar{\psi}}(o_Y, \xi_t) - C > T_0.$$

Set

$$T := 2\mathbf{d}_{\bar{\psi}}(o_Y, \xi_t) - C.$$

By Proposition 4.3, $\psi(\mu(g)) \approx \bar{\psi}(\mu(g))$ for $g \in \mathbf{P}$. Since $\delta_{\psi_{\text{oi}}}(\Gamma) = \delta_\psi(\Gamma) = 1$, it follows that $\delta_{\bar{\psi}}(\Gamma) \leq 1$. Together with the entropy gap (Theorem 5.4), this implies $\delta_{\bar{\psi}}(\mathbf{P}) < 1$. Hence Theorem 5.1 gives

$$\begin{aligned} &\sum_{\substack{g \in \mathbf{P} \\ \bar{\psi}(\mu(g)) \geq T}} e^{-\psi(\mu(g))} \\ &\ll \sum_{n=0}^{\infty} e^{-(T+kn)} \#\{g \in \mathbf{P} : T+kn \leq \bar{\psi}(\mu(g)) < T+k(n+1)\} \\ &\ll \sum_{n=0}^{\infty} e^{(\delta_{\bar{\psi}}(\mathbf{P})-1)(T+kn)} (1+T+kn)^{a_{\bar{\psi}}(\mathbf{P})} \\ &\asymp e^{(\delta_{\bar{\psi}}(\mathbf{P})-1)T} (1+T)^{a_{\bar{\psi}}(\mathbf{P})} \\ &\asymp e^{2(\delta_{\bar{\psi}}(\mathbf{P})-1)\mathbf{d}_{\bar{\psi}}(o_Y, \xi_t)} (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(o_Y, \xi_t))^{a_{\bar{\psi}}(\mathbf{P})}. \end{aligned}$$

The lower bound is proved in the same way. Thus by Proposition 6.7,

$$\nu_{\xi_t}(O_R(o_Y, \xi_t)) \asymp e^{\mathbf{d}_\psi(o_Y, \xi_t)} e^{2(\delta_{\bar{\psi}}(\mathbf{P})-1)\mathbf{d}_{\bar{\psi}}(o_Y, \xi_t)} \cdot (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(o_Y, \xi_t))^{a_{\bar{\psi}}(\mathbf{P})}.$$

Applying Lemma 6.4 proves the theorem in this case.

It remains to consider the case when $2d_{\bar{\psi}}(o_Y, \xi_t) - C \leq T_0$. Since $\bar{\psi}$ is positive on $\mathcal{L}_f - \{0\}$, this implies that $d(o_Y, \xi_t)$ is uniformly bounded. Hence, after increasing R by a uniform amount, $O_R(o_Y, \xi_t) = \partial Y$, and the desired estimate is trivial. \square

Proof of Theorem 6.3. Let $\xi = \gamma_0 \xi_P$ for some $\gamma_0 \in \Gamma$, and let $B_\xi \in \mathcal{B}$ be the horoball based at ξ . Then $\text{Stab}_\Gamma(B_\xi) = \gamma_0 P \gamma_0^{-1}$. Since $\gamma_0 P \gamma_0^{-1}$ acts cocompactly on ∂B_ξ , there exist a uniform constant $c > 0$, an element $\gamma_1 \in P$, and $t_0 \geq 0$ such that

$$d(\xi_{t_0}, \gamma_0 \gamma_1 o_Y) < c.$$

Set $\gamma := \gamma_0 \gamma_1$.

For $s \geq 0$, let $\eta_s \in [\gamma o_Y, \xi]$ be the point satisfying $d(\gamma o_Y, \eta_s) = s$. Increasing c by a uniform amount, if necessary, we have

$$d(\xi_t, \eta_{t-t_0}) < c \quad \text{for all } t \geq t_0.$$

Thus, there exists $c' > 0$ so that for all sufficiently large R ,

$$(6.3) \quad O_{R-c'}(\gamma o_Y, \eta_{t-t_0}) \subset O_R(o_Y, \xi_t) \subset O_{R+c'}(\gamma o_Y, \eta_{t-t_0})$$

for all large $t \geq t_0$.

Moreover,

$$O_R(o_Y, \xi_t) \subset O_{R+c}(o_Y, \gamma o_Y).$$

Using Lemma 6.5, we obtain

$$\begin{aligned} \nu(O_R(o_Y, \xi_t)) &= \nu(\gamma \gamma^{-1} O_R(o_Y, \xi_t)) \\ &= \int_{\gamma^{-1} O_R(o_Y, \xi_t)} e^{\psi(\beta_x^\theta(e, \gamma^{-1}))} d\nu(x) \\ &= \int_{\gamma^{-1} O_R(o_Y, \xi_t)} e^{-\psi(\beta_{\gamma x}^\theta(e, \gamma))} d\nu(x) \\ &\asymp e^{-\psi(\mu(\gamma))} \nu(\gamma^{-1} O_R(o_Y, \xi_t)). \end{aligned}$$

By (6.3) and Theorem 6.2,

$$\nu(O_R(o_Y, \xi_t)) \asymp e^{-\psi(\mu(\gamma))} e^{2(\delta_{\bar{\psi}}(P)-1)d_{\bar{\psi}}(\gamma o_Y, \eta_{t-t_0})} \cdot (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(\gamma o_Y, \eta_{t-t_0}))^{a_{\bar{\psi}}(P)}.$$

Since (6.1) holds, the factor $C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}$ may be replaced, up to multiplicative constants, by $C + \mathbf{d}_{\bar{\psi}}$ for any $C \geq C_{\bar{\psi}}$. Finally, ξ_t and η_{t-t_0} are uniformly close, so Lemma 2.1 gives

$$\mathbf{d}_{\bar{\psi}}(\gamma o_Y, \eta_{t-t_0}) \approx \mathbf{d}_{\bar{\psi}}(\gamma o_Y, \xi_t).$$

This proves the theorem. \square

Complements of parabolic shadows. We also need an estimate for the complement of a shadow based at a parabolic limit point.

Proposition 6.8. *Let $\xi = \xi_P$ for some $P \in \mathcal{P}$. For all sufficiently large $R > 0$ and all sufficiently large $t \geq 0$,*

$$\nu_{\xi_t}(\partial Y - O_R(o_Y, \xi_t)) \asymp e^{-d_\psi(\xi_t, o_Y)} e^{2\delta_{\bar{\psi}}(P)d_{\bar{\psi}}(o_Y, \xi_t)} \cdot (C_{\bar{\psi}} + d_{\bar{\psi}}(o_Y, \xi_t))^{a_{\bar{\psi}}(P)}.$$

Proof. Let $R > 0$ and $Q \subset \partial Y - \{\xi\}$ be as in the proof of Proposition 6.7. Since $PQ = \partial Y - \{\xi\}$, it follows from (6.2) that

$$\bigcup_{\substack{g \in P \\ \frac{1}{2}\bar{\psi}(\mu(g)) < d_{\bar{\psi}}(o_Y, \xi_t) - c - c_1}} gQ \subset \partial Y - O_R(o_Y, \xi_t) \subset \bigcup_{\substack{g \in P \\ \frac{1}{2}\bar{\psi}(\mu(g)) < d_{\bar{\psi}}(o_Y, \xi_t) + c + c_2}} gQ.$$

The thin-triangle property implies that there exists $R' > 0$ such that, if

$$gQ \cap (\partial Y - O_R(o_Y, \xi_t)) \neq \emptyset,$$

then

$$d(\xi_t, g\xi_t) < R'.$$

For such $g \in P$, we therefore have

$$\nu_{\xi_t}(gQ) \asymp \nu_{\xi_t}(Q).$$

After increasing R by a uniform amount, we may also assume that

$$Q \subset O_R(\xi_t, o_Y).$$

Hence Lemma 6.5 gives

$$\nu_{\xi_t}(Q) \asymp e^{-d_\psi(\xi_t, o_Y)} \nu(Q).$$

Using Proposition 5.2, we obtain

$$\nu_{\xi_t}(\partial Y - O_R(o_Y, \xi_t)) \asymp e^{-d_\psi(\xi_t, o_Y)} e^{2\delta_{\bar{\psi}}(P)d_{\bar{\psi}}(o_Y, \xi_t)} (C_{\bar{\psi}} + d_{\bar{\psi}}(o_Y, \xi_t))^{a_{\bar{\psi}}(P)}$$

as claimed. \square

7. GENERAL FORM OF THE GLOBAL SHADOW LEMMA

Continuing with the setting of section 6, we now prove the general form of the global shadow lemma. The estimate below should be viewed as the cusp version of the usual shadow lemma: when the point under consideration stays outside the horoballs in \mathcal{B} , one recovers the ordinary orbit-shadow estimate (Theorem 1.3).

Recall that, for $\xi \in \partial Y$ and $t \geq 0$, we write $\xi_t \in [o_Y, \xi]$ for the point satisfying $d(o_Y, \xi_t) = t$.

Theorem 7.1 (Global Shadow Lemma). *For all sufficiently large $R > 0$, the following estimate holds uniformly. Let $\xi \in \partial Y$ and suppose that $\xi_t \in B_\eta$ for some $t \geq 0$ and some horoball $B_\eta \in \mathcal{B}$ based at $\eta \in \Gamma\xi_P$, where $P \in \mathcal{P}$.*

Let $\gamma \in \Gamma$ be such that γo_Y is a closest orbit point to ξ_t in the orbit Γo_Y , with respect to the metric d on Y . Then

$$\begin{aligned} \nu(O_R(o_Y, \xi_t)) &\asymp e^{-\mathbf{d}_\psi(o_Y, \xi_t)} e^{\mathbf{d}_\psi(\gamma o_Y, \xi_t)} e^{2(\delta_{\bar{\psi}}(\mathbf{P})-1)\mathbf{d}_{\bar{\psi}}(\gamma o_Y, \xi_t)} \\ &\quad \cdot (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(\gamma o_Y, \xi_t))^{a_{\bar{\psi}}(\mathbf{P})}. \end{aligned}$$

Equivalently,

$$\begin{aligned} \nu(O_R(o_Y, \xi_t)) &\asymp e^{-\mathbf{d}_\psi(o_Y, \xi_t)} e^{\mathbf{d}_\psi(\Gamma o_Y, \xi_t)} e^{2(\delta_{\bar{\psi}}(\mathbf{P})-1)\mathbf{d}_{\bar{\psi}}(\Gamma o_Y, \xi_t)} \\ &\quad \cdot (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(\Gamma o_Y, \xi_t))^{a_{\bar{\psi}}(\mathbf{P})}. \end{aligned}$$

The implied constants are independent of ξ , t , η , and γ .

Proof. Let $A > 0$ be a sufficiently large constant depending only on the hyperbolicity constant of Y and on the uniform constants fixed above. We allow A to increase finitely many times during the proof.

First note that the endpoint of

$$[o_Y, \xi] \cap \partial B_\eta$$

closest to ξ_t is uniformly close to some orbit point of Γo_Y . Consequently, if γo_Y is chosen to be a closest orbit point to ξ_t , for some $\gamma \in \Gamma$, then Lemma 4.11 gives

$$\mathbf{d}_\psi(\gamma o_Y, \xi_t) \approx \mathbf{d}_\psi(\Gamma o_Y, \xi_t) \quad \text{and} \quad \mathbf{d}_{\bar{\psi}}(\gamma o_Y, \xi_t) \approx \mathbf{d}_{\bar{\psi}}(\Gamma o_Y, \xi_t).$$

Thus it suffices to prove the estimate with this choice of γ .

We divide the proof into three cases.

Case 1: $\eta \in O_{R-A}(o_Y, \xi_t)$. In this case, there exists $x \in [o_Y, \eta]$ such that

$$d(x, \xi_t) < R - A.$$

This implies

$$O_A(o_Y, x) \subset O_R(o_Y, \xi_t) \subset O_{2R-A}(o_Y, x).$$

Note that we may also choose $\gamma \in \Gamma$ so that γo_Y is uniformly close to $[o_Y, \eta]$, in this case. Applying Theorem 6.3 to the parabolic point η , and then using Lemma 2.1 to replace x by ξ_t , we obtain

$$\nu(O_R(o_Y, \xi_t)) \asymp e^{-\psi(\mu(\gamma))} e^{2(\delta_{\bar{\psi}}(\mathbf{P})-1)\mathbf{d}_{\bar{\psi}}(\gamma o_Y, \xi_t)} \cdot (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(\gamma o_Y, \xi_t))^{a_{\bar{\psi}}(\mathbf{P})}.$$

Since ξ_t lies in B_η and is uniformly close to the ray $[o_Y, \eta]$, Lemmas 4.11 and 4.5 imply

$$-\psi(\mu(\gamma)) = -\mathbf{d}_\psi(o_Y, \gamma o_Y) \approx -\mathbf{d}_\psi(o_Y, \xi_t) + \mathbf{d}_\psi(\gamma o_Y, \xi_t).$$

Substituting this into the preceding estimate proves the theorem in this case.

Case 2: $\eta \notin O_{R+A}(o_Y, \xi_t)$. In particular, we have $\xi \neq \eta$. Let $o' \in [o_Y, \xi] \cap \partial B_\eta$ be the endpoint farthest from o_Y ; equivalently, o' is the point at which the ray $[o_Y, \xi]$ exits the horoball B_η . Set

$$t' := d(o', \xi_t).$$

We first compare the shadow $O_R(o_Y, \xi_t)$ with complements of shadows based at o' . If $\zeta \in O_R(o_Y, \xi_t)$, then $d([o_Y, \zeta], \xi_t) < R$, and hence

$$d(o', [o_Y, \zeta]) \leq d(o', \xi_t) + R.$$

Therefore

$$(7.1) \quad O_R(o_Y, \xi_t) \subset \partial Y - \{\zeta \in \partial Y : d(o', [o_Y, \zeta]) > d(o', \xi_t) + R\}.$$

Conversely, if

$$d(o', [o_Y, \zeta]) \leq d(o', \xi_t) + R - O(\delta),$$

then by the Gromov hyperbolicity, this implies that $[o_Y, \zeta]$ passes within distance R of ξ_t , provided $O(\delta)$ is sufficiently large. Hence

$$(7.2) \quad \partial Y - \{\zeta \in \partial Y : d(o', [o_Y, \zeta]) > d(o', \xi_t) + R - O(\delta)\} \subset O_R(o_Y, \xi_t).$$

For $s \geq 0$, let $\eta_s \in [o', \eta]$ be the point satisfying $d(o', \eta_s) = s$. We claim that

$$(7.3) \quad O_R(o_Y, \xi_t) \subset \partial Y - O_{A/2}(o', \eta_{t'+R+A})$$

and

$$(7.4) \quad \partial Y - O_{A/2}(o', \eta_{t'+R}) \subset O_R(o_Y, \xi_t).$$

To prove (7.3), let

$$\zeta \in O_{A/2}(o', \eta_{t'+R+A}).$$

By Lemma 3.3,

$$\langle \zeta, \eta \rangle_{o'} \geq t' + R + A/2 - O(\delta).$$

Since $\eta \notin O_{R+A}(o_Y, \xi_t)$, applying (7.2) with $R + A$ in place of R gives

$$d(o', [o_Y, \eta]) > t' + R + A - O(\delta).$$

The Gromov product inequality (3.1) then implies

$$\langle o_Y, \zeta \rangle_{o'} \geq t' + R + A/2 - O(\delta).$$

For A sufficiently large, this and (7.1) imply that $\zeta \notin O_R(o_Y, \xi_t)$. This proves (7.3).

The proof of (7.4) is similar. If $\zeta \notin O_R(o_Y, \xi_t)$, then by (7.2),

$$d(o', [o_Y, \zeta]) > t' + R - O(\delta).$$

Together with the estimate for $d(o', [o_Y, \eta])$ above, the Gromov product inequality gives

$$\langle \zeta, \eta \rangle_{o'} \geq t' + R - O(\delta).$$

By Lemma 3.3, this implies

$$\zeta \in O_{A/2}(o', \eta_{t'+R})$$

after increasing A , if necessary. This proves (7.4).

We now estimate the measure. The hypothesis $\eta \notin O_{R+A}(o_Y, \xi_t)$ implies, by thin triangles, that $\eta_{t'}$ and ξ_t are uniformly close. Using (7.3), moving

the basepoint from $\eta_{t'}$ to $\eta_{t'+R+A}$, and absorbing the resulting multiplicative constant into \asymp , we get

$$\begin{aligned} \nu_{\xi_t}(O_R(o_Y, \xi_t)) &\ll \nu_{\eta_{t'}}(\partial Y - O_{A/2}(o', \eta_{t'+R+A})) \\ &\ll \nu_{\eta_{t'+R+A}}(\partial Y - O_{A/2}(o', \eta_{t'+R+A})). \end{aligned}$$

Since $o' \in \partial B_\eta$, there exists $\gamma_0 \in \Gamma$ such that $\eta = \gamma_0 \xi_P$ and $d(o', \gamma_0 o_Y)$ is uniformly bounded. By equivariance of the measures ν_x ,

$$\begin{aligned} &\nu_{\eta_{t'+R+A}}(\partial Y - O_{A/2}(o', \eta_{t'+R+A})) \\ &\asymp \nu_{\gamma_0^{-1} \eta_{t'+R+A}}(\partial Y - O_{A/2}(o_Y, \gamma_0^{-1} \eta_{t'+R+A})). \end{aligned}$$

Applying Proposition 6.8, we obtain

$$\begin{aligned} &\nu_{\gamma_0^{-1} \eta_{t'+R+A}}(\partial Y - O_{A/2}(o_Y, \gamma_0^{-1} \eta_{t'+R+A})) \\ &\asymp e^{-\mathbf{d}_\psi(\gamma_0^{-1} \eta_{t'+R+A}, o_Y)} e^{2\delta_{\bar{\psi}}(\mathbf{P}) \mathbf{d}_{\bar{\psi}}(o_Y, \gamma_0^{-1} \eta_{t'+R+A})} \\ &\quad \cdot (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(o_Y, \gamma_0^{-1} \eta_{t'+R+A}))^{a_{\bar{\psi}}(\mathbf{P})} \\ &\asymp e^{\mathbf{d}_\psi(\gamma_0 o_Y, \eta_{t'+R+A})} e^{2(\delta_{\bar{\psi}}(\mathbf{P})-1) \mathbf{d}_{\bar{\psi}}(\gamma_0 o_Y, \eta_{t'+R+A})} \\ &\quad \cdot (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(\gamma_0 o_Y, \eta_{t'+R+A}))^{a_{\bar{\psi}}(\mathbf{P})}. \end{aligned}$$

Combining altogether gives

$$\begin{aligned} \nu_{\xi_t}(O_R(o_Y, \xi_t)) &\ll e^{\mathbf{d}_\psi(\gamma_0 o_Y, \eta_{t'+R+A})} e^{2(\delta_{\bar{\psi}}(\mathbf{P})-1) \mathbf{d}_{\bar{\psi}}(\gamma_0 o_Y, \eta_{t'+R+A})} \\ &\quad \cdot (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(\gamma_0 o_Y, \eta_{t'+R+A}))^{a_{\bar{\psi}}(\mathbf{P})}. \end{aligned}$$

The points $\eta_{t'+R+A}$ and ξ_t are at uniformly bounded distance depending only on R and A . Therefore, by Lemmas 2.1 and 4.11,

$$\mathbf{d}_\psi(\gamma_0 o_Y, \eta_{t'+R+A}) \approx \mathbf{d}_\psi(\gamma o_Y, \xi_t),$$

and similarly for $\mathbf{d}_{\bar{\psi}}$. Hence

$$\nu_{\xi_t}(O_R(o_Y, \xi_t)) \ll e^{\mathbf{d}_\psi(\gamma o_Y, \xi_t)} e^{2(\delta_{\bar{\psi}}(\mathbf{P})-1) \mathbf{d}_{\bar{\psi}}(\gamma o_Y, \xi_t)} \cdot (C_{\bar{\psi}} + \mathbf{d}_{\bar{\psi}}(\gamma o_Y, \xi_t))^{a_{\bar{\psi}}(\mathbf{P})}.$$

The reverse inequality is obtained in the same way using (7.4) instead of (7.3).

Finally, Lemma 6.4 gives

$$\nu(O_R(o_Y, \xi_t)) \asymp e^{-\mathbf{d}_\psi(o_Y, \xi_t)} \nu_{\xi_t}(O_R(o_Y, \xi_t)),$$

and the desired estimate follows.

Case 3: $\eta \notin O_{R-A}(o_Y, \xi_t)$ and $\eta \in O_{R+A}(o_Y, \xi_t)$. This is the transition region between Cases 1 and 2. Moving a uniformly bounded distance farther along the ray $[o_Y, \xi]$, we obtain a point $\xi_{t'}$ for which Case 2 applies. Since $d(\xi_t, \xi_{t'})$ is uniformly bounded, the corresponding shadows are comparable after changing R by a uniform amount, and all \mathbf{d}_{ψ^-} - and $\mathbf{d}_{\bar{\psi}^-}$ -terms change only by a uniform additive error. Hence the estimate follows from Case 2.

The proof is complete. \square

Remark 7.2. When $\text{rank } G = 1$, the relatively Morse condition coincides with geometric finiteness. Moreover, after choosing the unit vector $H_0 \in \mathfrak{a}^+$, we identify \mathfrak{a} with $\mathbb{R}H_0$ and take $\psi(tH_0) = \delta_\Gamma t$. Then $\psi(\mu(g)) = \delta_\Gamma d(o, g o)$ and $\delta_\psi(\Gamma) = 1$. Since the opposition involution is trivial in rank one, $\bar{\psi} = \psi$; and by Theorem 5.5, we have $a_\psi(\mathbf{P}) = a_{\bar{\psi}}(\mathbf{P}) = 0$. Therefore Theorem 7.1 specializes to Theorem 1.2. In this sense, Theorem 7.1 is the higher-rank relatively Morse analogue of the global shadow lemma of Stratmann–Velani [23]. The condition $\delta_\psi(\Gamma) = 1$ in Theorem 7.1 is merely a normalization of the Patterson–Sullivan parameter; in the rank-one specialization above, it is achieved by the choice $\psi(tH_0) = \delta_\Gamma t$.

8. LOCAL PROPERTIES OF PATTERSON-SULLIVAN MEASURES

In this section, we apply the global shadow lemma to local properties of Patterson-Sullivan measures. As before, let $\Gamma < G$ be θ -Morse relative to \mathcal{P} , with Morse embedding $f : Y \rightarrow X$ of a Gromov model (Y, d) for (Γ, \mathcal{P}) . Let $\psi \in \mathfrak{a}_\theta^*$ be such that

$$\psi > 0 \quad \text{on } \mathcal{L}_f - \{0\} \quad \text{and} \quad \delta_\psi(\Gamma) = 1.$$

Let ν be a (Γ, ψ) -Patterson-Sullivan measure on Λ_θ . Throughout this section, we also assume that

$$\theta = i(\theta) \quad \text{and} \quad \psi = \psi \circ i.$$

Thus $\bar{\psi} = \psi$.

The visual quasi-metric. We first define the higher-rank visual quasi-metric associated to ψ . For distinct $\xi, \eta \in \Lambda_\theta$, set

$$(8.1) \quad d_\psi(\xi, \eta) := e^{-\psi(\mathcal{G}^\theta(\xi, \eta))},$$

and put $d_\psi(\xi, \xi) = 0$. Note that this is not the same object as the metric-like function \mathbf{d}_ψ on Y . This function behaves like a metric: there exists $c > 0$ such that

$$(8.2) \quad d_\psi(\xi, \eta) \leq c(d_\psi(\xi, \zeta) + d_\psi(\zeta, \eta))$$

for all $\xi, \eta, \zeta \in \Lambda_\theta$. For Anosov groups, this was proved in [18, Proposition 5.3]; the same argument applies to a general Morse embedding.

For $r > 0$ and $\xi \in \Lambda_\theta$, let

$$B_\psi(\xi, r) := \{\eta \in \Lambda_\theta : d_\psi(\xi, \eta) < r\}.$$

We identify ∂Y with Λ_θ via the Γ -equivariant homeomorphism $f : \partial Y \rightarrow \Lambda_\theta$. Since $\psi = \bar{\psi}$, Proposition 4.12 implies that, for all sufficiently large $R > 0$, there exist constants $c_1, c_2 > 0$ such that

$$(8.3) \quad B_\psi(\xi, c_1 e^{-\mathbf{d}_\psi(o_Y, \xi_t)}) \subset O_R(o_Y, \xi_t) \subset B_\psi(\xi, c_2 e^{-\mathbf{d}_\psi(o_Y, \xi_t)})$$

for all $\xi \in \Lambda_\theta$ and all $t \geq 0$, where $\xi_t \in [o_Y, \xi]$ is the point satisfying $d(o_Y, \xi_t) = t$.

For later use, we rewrite the global shadow lemma in a compact form. If $x \in Y$ lies in a horoball based at a point of $\Gamma\xi_{\mathbf{P}}$, choose $\gamma_x \in \Gamma$ so that $\gamma_x o_Y$ is a closest orbit point to x , and set

$$h(x) := \mathbf{d}_\psi(\gamma_x o_Y, x), \quad \delta(x) := \delta_\psi(\mathbf{P}), \quad \text{and} \quad a(x) := a_\psi(\mathbf{P}).$$

If x lies outside the horoballs, we set

$$h(x) = 0, \quad \delta(x) = 0, \quad \text{and} \quad a(x) = 0.$$

By the hypothesis $\psi = \psi \circ \mathbf{i}$, the global shadow lemma, together with the ordinary shadow estimate in the thick part, gives the uniform estimate

$$(8.4) \quad \nu(O_R(o_Y, x)) \asymp e^{-\mathbf{d}_\psi(o_Y, x)} e^{(2\delta(x)-1)h(x)} (C_\psi + h_\psi(x))^{a(x)}.$$

The implied constants are independent of x .

Local doubling. We first prove that ν is locally doubling with respect to the visual quasi-metric d_ψ .

Theorem 8.1. *For every $L > 1$, there exists $\varepsilon_L > 0$ such that*

$$\nu(B_\psi(\xi, r)) \geq \varepsilon_L \nu(B_\psi(\xi, Lr))$$

for all $\xi \in \Lambda_\theta$ and all $r > 0$ with $Lr \leq 1$.

Proof. Let $c_1, c_2 > 0$ be the constants in (8.3). Choose $t \geq 0$ so that

$$c_2 e^{-\mathbf{d}_\psi(o_Y, \xi_t)} < r$$

with t minimal. Then (8.3) gives

$$O_R(o_Y, \xi_t) \subset B_\psi(\xi, r).$$

Similarly, choose $t_L \geq 0$ so that

$$Lr < c_1 e^{-\mathbf{d}_\psi(o_Y, \xi_{t_L})}$$

with t_L maximal. Then

$$B_\psi(\xi, Lr) \subset O_R(o_Y, \xi_{t_L}).$$

Hence, by (8.4),

$$\frac{\nu(B_\psi(\xi, r))}{\nu(B_\psi(\xi, Lr))} \gg \frac{e^{-\mathbf{d}_\psi(o_Y, \xi_t)} e^{(2\delta(\xi_t)-1)h_\psi(\xi_t)} (C_\psi + h_\psi(\xi_t))^{a(\xi_t)}}{e^{-\mathbf{d}_\psi(o_Y, \xi_{t_L})} e^{(2\delta(\xi_{t_L})-1)h_\psi(\xi_{t_L})} (C_\psi + h_\psi(\xi_{t_L}))^{a(\xi_{t_L})}}.$$

By the choice of t and t_L ,

$$-\mathbf{d}_\psi(o_Y, \xi_t) \approx \log r \quad \text{and} \quad -\mathbf{d}_\psi(o_Y, \xi_{t_L}) \approx \log L + \log r.$$

This gives a lower bound for the factor $\frac{e^{-\mathbf{d}_\psi(o_Y, \xi_t)}}{e^{-\mathbf{d}_\psi(o_Y, \xi_{t_L})}}$ depending only on L .

Note also that by Lemma 4.5, the above implies that $\mathbf{d}_\psi(\xi_t, \xi_{t_L}) \approx \log L$, and hence $d(\xi_t, \xi_{t_L})$ is bounded above by a constant depending only on L .

We now compare remaining factors. If ξ_t and ξ_{t_L} lie in different horoballs, then, since the horoballs are disjoint and the two points are within bounded distance of each other, both points are within uniformly bounded distance,

depending on L , of Γ_{o_Y} . Hence both $h_\psi(\xi_t)$ and $h_\psi(\xi_{t_L})$ are bounded in terms of L , and therefore we obtain the desired estimate.

Now suppose that ξ_t and ξ_{t_L} lie in the same horoball $B_\eta \in \mathcal{B}$, for some parabolic limit point $\eta \in \partial Y$. In particular, $\delta(\xi_t) = \delta(\xi_{t_L})$ and $a(\xi_t) = a(\xi_{t_L})$. Hence, it suffices to show that $|h_\psi(\xi_t) - h_\psi(\xi_{t_L})|$ is uniformly bounded from above by a constant determined by L .

If the closest orbit points are the same, then it follows from Lemma 4.5 that

$$|h_\psi(\xi_t) - h_\psi(\xi_{t_L})|$$

is bounded from above by a constant determined by L , since $d(\xi_t, \xi_{t_L})$ is bounded from above by a constant depending on L .

The remaining possibility is that the closest orbit points are uniformly close to opposite endpoints of the segment

$$[o_Y, \xi] \cap B_\eta.$$

In this case, the midpoint y of this segment lies between ξ_t and ξ_{t_L} . We then choose $\gamma_L, \gamma \in \Gamma$ so that $\gamma_L o_Y$ and γo_Y are the closest orbit points to ξ_{t_L} and ξ_t , respectively. Note that we may assume that $\gamma_L^{-1} \gamma \in \mathbf{P}$. Then as in the proof of Lemma 4.8, it follows from Lemmas 4.7 and 4.4 that $\mathbf{d}_\psi(\gamma_L o_Y, y) \approx \psi(\mathcal{G}^\theta(\xi_{\mathbf{P}}, \gamma_L^{-1} \gamma \zeta))$ and $\mathbf{d}_\psi(\gamma o_Y, y) \approx \psi(\mathcal{G}^\theta(\xi_{\mathbf{P}}, \gamma^{-1} \gamma_L \zeta))$ for some $\zeta \in \Lambda_\theta$ such that $[\zeta, \xi_{\mathbf{P}}] \subset Y$ is uniformly close to o_Y . Here, the implied constant does not depend on L . By Proposition 4.3, $\psi(\mathcal{G}^\theta(\xi_{\mathbf{P}}, \gamma_L^{-1} \gamma \zeta)) \approx \frac{1}{2} \psi(\mu(\gamma_L^{-1} \gamma)) = \frac{1}{2} \psi(\mu(\gamma^{-1} \gamma_L)) \approx \psi(\mathcal{G}^\theta(\xi_{\mathbf{P}}, \gamma^{-1} \gamma_L \zeta))$ with the implied constant independent of L . Combining altogether, $\mathbf{d}_\psi(\gamma_L o_Y, y) \approx \mathbf{d}_\psi(\gamma o_Y, y)$.

Now by Lemma 4.5,

$$\begin{aligned} h_\psi(\xi_{t_L}) + \mathbf{d}_\psi(\xi_{t_L}, y) &\approx \mathbf{d}_\psi(\gamma_L o_Y, y) \\ &\approx \mathbf{d}_\psi(\gamma o_Y, y) \\ &\approx h_\psi(\xi_t) + \mathbf{d}_\psi(\xi_t, y). \end{aligned}$$

Hence,

$$|h_\psi(\xi_{t_L}) - h_\psi(\xi_t)| \approx |\mathbf{d}_\psi(\xi_{t_L}, y) - \mathbf{d}_\psi(\xi_t, y)|.$$

Since $\mathbf{d}_\psi(\xi_{t_L}, \xi_t) \approx \mathbf{d}_\psi(\xi_{t_L}, y) + \mathbf{d}_\psi(\xi_t, y)$ by Lemma 4.5 and $|\mathbf{d}_\psi(\xi_{t_L}, \xi_t)|$ is bounded from above by $\log L$ up to a uniform additive error,

$$|h_\psi(\xi_{t_L}) - h_\psi(\xi_t)|$$

is bounded by $\log L$ up to a uniform additive error as well.

Therefore, in any case, $|h_\psi(\xi_{t_L}) - h_\psi(\xi_t)|$ is uniformly bounded, and hence the ratio $\frac{\nu(B_\psi(\xi, r))}{\nu(B_\psi(\xi, Lr))}$ is bounded below by a positive constant depending only on L . This proves the theorem. \square

Local reverse doubling. The next result is a local reverse doubling estimate: after shrinking the radius by a sufficiently large factor, the mass drops by any prescribed factor, uniformly in the center and the scale.

Theorem 8.2. *For every $\kappa \geq 1$, there exists $L > 1$ such that*

$$\nu(B_\psi(\xi, r/L)) \leq \kappa^{-1} \nu(B_\psi(\xi, r))$$

for all $\xi \in \Lambda_\theta$ and all $0 < r \leq 1$.

Proof. We regard $L > 1$ as a parameter to be chosen. Let $c_1, c_2 > 0$ be the constants in (8.3). Choose $t \geq 0$ so that

$$c_2 e^{-d_\psi(o_Y, \xi_t)} < r$$

with t minimal. Then

$$O_R(o_Y, \xi_t) \subset B_\psi(\xi, r).$$

Choose $t_L \geq 0$ so that

$$r/L < c_1 e^{-d_\psi(o_Y, \xi_{t_L})}$$

with t_L maximal. Then

$$B_\psi(\xi, r/L) \subset O_R(o_Y, \xi_{t_L}).$$

Using (8.4), we obtain

$$\frac{\nu(B_\psi(\xi, r))}{\nu(B_\psi(\xi, r/L))} \gg \frac{e^{-d_\psi(o_Y, \xi_t)}}{e^{-d_\psi(o_Y, \xi_{t_L})}} \cdot \frac{e^{(2\delta(\xi_t)-1)h_\psi(\xi_t)}(C_\psi + h_\psi(\xi_t))^{a(\xi_t)}}{e^{(2\delta(\xi_{t_L})-1)h_\psi(\xi_{t_L})}(C_\psi + h_\psi(\xi_{t_L}))^{a(\xi_{t_L})}}.$$

By the choice of t and t_L ,

$$-d_\psi(o_Y, \xi_t) \approx \log r \quad \text{and} \quad -d_\psi(o_Y, \xi_{t_L}) \approx \log r - \log L,$$

and hence

$$\frac{e^{-d_\psi(o_Y, \xi_t)}}{e^{-d_\psi(o_Y, \xi_{t_L})}} \asymp L.$$

We now estimate the remaining factor from below. Since $0 < \delta_\psi(\mathbf{P}) < 1$ for every $\mathbf{P} \in \mathcal{P}$ by Theorem 5.4, and since \mathcal{P} is finite, there exists $0 < \sigma < 1$ such that

$$|2\delta_\psi(\mathbf{P}) - 1| \leq \sigma \quad \text{for all } \mathbf{P} \in \mathcal{P}.$$

Let

$$a_0 := \max_{\mathbf{P} \in \mathcal{P}} a_\psi(\mathbf{P}).$$

If ξ_t and ξ_{t_L} lie in different horoballs, then $h_\psi(\xi_t) + h_\psi(\xi_{t_L})$ is bounded above by $\log L$, up to a uniform additive error, by Lemmas 4.11 and 4.5. Hence

$$\frac{e^{(2\delta(\xi_t)-1)h_\psi(\xi_t)}}{e^{(2\delta(\xi_{t_L})-1)h_\psi(\xi_{t_L})}} \gg L^{-\sigma}.$$

We also have that

$$\frac{(C_\psi + h_\psi(\xi_t))^{a(\xi_t)}}{(C_\psi + h_\psi(\xi_{t_L}))^{a(\xi_{t_L})}} \gg (1 + \log L)^{-a_0}.$$

Therefore, combining altogether, we have

$$\frac{\nu(B_\psi(\xi, r))}{\nu(B_\psi(\xi, r/L))} \gg L^{1-\sigma} (1 + \log L)^{-a_0}.$$

If ξ_t and ξ_{t_L} lie in the same horoball in \mathcal{B} , the same argument as in the proof of Theorem 8.1 gives that

$$|h_\psi(\xi_t) - h_\psi(\xi_{t_L})|$$

is bounded from above by $\log L$, up to a uniform additive error. Since $\delta(\xi_t) = \delta(\xi_{t_L})$ and $a(\xi_t) = a(\xi_{t_L})$ in this case, this implies that the same lower bound holds:

$$\frac{\nu(B_\psi(\xi, r))}{\nu(B_\psi(\xi, r/L))} \gg L^{1-\sigma}(1 + \log L)^{-a_0}.$$

Now in any case, since $\sigma < 1$, the right-hand side tends to infinity as $L \rightarrow \infty$. We may therefore choose $L > 1$ large enough so that

$$\frac{\nu(B_\psi(\xi, r))}{\nu(B_\psi(\xi, r/L))} \geq \kappa$$

uniformly in ξ and r . This proves the theorem. \square

9. HAUSDORFF MEASURE

In this section, we characterize when Patterson-Sullivan measures are Hausdorff measures, using the global shadow lemma we obtained. As in the previous sections, let $\Gamma < G$ be θ -Morse relative to \mathcal{P} , with Morse embedding

$$f : Y \rightarrow X$$

from a Gromov model (Y, d) for (Γ, \mathcal{P}) . Let $\psi \in \mathfrak{a}_\theta^*$ satisfy

$$\psi > 0 \quad \text{on } \mathcal{L}_f - \{0\} \quad \text{and} \quad \delta_\psi(\Gamma) = 1.$$

Let ν be a (Γ, ψ) -Patterson-Sullivan measure on Λ_θ . Throughout this section, we also assume that

$$\theta = i(\theta) \quad \text{and} \quad \psi = \psi \circ i.$$

We equip Λ_θ with the visual quasi-metric d_ψ defined in (8.1). Since d_ψ satisfies the triangle inequality up to a multiplicative constant, as in (8.2), the Vitali covering lemma holds for d_ψ by the standard proof; see, for instance, [18, Lemma 6.12].

For $s > 0$, $\varepsilon > 0$, and $B \subset \Lambda_\theta$, define

$$\mathcal{H}_{\psi, \varepsilon}^s(B) := \inf \left\{ \sum_i (\text{diam}_\psi U_i)^s : B \subset \bigcup_i U_i, \sup_i \text{diam}_\psi U_i \leq \varepsilon \right\},$$

where $\text{diam}_\psi U := \sup_{\xi, \eta \in U} d_\psi(\xi, \eta)$. Then

$$\mathcal{H}_\psi^s(B) := \lim_{\varepsilon \rightarrow 0} \mathcal{H}_{\psi, \varepsilon}^s(B)$$

defines an outer measure and hence a Borel measure on Λ_θ ; see [12] and [10, Appendix A]. We call \mathcal{H}_ψ^s the s -dimensional Hausdorff measure associated to d_ψ . For $s = 1$, we write simply

$$\mathcal{H}_\psi := \mathcal{H}_\psi^1.$$

Theorem 9.1. *Suppose that, for every $P \in \mathcal{P}$, one of the following holds:*

- (1) $\delta_\psi(P) < 1/2$;
- (2) $\delta_\psi(P) = 1/2$ and $a_\psi(P) = 0$.

Then ν is a positive multiple of \mathcal{H}_ψ .

Remark 9.2. This theorem generalizes Sullivan's Hausdorff-measure criterion for geometrically finite Kleinian groups [25]. In the real hyperbolic case, if P is a rank- k parabolic subgroup, then its critical exponent is $\delta_P = k/2$, and after the normalization $\delta_\psi(\Gamma) = 1$ we have

$$\delta_\psi(P) = \frac{\delta_P}{\delta_\Gamma}.$$

Thus the classical condition $k \leq \delta_\Gamma$ is exactly condition $\delta_\psi(P) \leq 1/2$. Recalling also that $a_\psi(P)$ is always 0 in rank one, Theorem 9.1 may therefore be viewed as a higher-rank generalization of Sullivan's criterion on Patterson-Sullivan measures to be Hausdorff measures.

Recall also that Anosov groups are special cases of relatively Morse groups, with trivial peripheral subgroups, and hence Theorem 9.1 generalizes [11, Theorem 1.1].

Remark 9.3. We also note that the hypothesis $\psi = \psi \circ i$ is necessary, as in the Anosov case [11, Theorem 1.3]. Indeed, although $\psi \neq \bar{\psi}$, Lemma 4.4 implies that the identity map between (Λ_θ, d_ψ) and $(\Lambda_\theta, d_{\bar{\psi}})$ is bi-Lipschitz, and hence their Hausdorff measures are mutually absolutely continuous to each other. On the other hand, associated Patterson-Sullivan measures are singular [15]. Hence, when the Patterson-Sullivan measure for $\bar{\psi}$ is the Hausdorff measure for $(\Lambda_\theta, d_{\bar{\psi}})$ as in Theorem 9.1, the Patterson-Sullivan measure for ψ cannot be the Hausdorff measure for (Λ_θ, d_ψ) .

The rest of this section is devoted to the proof of Theorem 9.1. Recall the hypothesis that $\delta_\psi(\Gamma) = 1$. First, the Hausdorff measure \mathcal{H}_ψ has the same conformality rule as the Patterson-Sullivan measure: for $\gamma \in \Gamma$,

$$\frac{d\gamma_*\mathcal{H}_\psi}{d\mathcal{H}_\psi}(\xi) = e^{\psi(\beta_\xi^\theta(e, \gamma))}.$$

This was proved in [11, Lemma 9.7] for Anosov subgroups, and the same proof applies in the present relatively Morse setting. Therefore, by the uniqueness of the (Γ, ψ) -Patterson-Sullivan measure (Theorem 6.1), it suffices to prove that

$$0 < \mathcal{H}_\psi(\Lambda_\theta) < \infty.$$

Finite positive Hausdorff measure. We first establish local upper and lower estimates for ν with respect to the visual quasi-metric. We identify ∂Y with Λ_θ via $f : \partial Y \rightarrow \Lambda_\theta$, and denote by $\Lambda_\theta^{\text{con}}$ the f -image of the conical limit set in ∂Y .

Lemma 9.4. *Assume that, for every $P \in \mathcal{P}$, either $\delta_\psi(P) < 1/2$, or $\delta_\psi(P) = 1/2$ and $a_\psi(P) = 0$. Then there exists $C > 1$ such that:*

(1) for every $\xi \in \Lambda_\theta$ and every $r > 0$,

$$\nu(B_\psi(\xi, r)) \leq Cr;$$

(2) for every conical limit point $\xi \in \Lambda_\theta^{\text{con}}$, there exists a sequence $r_i \rightarrow 0$ such that

$$\nu(B_\psi(\xi, r_i)) \geq C^{-1}r_i \quad \text{for all } i.$$

Proof. We first prove the upper bound. Let $R > 0$ be large enough so that the shadow-ball compatibility (8.3) and the global shadow lemma hold. By the hypothesis on the parabolic subgroups, the cusp correction factor in the global shadow lemma is uniformly bounded above. Indeed, if x lies in a horoball associated to \mathbf{P} , then the correction factor is

$$e^{(2\delta_\psi(\mathbf{P})-1)\mathbf{d}_\psi(\Gamma o_Y, x)} (C_\psi + \mathbf{d}_\psi(\Gamma o_Y, x))^{a_\psi(\mathbf{P})}.$$

This is uniformly bounded when $\delta_\psi(\mathbf{P}) < 1/2$, and also when $\delta_\psi(\mathbf{P}) = 1/2$ and $a_\psi(\mathbf{P}) = 0$. In the thick part, the usual shadow estimate gives the same conclusion. Hence

$$\nu(O_R(o_Y, x)) \ll e^{-\mathbf{d}_\psi(o_Y, x)} \quad \text{uniformly for all } x \in Y.$$

Now fix $\xi \in \Lambda_\theta$ and $0 < r \leq 1$. Choose $x \in [o_Y, \xi]$ so that $e^{-\mathbf{d}_\psi(o_Y, x)} \asymp r$ and $B_\psi(\xi, r) \subset O_R(o_Y, x)$, which is possible by Lemma 4.10 and (8.3), after changing the implicit constants. Then

$$\nu(B_\psi(\xi, r)) \leq \nu(O_R(o_Y, x)) \ll e^{-\mathbf{d}_\psi(o_Y, x)} \asymp r.$$

After increasing the constant, the same bound holds for all $r > 0$, since ν is a probability measure.

We now prove the lower bound at conical limit points. Let $\xi \in \Lambda_\theta^{\text{con}}$. By conicality, there exist $D > 0$ and a sequence $\gamma_i \in \Gamma$ with

$$d(\gamma_i o_Y, [o_Y, \xi]) \leq D \quad \text{and} \quad d(o_Y, \gamma_i o_Y) \rightarrow \infty.$$

Choose $x_i \in [o_Y, \xi]$ with $d(x_i, \gamma_i o_Y) \leq D$. Then

$$\mathbf{d}_\psi(o_Y, x_i) \rightarrow \infty.$$

By the ordinary orbit-shadow lemma, together with Lemma 2.1,

$$\nu(O_R(o_Y, x_i)) \asymp e^{-\mathbf{d}_\psi(o_Y, x_i)}.$$

Using (8.3), choose $r_i \asymp e^{-\mathbf{d}_\psi(o_Y, x_i)}$ so that

$$O_R(o_Y, x_i) \subset B_\psi(\xi, r_i).$$

Then $r_i \rightarrow 0$ and

$$\nu(B_\psi(\xi, r_i)) \geq \nu(O_R(o_Y, x_i)) \gg e^{-\mathbf{d}_\psi(o_Y, x_i)} \asymp r_i.$$

This proves the lemma. \square

Now the following finishes the proof of Theorem 9.1.

Proposition 9.5. *Assume that, for every $P \in \mathcal{P}$, either $\delta_\psi(P) < 1/2$, or $\delta_\psi(P) = 1/2$ and $a_\psi(P) = 0$. Then*

$$0 < \mathcal{H}_\psi(\Lambda_\theta) < \infty.$$

Proof. We first prove positivity. Fix $\varepsilon > 0$ and let $\{U_i\}_{i \in \mathbb{N}}$ be a countable cover of Λ_θ with $\text{diam}_\psi U_i \leq \varepsilon$ for all i . For each i , choose $\xi_i \in U_i$ and $\rho_i > \text{diam}_\psi U_i$ such that

$$\sum_i \rho_i \leq \varepsilon + \sum_i \text{diam}_\psi U_i.$$

Then

$$U_i \subset B_\psi(\xi_i, \rho_i).$$

By Lemma 9.4,

$$1 = \nu(\Lambda_\theta) \leq \sum_i \nu(B_\psi(\xi_i, \rho_i)) \leq C \sum_i \rho_i \leq C \left(\varepsilon + \sum_i \text{diam}_\psi U_i \right).$$

Taking the infimum over all such covers and then letting $\varepsilon \rightarrow 0$, we obtain

$$\mathcal{H}_\psi(\Lambda_\theta) > 0.$$

We now prove finiteness. Since the set of parabolic limit points is countable, it has \mathcal{H}_ψ -measure zero. It therefore suffices to show that $\mathcal{H}_\psi(\Lambda_\theta^{\text{con}}) < \infty$. Fix $\varepsilon > 0$. By Lemma 9.4, for every $\xi \in \Lambda_\theta^{\text{con}}$ we may choose $0 < r_\xi < \varepsilon$ such that $\nu(B_\psi(\xi, r_\xi)) \geq C^{-1}r_\xi$. Applying the Vitali covering lemma to the family $\{B_\psi(\xi, r_\xi) : \xi \in \Lambda_\theta^{\text{con}}\}$, there exists a countable disjoint subcollection $\{B_\psi(\xi_n, r_n) : n \in \mathbb{N}\}$ such that

$$\Lambda_\theta^{\text{con}} \subset \bigcup_n B_\psi(\xi_n, \lambda r_n)$$

for some uniform constant $\lambda > 1$. Since d_ψ satisfies the triangle inequality up to a multiplicative constant (8.2), there exists $D > 0$ such that

$$\text{diam}_\psi B_\psi(\xi_n, \lambda r_n) \leq D r_n \quad \text{for all } n.$$

Hence

$$\mathcal{H}_{\psi, D\varepsilon}(\Lambda_\theta^{\text{con}}) \leq \sum_n D r_n \leq DC \sum_n \nu(B_\psi(\xi_n, r_n)) \leq DC \nu(\Lambda_\theta).$$

Since $\varepsilon > 0$ is arbitrary, this proves $\mathcal{H}_\psi(\Lambda_\theta^{\text{con}}) < \infty$. This completes the proof. \square

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