MEASURABLE BOUNDARY MAPS AND PATTERSON–SULLIVAN MEASURES FOR NON-BOREL ANOSOV GROUPS ON THE FURSTENBERG BOUNDARY

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Abstract. In this paper we develop a theory for Patterson–Sullivan measures for non-Borel Anosov groups on the Furstenberg boundary. Previously, such a theory has been successfully developed for measures supported on the partial flag manifold associated to the Anosov condition, which coincides with the Furstenberg boundary only under the strongest Anosov condition, Borel Anosov. We establish existence, uniqueness, and ergodicity results for the measures on the Furstenberg boundary under arbitrary Anosov conditions; we show ergodicity of Bowen–Margulis–Sullivan measures on the homogeneous space; and we establish strict convexity results for the critical exponent associated to functionals on the entire Cartan subspace. Using this strict convexity, we establish an entropy rigidity result for Anosov groups with Lipschitz limit set.

A key tool we develop is a new sufficient condition for the existence of a measurable boundary map associated to a Zariski dense representation. This result not only applies to Anosov groups, but also to transverse groups, mapping class groups, and discrete subgroups of the isometry groups of Gromov hyperbolic spaces.

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1. Introduction

Throughout this paper G will be a semisimple Lie group with finite center and no compact factors. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ of the Lie algebra, a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$, and a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Let $\Delta \subset \mathfrak{a}^*$ denote the system of simple restricted roots corresponding to the choice of $\mathfrak{a}^+$ and let $\kappa : \mathsf{G} \to \mathfrak{a}^+$ denote the Cartan projection.

Given a non-empty subset $\theta \subset \Delta$, let $\mathsf{P}_\theta < \mathsf{G}$ denote the associated parabolic subgroup. A discrete subgroup $\Gamma < \mathsf{G}$ is P_θ-*Anosov* if Γ is word hyperbolic as an abstract group and its Gromov boundary nicely embeds into the partial flag manifold $\mathcal{F}_\theta := \mathsf{G} / \mathsf{P}_\theta$ (see Section 3.8 for a precise definition).

For each $\theta \subset \Delta$, there is a partial Cartan subspace $\mathfrak{a}_\theta \subset \mathfrak{a}$ and a cocycle $B_\theta^{IW} : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ called the *(partial) Iwasawa cocycle*. This cocycle can be used to define Patterson–Sullivan measures as follows.

Definition 1.1. Given a subgroup $\Gamma < \mathsf{G}$, $\theta \subset \Delta$, $\phi \in \mathfrak{a}_\theta^*$, and $\delta \geq 0$, a Borel probability measure μ on $\mathcal{F}_\theta$ is a (Γ, ϕ, δ)-*Patterson–Sullivan measure* if for every $\gamma \in \Gamma$ the measures μ, $\gamma_*\mu$ are absolutely continuous and

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-\delta\phi B_\theta^{IW}(\gamma^{-1}, x)} \quad \mu\text{-a.e.}$$

When G is of rank one, the above definition (with an appropriate choice of functional) coincides with the classical Patterson–Sullivan measures introduced by Patterson [Pat76] and Sullivan [Sul79]. In higher rank, the above definition is due to Quint [Qui02].

The theory of Patterson–Sullivan measures on $\mathcal{F}_\theta$ for P_θ-Anosov groups has been extensively developed and has become a useful tool for studying Anosov groups. In this case, the definition of an Anosov group implies that a P_θ-Anosov group acts very nicely on $\mathcal{F}_\theta = \mathsf{G} / \mathsf{P}_\theta$, but in general the action of a P_θ-Anosov group can be very complicated on larger flag manifolds, especially the Furstenberg boundary $\mathcal{F}_\Delta$.

In this paper we use and extend ideas from our earlier work [KZ25] to develop a theory of Patterson–Sullivan measures on the Furstenberg boundary $\mathcal{F}_\Delta$ for Zariski dense P_θ-Anosov groups, even though $\theta \neq \Delta$. Having such a Patterson–Sullivan theory on the Furstenberg boundary enables us to obtain several applications as discussed below, which was previously known only for P_Δ-Anosov groups (or, Borel Anosov groups). Recall that the P_Δ-Anosov condition is very restrictive; in many cases, the group must either be virtually a free group or a surface group [CT20, Tso20, Dey25, DGR24].

Our results hold for the more general class of transverse groups (which also contain the relatively Anosov groups), for any partial flag manifold containing $\mathcal{F}_\theta$, and for weaker irreducibility conditions than Zariski dense. However, for simplicity **in the introduction we only state our results for Zariski dense Anosov groups** and Patterson–Sullivan measures on the Furstenberg boundary.

Before stating our results for discrete subgroups of Lie groups, we describe one of the tools we develop.

1.1. **Measurable boundary maps.** The existence of boundary maps for representations of lattices played a significant role in the proof of Mostow's Rigidity [Mos68, Mos73, Pra73] and Margulis' Superrigidity [Mar91]. A key tool in this work is constructing measurable boundary maps associated to Zariski dense representations.

The domains in our boundary map theorem are "Patterson–Sullivan systems," a notion introduced in our earlier work [KZ25]. A precise definition is given in Section 2 below, but informally these consist of a compact metric space M, a Borel probability measure μ on M, and an action $\Gamma \curvearrowright M$ which preserves the measure class of μ which satisfy certain assumptions that allowed us to extend the classical theory of Patterson–Sullivan measures. We note that a different framework for abstract Patterson–Sullivan-like measures was given in [BCZZ24].

Amongst "Patterson–Sullivan systems," we further identified a special class called "well-behaved Patterson–Sullivan systems" and for these systems defined conical limit sets (again see Section 2 for a precise definition). Under the assumption that the conical limit set has full measure, in this paper we prove the following existence theorem for boundary maps.

Theorem 1.2 (see Theorem 6.1 below)**.** *Suppose (M, Γ, σ, μ) is a well-behaved Patterson–Sullivan system with respect to a hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$, the $\mathcal{H}$-conical limit set has full μ-measure, and the Γ-action on (M, μ) is amenable.*

If $\rho : \Gamma \to \mathsf{G}$ is a Zariski dense representation, then there exists a unique ρ-equivariant μ-a.e. defined measurable map $f : M \to \mathcal{F}_\Delta$.

Remark 1.3. The above is actually a special case of Theorem 6.1, which assumes weaker conditions than Zariski dense, provides the existence of μ-a.e. conical limits, and shows that f maps into the "contracting conical limit set."

We highlight three examples of well-behaved Patterson–Sullivan systems:

(1) If $\Gamma < \mathsf{G}$ is P_θ-Anosov and μ is a Patterson–Sullivan measure (in the sense of Definition 1.1) supported on the limit set of Γ in $\mathcal{F}_\theta$, then $\mathcal{F}_\theta$, Γ, μ are part of a well-behaved Patterson–Sullivan system where the conical limit set has full μ-measure. In fact it suffices to assume Γ is a P_θ-transverse group and the Poincaré series associated to μ diverges at the critical exponent, see Section 4.2 for more details. When Γ is P_θ-Anosov, amenability follows from the work of Adams [Ada94]. For P_θ-transverse groups, we show amenability in Section 7. See Theorem 8.1 for more details.

(2) Let X be a proper Gromov hyperbolic metric space, let ∂X denote the Gromov boundary of X, let $\Gamma < \mathsf{Isom}(X)$ be a discrete subgroup, let $\delta_X(\Gamma)$ denote the critical exponent of Γ, and let μ be a Patterson–Sullivan measure of dimension $\delta_X(\Gamma)$. Then ∂X, Γ, μ are part of a well-behaved Patterson–Sullivan system. Further, if
$$\sum_{\gamma \in \Gamma} e^{-\delta_X(\Gamma)\, \mathrm{d}(o, \gamma o)} = +\infty$$
for some (any) $o \in X$, then the conical limit set has full μ-measure. When X has exponential bounded geometry, Kaimanovich [Kai04] showed that the Γ-action on $(\partial X, \mu)$ is amenable. See Theorem 13.2 for more details.

(3) Let S be a connected orientable surface of finite type with negative Euler characteristic, let $\mathsf{Mod}(S)$ denote its mapping class group, let $\mathcal{PML}$ denote the space of projective measured laminations on S, and let Leb denote the natural Lebesgue measure class on $\mathcal{PML}$. Then $\mathcal{PML}$, $\mathsf{Mod}(S)$, Leb are part of a well-behaved Patterson–Sullivan system where the conical limit set has full Leb-measure. Further, the work of Hamenstädt [Ham09a] implies that the $\mathsf{Mod}(S)$-action on $(\mathcal{PML}, \mathrm{Leb})$ is amenable. See Theorem 12.1 for more details.

We note that Bader–Furman [BF14, BF25] have developed a different abstract setting that leads to the existence of boundary maps. Many examples seem to satisfy both setting, but there does not seem to be any obvious implication between the two abstract settings. Further some examples, like Poisson boundaries of countable groups, satisfy the Bader–Furman assumptions but not ours. Other examples, like Kleinian groups of divergent type with infinite Bowen–Margulis–Sullivan (BMS) measure, satisfy our assumptions but do not seem to satisfy the Bader–Furman assumptions.

In this paper our main motivation is the study of P_θ-transverse groups. These have natural flow spaces which can have infinite BMS-measures and thus it seems that they do not satisfy the Bader–Furman assumptions (in fact any Kleinian group is a transverse group).

One idea in the proof of Theorem 1.2 is to adapt the "standard argument" for almost sure convergence of random walks on symmetric spaces (e.g. as in [Mar91, Chapter VI, Section 2] or [BQ16, Section 4.2]) where we replace the use of the Martingale convergence theorem with a technical continuity result along "conical sequences" (see Section 5).

1.2. **Existence and uniqueness of PS-measures.** For $\alpha \in \Delta$, let ω_α denote the associated fundamental weight. The dual space of the partial Cartan subspace $\mathfrak{a}_\theta \subset \mathfrak{a}$ can be identified with $\operatorname{span}\{\omega_\alpha\}_{\alpha\in\theta}$. Then given $\phi \in \mathfrak{a}_\theta^* := \operatorname{span}\{\omega_\alpha\}_{\alpha\in\theta}$, the *$\phi$-critical exponent* of a discrete subgroup $\Gamma < \mathsf{G}$ is the exponential growth rate

$$\delta^\phi(\Gamma) := \limsup_{T\to+\infty} \frac{1}{T} \log \#\{\gamma \in \Gamma : \phi(\kappa(\gamma)) \le T\} \in [0, +\infty],$$

equivalently

$$\delta^\phi(\Gamma) = \inf\left\{ s > 0 : \sum_{\gamma\in\Gamma} e^{-s\phi(\kappa(\gamma))} < +\infty \right\} \in [0, +\infty]. \tag{1}$$

We establish existence and uniqueness of Patterson–Sullivan (PS) measures on the Furstenberg boundary. Further, we show that the measures are supported on the "contracting conical limit set" which is a smaller subset than the usual conical limit set defined in terms of shadows in the symmetric space. See Section 4.2 for details.

Theorem 1.4 (PS-measures on the Furstenberg boundary)**.** *Suppose $\Gamma < \mathsf{G}$ is Zariski dense and P_θ-Anosov. If $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$, then there exists a unique $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure on the Furstenberg boundary $\mathcal{F}_\Delta$. Moreover, this unique Patterson–Sullivan measure is supported on the contracting conical limit set of Γ in $\mathcal{F}_\Delta$.*

Previously, existence and uniqueness for Patterson–Sullivan measures for P_θ-Anosov groups were only established on the partial flag manifold $\mathcal{F}_\theta$. The limit set in $\mathcal{F}_\theta$ compactifies the group (in fact identifies with the Gromov boundary of the group) which makes the construction on $\mathcal{F}_\theta$ much more straightforward.

We construct the Patterson–Sullivan measure on $\mathcal{F}_\Delta$ by using Theorem 1.2 to construct a measurable section $\mathcal{F}_\theta \to \mathcal{F}_\Delta$ and then pushing forward the Patterson–Sullivan measure on $\mathcal{F}_\theta$.

Remark 1.5. Quint [Qui02] proved that for a general Zariski dense discrete subgroup $\Gamma < \mathsf{G}$, $\phi \in \mathfrak{a}^*$, and $\delta \geq 0$, a (Γ, ϕ, δ)-Patterson–Sullivan measure on $\mathcal{F}_\Delta$ exists if $\delta \cdot \phi$ is tangent to the so-called growth indicator of Γ at the direction in the interior $\operatorname{int} \mathfrak{a}^+$. We suspect that this condition can fail to hold for our most general existence theorem (see Theorem 8.1) and in particular when the flow space associated to the transverse group has infinite BMS-measure. Indeed, in general, it is a priori possible that a tangent direction is on the wall $\mathfrak{a}^+ \cap \mathfrak{a}_\theta$. See Figure 1 below.

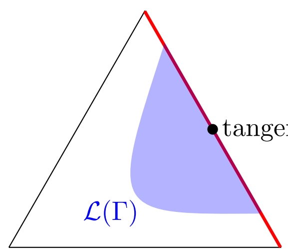

FIGURE 1. Slice of $\mathfrak{a}^+$ along its unit sphere, when $\operatorname{rank} \mathsf{G} = 3$ and $\#\theta = 2$. $\mathcal{L}(\Gamma) \subset \mathfrak{a}^+$ denotes the asymptotic cone of $\kappa(\Gamma)$, and there is a tangent direction for $\delta^\phi(\Gamma) \cdot \phi$ on $\mathcal{L}(\Gamma) \cap \mathfrak{a}_\theta$.

Remark 1.6. In particular, if $\Gamma < \mathsf{G}$ is Zariski dense and P_Δ-Anosov, then the existence of PS-measures on $\mathcal{F}_\Delta$ follows from Quint's work as in Remark 1.5. In this case, the uniqueness was proved by Edwards–Lee–Oh [ELO22] when $\operatorname{rank} \mathsf{G} \leq 3$, and by Lee–Oh [LO24] without rank assumption. On the other hand, for P_θ-Anosov case with $\theta \neq \Delta$, the existence and uniqueness of PS-measures on $\mathcal{F}_\Delta$ were not known, to the best of our knowledge.

We also show that these Patterson–Sullivan measures are singular to every other Patterson–Sullivan measure.

Theorem 1.7. *Suppose $\Gamma < \mathsf{G}$ is Zariski dense and P_θ-Anosov. Assume*

- $\phi_1 \in \mathfrak{a}_\theta^*$, $\delta^{\phi_1}(\Gamma) < +\infty$, *and μ_1 is the $(\Gamma, \phi_1, \delta^{\phi_1}(\Gamma))$-Patterson–Sullivan measure on the Furstenberg boundary $\mathcal{F}_\Delta$.*
- $\phi_2 \in \mathfrak{a}^*$, $\delta \geq 0$, *and μ_2 is a (Γ, ϕ_2, δ)–Patterson–Sullivan measure on the Furstenberg boundary $\mathcal{F}_\Delta$.*

Then,

$$\mu_1 \text{ and } \mu_2 \text{ are non-singular} \iff \mu_1 = \mu_2 \iff \delta^{\phi_1}(\Gamma) \cdot \phi_1 = \delta \cdot \phi_2.$$

Notice that the functional ϕ_2 is not assumed to be in $\mathfrak{a}_\theta^*$. Previously, singularity results of this form were only established for Patterson–Sullivan measures associated to functionals in $\mathfrak{a}_\theta^*$ on the partial flag manifold $\mathcal{F}_\theta$. The Anosov case was established in [LO23, Sam24] and the transverse group case was established in [BCZZ24, Kim24].

1.3. **Double ergodicity and Bowen–Margulis–Sullivan measures.** Let $\mathsf{K} < \mathsf{G}$ denote the maximal compact subgroup with Lie algebra $\mathfrak{k}$, let $\mathsf{A} < \mathsf{G}$ denote the subgroup with Lie algebra $\mathfrak{a}$, and let $\mathsf{M} < \mathsf{K}$ denote the centralizer of A in K.

When Γ is Zariski dense and P_Δ-Anosov, the dynamics of the A-action on the homogeneous space $\Gamma\backslash\mathsf{G}/\mathsf{M}$ has been extensively studied; see for instance [CS23, ELO23, BLLO23, Sam24, LO24, KOW25b, CZZ25, KOW25a]. However, very little is known when Γ is only P_θ-Anosov and $\theta \neq \Delta$. One reason for this is that to construct Bowen–Margulis–Sullivan measures, one needs to start with appropriate Patterson–Sullivan measures on the Furstenberg boundary which previously were only known to exist when Γ is P_Δ-Anosov, or under extra hypotheses as in Remark 1.5.

Using Theorem 1.4 we can now construct Bowen–Margulis–Sullivan measures on $\Gamma\backslash\mathsf{G}/\mathsf{M}$ when $\Gamma < \mathsf{G}$ is Zariski dense and P_θ-Anosov. We briefly describe the construction here, for more details see Section 9.

Suppose $\Gamma < \mathsf{G}$ is Zariski dense P_θ-Anosov and $\phi \in \mathfrak{a}_\theta^*$ satisfies $\delta^\phi(\Gamma) < +\infty$. Let $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ denote the opposite involution. The adjoint $\mathrm{i}^* : \mathfrak{a}^* \to \mathfrak{a}^*$ preserves the set of simple roots and we write $\mathrm{i}^*\theta := \{\mathrm{i}^*\alpha : \alpha \in \theta\}$. Then $\mathrm{i}^*\phi \in \mathfrak{a}_{\mathrm{i}^*\theta}^*$ and Γ is $\mathsf{P}_{\mathrm{i}^*\theta}$-Anosov. Since

$$\mathrm{i}^*\phi(\kappa(g)) = \phi(\kappa(g^{-1})) \quad \text{for all} \quad g \in \mathsf{G},$$

we have $\delta := \delta^\phi(\Gamma) = \delta^{\mathrm{i}^*\phi}(\Gamma)$. Then, by Theorem 1.4 there exist unique (Γ, ϕ, δ) and $(\Gamma, \mathrm{i}^*\phi, \delta)$-Patterson–Sullivan measures μ_ϕ and $\mu_{\mathrm{i}^*\phi}$ on $\mathcal{F}_\Delta$, respectively.

Next let $\mathcal{F}_\Delta^{(2)}$ denote the space of ordered transverse pairs and let $\mathcal{G}_\Delta : \mathcal{F}_\Delta^{(2)} \to \mathfrak{a}$ denote the Gromov product (see Equation (12) for a definition). One can show that the measure

$$d\nu_\phi(x,y) := e^{\delta\phi\,\mathcal{G}_\Delta(x,y)} d\mu_\phi(x) \otimes d\mu_{\mathrm{i}^*\phi}(y) \quad \text{on} \quad \mathcal{F}_\Delta^{(2)}$$

is Γ-invariant. There is a natural equivariant identification of the homogeneous space G/M with $\mathcal{F}_\Delta^{(2)} \times \mathfrak{a}$ where the left multiplication of G on G/M descends to the G-action on $\mathcal{F}_\Delta^{(2)} \times \mathfrak{a}$ by

$$g \cdot (x, y, u) = (gx, gy, u + B_\Delta^{IW}(g, x)).$$

Since M commutes with A, we also have a right A-action on G/M, which corresponds to the $\mathfrak{a}$-action on $\mathcal{F}_\Delta^{(2)} \times \mathfrak{a}$ by translation on the $\mathfrak{a}$-component. Further, if $\mathrm{Leb}_\mathfrak{a}$ denotes the Lebesgue measure on $\mathfrak{a}$, then the measure $\tilde{\mathsf{m}}_\phi := \nu_\phi \otimes \mathrm{Leb}_\mathfrak{a}$ on G/M is Γ-invariant, and hence it descends to a Radon measure

$$\mathsf{m}_\phi \quad \text{on} \quad \Gamma\backslash\mathsf{G}/\mathsf{M}$$

called the *Bowen–Margulis–Sullivan measure*, which is invariant under the A-action.

Theorem 1.8 (Ergodicity on homogeneous spaces)**.** *With the notations above,*

1. *(1) The measure ν_ϕ is non-zero and the Γ-action on $(\mathcal{F}_\Delta^{(2)}, \nu_\phi)$ is ergodic.*
2. *(2) The Bowen–Margulis–Sullivan measure m_ϕ is non-zero and the A-action on $(\Gamma\backslash\mathsf{G}/\mathsf{M}, \mathsf{m}_\phi)$ is ergodic.*

More generally, we obtain an ergodic dichotomy for transverse groups in Theorem 9.1, generalizing the classical Hopf–Tsuji–Sullivan dichotomy [Tsu59, Hop71, Sul79, AS84, Rob03]. Similar ergodicity results were previously known only for certain abstract flow spaces associated to $\theta \subset \Delta$ [Sam24, CZZ24, KOW25b], which are not homogeneous spaces unless $\theta = \Delta$. When $\theta = \Delta$, the ergodic dichotomy for P_Δ-Anosov groups was first proved in [LO24].

To the best of our knowledge, Theorem 1.8 is the first ergodicity result on the homogeneous space for P_θ-Anosov groups when $\theta \neq \Delta$.

Remark 1.9. One might also ask for the ergodicity as in Theorem 1.8 when $\phi \notin \mathfrak{a}_\theta^*$. However, this is not true in general. Indeed, consider $\mathsf{G} = \mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{C})$ and

$$\Gamma := \{(\rho_1(\gamma), \rho_2(\gamma)) \in \mathsf{G} : \gamma \in \pi_1(S)\}$$

where S is a closed surface of genus at least two, $\rho_1 : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$ is a cocompact representation, and $\rho_2 : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{C})$ is a discrete faithful representation such that $\rho_2(\pi_1(S))\backslash \mathbb{H}^3$ has one geometrically finite end and one geometrically infinite end.

In this case, we can assume that $\Delta = \{\alpha_1, \alpha_2\}$ where α_1 is the simple root for $\mathsf{PSL}(2,\mathbb{R})$ and α_2 is the one for $\mathsf{PSL}(2,\mathbb{C})$, and $\mathcal{F}_\Delta = \partial\,\mathbb{H}^2 \times \partial\,\mathbb{H}^3$. In addition, Γ is Zariski dense and $\mathsf{P}_{\{\alpha_1\}}$-Anosov.

On the other hand, if one produces a measure ν_{α_2} on $\mathcal{F}_\Delta^{(2)}$ associated to α_2, then ν_{α_2} is not ergodic under the Γ-action. This is because, if μ is a $(\Gamma, \alpha_2, \delta)$-Patterson–Sullivan measure on $\mathcal{F}_\Delta$, then passing to the projection $\pi : \mathcal{F}_\Delta \to \partial\,\mathbb{H}^3$, the measure $\pi_*\mu$ is a $(\rho_2(\pi_1(S)), \alpha_2, \delta)$-Patterson–Sullivan measure on $\partial\,\mathbb{H}^3$. Hence, the Γ-ergodicity of ν_{α_2} implies that the $\rho_2(\pi_1(S))$-action on $\partial\,\mathbb{H}^3 \times \partial\,\mathbb{H}^3 \smallsetminus \mathrm{diag}$ with respect to the measure $\pi_*\mu \otimes \pi_*\mu$ is ergodic, which is impossible due to the work of Canary [Can93]. A similar discussion holds for Bowen–Margulis–Sullivan measures.

Example 1.10. Consider the case that $\mathsf{G} = \mathsf{G}_1 \times \mathsf{G}_2$ for some semisimple Lie groups G_1 and G_2. Then for any Anosov subgroup $\Gamma_1 < \mathsf{G}_1$ and a representation $\rho : \Gamma_1 \to \mathsf{G}_2$, the subgroup

$$\Gamma := \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma_1\} < \mathsf{G}$$

is P_θ-Anosov for some $\theta \subset \Delta$ consisting of simple roots for G_1. While ρ can even be an indiscrete representation, Theorem 1.8 applies to $\Gamma\backslash \mathsf{G}/\mathsf{M}$ and implies the A-ergodicity when Γ is Zariski dense.

A typical example is when $\mathsf{G}_1 = \mathsf{G}_2 = \mathsf{PSL}(2,\mathbb{R})$, $\Gamma_1 < \mathsf{PSL}(2,\mathbb{R})$ discrete, and $\rho : \Gamma_1 \to \mathsf{PSL}(2,\mathbb{R})$ indiscrete. Such examples arise as Kobayashi geodesic curves in Hilbert modular varieties.

1.4. **Strict convexity of entropy.** Using the theory developed here, we establish new strict convexity results for variations of critical exponent.

Given a discrete subgroup $\Gamma < \mathsf{G}$, one can show that the subset

$$\mathcal{Q}_\theta(\Gamma) := \{\phi \in \mathfrak{a}_\theta^* : \delta^\phi(\Gamma) \leq 1\} \tag{2}$$

is convex. Further, when Γ is P_θ-Anosov the set $\mathcal{Q}_\theta(\Gamma)$ is strictly convex [Sam24, Corollary 5.9.1] (see [CZZ24, Corollary 13.2] for the P_θ-transverse case). In this paper, we will establish the following strict convexity result for $\mathcal{Q}_\Delta(\Gamma)$.

Theorem 1.11 (Strict convexity in non-Anosov directions I)**.** *If $\Gamma < \mathsf{G}$ is Zariski dense and P_θ-Anosov, then*
$$\{\phi \in \mathfrak{a}_\theta^* : \delta^\phi(\Gamma) = 1\}$$
are extreme points of $\mathcal{Q}_\Delta(\Gamma)$.

Theorem 1.11 is a consequence of the following estimate for critical exponent. We also establish a more general version for transverse groups (see Section 10).

Theorem 1.12 (Strict convexity in non-Anosov directions II)**.** *Suppose $\Gamma < \mathsf{G}$ is a Zariski dense P_θ-Anosov group, $\phi \in \mathfrak{a}_\theta^*$, and $\delta^\phi(\Gamma) < +\infty$. If $\phi_1, \phi_2 \in \mathfrak{a}^*$ are linearly independent with $\delta^{\phi_1}(\Gamma), \delta^{\phi_2}(\Gamma) < +\infty$ and $\phi = c_1\phi_1 + c_2\phi_2$ for some $c_1, c_2 > 0$, then*
$$\delta^\phi(\Gamma) < \frac{1}{\frac{c_1}{\delta^{\phi_1}(\Gamma)} + \frac{c_2}{\delta^{\phi_2}(\Gamma)}}.$$

The proof of strict convexity is based on the following proof strategy: if the critical exponent does not drop under convex combination, then the associated Patterson–Sullivan measures should be non-singular, which should imply that the associated lengths are equal. Implementing this strategy has several parts:

- Constructing Patterson–Sullivan measures. Prior to our work, it was known that there is a Patterson–Sullivan measure for ϕ on $\mathcal{F}_\theta$ and there was no guarantee that Patterson–Sullivan measures exist for ϕ_1, ϕ_2 (see Remark 1.5). In this paper we show that there is a Patterson–Sullivan measure for ϕ on $\mathcal{F}_\Delta$ (see Theorem 1.4). In a separate paper [KZ26], we show that there are Patterson–Sullivan measures for ϕ_1, ϕ_2 on a different boundary $\partial_\Delta X$ which contains $\mathcal{F}_\Delta$. The boundary $\partial_\Delta X$ is constructed as a vector-valued horofunction boundary of the symmetric space using Cartan projections, see Section 4 for details.
- Showing non-singularity of Patterson–Sullivan measures when the critical exponent does not drop. This will be a consequence of Proposition 8.3.
- Showing non-singularity of Patterson–Sullivan measures implies equality of lengths. This was the main result of our earlier work [KZ25] in the context of Patterson–Sullivan systems.

1.5. **Entropy rigidity.** As an application of our strict convexity results, we establish an entropy rigidity result for Anosov groups with Lipschitz limit sets.

In the following discussion, let $\mathfrak{a} = \{\mathrm{diag}(t_1, \dots, t_d) : t_1 + \cdots + t_d = 0\}$ denote the standard Cartan subspace for $\mathsf{SL}(d, \mathbb{R})$ and let $\Delta = \{\alpha_1, \dots, \alpha_{d-1}\}$ denote the standard simple roots defined by
$$\alpha_j(\mathrm{diag}(t_1, \dots, t_d)) = t_j - t_{j+1}.$$
For notation simplicity let $\mathsf{P}_k := \mathsf{P}_{\{\alpha_k\}}$, $\mathcal{F}_k := \mathcal{F}_{\{\alpha_k\}}$, and $\Lambda_k(\Gamma) := \Lambda_{\alpha_k}(\Gamma)$. Notice that we have a natural identification $\mathcal{F}_1 = \mathbb{P}(\mathbb{R}^d)$.

The *Hilbert functional* $\phi_{\mathrm{H}} \in \mathfrak{a}^*$ is
$$\phi_{\mathrm{H}}(\mathrm{diag}(t_1, \dots, t_d)) = \frac{1}{2}(t_1 - t_d)$$
and the *Hilbert critical exponent* of a discrete subgroup $\Gamma < \mathsf{SL}(d, \mathbb{R})$ is the critical exponent $\delta^{\phi_{\mathrm{H}}}(\Gamma)$ associated to ϕ_{H}. The motivation for this terminology comes from the fact that if Γ preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, then $\delta^{\phi_{\mathrm{H}}}(\Gamma)$

coincides with the critical exponent of Γ with respect to the Hilbert metric on Ω (this follows from [DGK24, Proposition 10.1]).

For a P_1-Anosov group $\Gamma < \mathsf{SL}(d,\mathbb{R})$ with Lipschitz limit set $\Lambda_1(\Gamma) \subset \mathbb{P}(\mathbb{R}^d)$, Pozzetti–Sambarino–Wienhard established the following upper bound for the Hilbert entropy.

Theorem 1.13 ([PSW23, consequence of Theorem A]). *Suppose $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is a P_1-Anosov group acting strongly irreducibly on $\mathbb{R}^d$ and on $\wedge^{p+1}\mathbb{R}^d$ for some $p \leq d-2$. If $\Lambda_1(\Gamma)$ is a Lipschitz p-manifold, then*

$$\delta^{\phi_{\mathrm{H}}}(\Gamma) \leq p.$$

Using Theorem 1.12 we prove rigidity in the equality case.

Theorem 1.14 (Entropy rigidity). *Suppose $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is a P_1-Anosov group acting strongly irreducibly on $\mathbb{R}^d$ and on $\wedge^{p+1}\mathbb{R}^d$ whose limit set $\Lambda_1(\Gamma)$ is a Lipschitz p-manifold for some $p \leq d-2$. Then*

$$\delta^{\phi_{\mathrm{H}}}(\Gamma) = p$$

if and only if Γ is conjugate to a uniform lattice in $\mathsf{SO}(d-1,1)$ and $p = d-2$.

Previously Pozzetti–Sambarino–Wienhard proved a variant of this rigidity result when Γ is also P_{p+1}-Anosov and $\Lambda_1(\Gamma)$ is a $\mathcal{C}^1$-smooth p-manifold [PSW23, Proposition 7.7]. In this case, the P_{p+1}-Anosov assumption implies that the set $\mathcal{Q}_{\{\alpha_1,\alpha_{p+1}\}}(\Gamma)$ in Equation (2) is strictly convex and this strict convexity is essential in their proof. Using our "strict convexity in non-Anosov directions" theorem, we are able to extend their argument to the non-P_{p+1}-Anosov case.

We also note that Theorem 1.14 generalizes a result of Crampon [Cra09] who proved that if $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is a discrete group which acts cocompactly on a strictly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, then

$$\delta_{\mathrm{Hil}}(\Gamma) \leq d-2$$

with equality if and only if Ω is an ellipsoid (and hence Γ is conjugate to a uniform lattice in $\mathsf{SO}(d-1,1)$). Under Crampon's hypothesis, Γ is P_1-Anosov and acts irreducibly on $\mathbb{R}^d$ and hence also $\wedge^{d-1}\mathbb{R}^d$, see [GW12, Section 6.2]. Further, $\Lambda_1(\Gamma)$ coincides with $\partial\Omega$ and is hence a Lipschitz $(d-2)$-manifold. So Theorem 1.14 does indeed generalize Crampon's result.

1.6. **Outline of Paper.** The first part of the paper is expository. In Section 2, we recall the definition and some properties of abstract Patterson–Sullivan systems, which were introduced in our earlier work [KZ25]. In Section 3, we fix the notation involving semisimple Lie groups that we will use throughout the paper. In Section 4, we recall the definition and some properties of vector-valued horofunction boundaries of symmetric spaces.

In the second part of the paper, we prove the existence of boundary maps for Zariski dense representations of groups in a Patterson–Sullivan system. In Section 5, we prove a technical continuity result for measurable maps into separable metric spaces whose domain is well-behaved Patterson–Sullivan systems. In Section 6, we establish the existence of a boundary map associated to a Zariski dense representation of a group which is part of a well-behaved Patterson–Sullivan system. Our boundary map existence result requires that the group action in the

Patterson–Sullivan system is amenable and in Section 7 we verify this for transverse groups.

In the final part of the paper, we apply the previous two parts to study Patterson–Sullivan measures for transverse groups, mapping class groups, and discrete subgroups of isometry groups of Gromov hyperbolic spaces. In Section 8, we establish existence and uniqueness results for Patterson–Sullivan measures on $\mathcal{F}_\Theta$ associated to P_θ-transverse groups when $\Theta \supset \theta$. In Section 9, we construct Bowen–Margulis–Sullivans measures on the homogeneous space $\Gamma \backslash \mathsf{G} / \mathsf{M}$ when Γ is a Zariski dense P_θ-transverse group and prove a version of the Hopf–Tsuji–Sullivan dichotomy for such measures. In Section 10, we prove the strict convexity of critical exponents in non-Anosov directions. In Section 11, we prove the entropy rigidity for Anosov subgroups with Lipschitz limit sets. In Section 12, we consider representations of mapping class groups and obtain measurable boundary maps from $\mathcal{PML}$ equipped with the Lebesgue measure class. In Section 13, we consider representations of discrete isometry groups of Gromov hyperbolic spaces with exponentially bounded geometry and also obtain measurable boundary maps from their Gromov boundaries. Finally, in Section 14, we explain how to establish the statements in the introduction from the results in the body of the paper.

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Part 1. **Background**

2. Patterson–Sullivan systems

In this section, we recall the definition and some properties of abstract Patterson–Sullivan systems, which were introduced in our earlier work [KZ25]. The main idea in this previous work was to identify the key features of a group action on a probability space that allows one to extend the theory of Patterson–Sullivan measures. We note that a different framework for abstract Patterson–Sullivan-like measures was given in [BCZZ24].

Given a compact metric space M, a subgroup $\Gamma < \mathsf{Homeo}(M)$, and $\kappa \geq 0$, a function $\sigma : \Gamma \times M \to \mathbb{R}$ is called a *κ-coarse-cocycle* if

$$|\sigma(\gamma_1\gamma_2, x) - (\sigma(\gamma_1, \gamma_2 x) + \sigma(\gamma_2, x))| \leq \kappa$$

for any $\gamma_1, \gamma_2 \in \Gamma$ and $x \in M$. Given such a coarse-cocycle and $\delta \geq 0$, a Borel probability measure μ on M is called *coarse* (Γ, σ, δ)*-Patterson–Sullivan measure* if there exists $C \geq 1$ such that for any $\gamma \in \Gamma$ the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$C^{-1} e^{-\delta\sigma(\gamma^{-1}, x)} \leq \frac{d\gamma_*\mu}{d\mu}(x) \leq C e^{-\delta\sigma(\gamma^{-1}, x)} \quad \text{for } \mu\text{-a.e. } x \in M. \tag{3}$$

When $C = 1$ and hence equality holds in Equation (3), we call μ a (σ, δ)*-Patterson–Sullivan measure.*

Now we recall the definition of Patterson–Sullivan systems.

Definition 2.1. A *Patterson–Sullivan-system (PS-system) of dimension* $\delta \geq 0$ consists of

- a coarse-cocycle $\sigma : \Gamma \times M \to \mathbb{R}$,
- a coarse (σ, δ)-Patterson–Sullivan measure (PS-measure) μ,
- for each $\gamma \in \Gamma$, a number $\|\gamma\|_\sigma \in \mathbb{R}$ called the *σ-magnitude of* γ, and
- for each $\gamma \in \Gamma$ and $R > 0$, a non-empty open set $\mathcal{O}_R(\gamma) \subset M$ called the *R-shadow of* γ

such that:

(PS1) For any $\gamma \in \Gamma$, there exists $c = c(\gamma) > 0$ such that $|\sigma(\gamma, x)| \leq c(\gamma)$ for μ-a.e. $x \in M$.

(PS2) For every $R > 0$ there is a constant $C = C(R) > 0$ such that
$$\|\gamma\|_\sigma - C \leq \sigma(\gamma, x) \leq \|\gamma\|_\sigma + C$$
for all $\gamma \in \Gamma$ and μ-a.e. $x \in \gamma^{-1}\mathcal{O}_R(\gamma)$.

(PS3) If $\{\gamma_n\} \subset \Gamma$, $R_n \to +\infty$, $Z \subset M$ is compact, and $[M \smallsetminus \gamma_n^{-1}\mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance, then for any $x \in Z$, there exists $g \in \Gamma$ such that
$$gx \notin Z.$$

We call the PS-system *well-behaved* with respect to a collection
$$\mathcal{H} := \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$$
of non-increasing subsets of Γ if the following additional properties hold:

(PS4) Γ is countable and for any $T > 0$, the set $\{\gamma \in \mathcal{H}(0) : \|\gamma\|_\sigma \leq T\}$ is finite.

(PS5) If $\{\gamma_n\} \subset \Gamma$, $R_n \to +\infty$, $Z \subset M$ is compact, and $[M \smallsetminus \gamma_n^{-1}\mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance, then for any $h_1, \dots, h_m \in \Gamma$ and $x \in Z$, there exists $g \in \Gamma$ such that
$$gx \notin \bigcup_{i=1}^m h_i Z.$$

(PS6) If $R_1 \leq R_2$ and $\gamma \in \mathcal{H}(0)$, then $\mathcal{O}_{R_1}(\gamma) \subset \mathcal{O}_{R_2}(\gamma)$.

(PS7) For any $R > 0$ there exist $C > 0$ and $R' > 0$ such that: if $\alpha, \beta \in \mathcal{H}(R)$, $\|\alpha\|_\sigma \leq \|\beta\|_\sigma$, and $\mathcal{O}_R(\alpha) \cap \mathcal{O}_R(\beta) \neq \emptyset$, then
$$\mathcal{O}_R(\beta) \subset \mathcal{O}_{R'}(\alpha)$$
and
$$\left| \|\beta\|_\sigma - (\|\alpha\|_\sigma + \left\|\alpha^{-1}\beta\right\|_\sigma) \right| \leq C.$$

(PS8) For every $R > 0$, there exists a set $M' \subset M$ of full μ-measure such that
$$\lim_{n \to +\infty} \operatorname{diam} \mathcal{O}_R(\gamma_n) = 0$$
whenever $\{\gamma_n\} \subset \mathcal{H}(R)$ is an escaping sequence and
$$x \in M' \cap \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n).$$

We call the collection $\mathcal{H}$ the *hierarchy* of the Patterson–Sullivan system.

For well-behaved PS-systems, there is a natural notion of conical limit set.

Definition 2.2. Let (M, Γ, σ, μ) be a PS-system.

- Given a subset $H \subset \Gamma$ and $R > 0$, let $\Lambda_R(H) \subset M$ be the set of points $x \in M$ where there exists an escaping sequence $\{\gamma_n\} \subset H$ such that
$$x \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n).$$
- If (M, Γ, σ, μ) is well-behaved with respect to a hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$, then the $\mathcal{H}$-*conical limit set* is

$$\Lambda^{\mathrm{con}}(\mathcal{H}) := \Gamma \cdot \bigcup_{R>0} \bigcap_{n \geq 1} \Lambda_R(\mathcal{H}(n)). \tag{4}$$

2.1. **Boundary rigidity.** One of the primary aims for developing the theory of PS-systems in [KZ25] was to establish a general setting where Tukia's measurable boundary rigidity theorem [Tuk89] holds.

Theorem 2.3 (Boundary rigidity, [KZ25, Theorem 1.28])**.** *Suppose*

- $(M_1, \Gamma_1, \sigma_1, \mu_1)$ *is a well-behaved PS-system of dimension* δ_1 *with respect to a hierarchy* $\mathcal{H}_1 = \{\mathcal{H}_1(R) \subset \Gamma_1 : R \geq 0\}$ *and*
$$\mu_1(\Lambda^{\mathrm{con}}(\mathcal{H}_1)) = 1.$$
- $(M_2, \Gamma_2, \sigma_2, \mu_2)$ *is a PS-system of dimension* δ_2.
- *There exists an onto homomorphism* $\rho : \Gamma_1 \to \Gamma_2$ *and a* μ_1*-a.e. defined measurable* ρ*-equivariant injective map* $f : M_1 \to M_2$.

If the measures $f_*\mu_1$ *and* μ_2 *are not singular, then*
$$\sup_{\gamma \in \Gamma_1} \left| \delta_1 \left\|\gamma\right\|_{\sigma_1} - \delta_2 \left\|\rho(\gamma)\right\|_{\sigma_2} \right| < +\infty.$$

2.2. **Properties of Patterson–Sullivan systems.** We now recall some useful properties of Patterson–Sullivan systems proved in [KZ25], which were key ingredients in the proof of Theorem 2.3. They will also be used in this paper.

We begin with the ergodic property of a well-behaved Patterson–Sullivan system.

Theorem 2.4 ([KZ25, Corollary 5.2])**.** *If* (M, Γ, σ, μ) *is a well-behaved PS-system with respect to a hierarchy* $\mathcal{H}$ *and* $\mu(\Lambda^{\mathrm{con}}(\mathcal{H})) = 1$, *then the* Γ*-action on* (M, μ) *is ergodic.*

The ergodicity as in Theorem 9.1 is based on the study of shadows and in particular a version of the Shadow Lemma.

Proposition 2.5 (Shadow Lemma, [KZ25, Proposition 3.1])**.** *Let* (M, Γ, σ, μ) *be a PS-system of dimension* $\delta \geq 0$. *For any* $R > 0$ *sufficiently large there exists* $C = C(R) > 1$ *such that*
$$\frac{1}{C} e^{-\delta \|\gamma\|_\sigma} \leq \mu(\mathcal{O}_R(\gamma)) \leq C e^{-\delta \|\gamma\|_\sigma}$$
for all $\gamma \in \Gamma$.

When the underlying PS-system is well-behaved, shadows also satisfy the following version of the Vitali covering lemma.

Lemma 2.6 (Vitali Covering, [KZ25, Lemma 3.2])**.** *Let* (M, Γ, σ, μ) *be a well-behaved PS-system with respect to a hierarchy* $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$. *Fix*

$R>0$ and let $R'>0$ be the constant satisfying Property (PS7) for R. Then for any $I\subset\mathcal{H}(R)$, there exists $J\subset I$ such that

$$\bigcup_{\gamma\in I}\mathcal{O}_R(\gamma)\subset\bigcup_{\gamma\in J}\mathcal{O}_{R'}(\gamma)$$

and the shadows $\{\mathcal{O}_R(\gamma):\gamma\in J\}$ are pairwise disjoint.

3. Notations for semisimple Lie groups

In this section we fix the notation involving semisimple Lie groups that we will use throughout the paper. Of particular importance for our arguments are the linear representations fixed in Section 3.6.

Recall from the introduction that G is a semisimple Lie group with finite center and no compact factors, $\mathfrak{g}=\mathfrak{p}+\mathfrak{k}$ is a fixed Cartan decomposition of the Lie algebra, $\mathfrak{a}\subset\mathfrak{p}$ is a fixed Cartan subspace, and $\mathfrak{a}^+\subset\mathfrak{a}$ is a fixed positive Weyl chamber. We use $\Sigma\subset\mathfrak{a}^*$ to denote the set of restricted roots and use $\Delta\subset\mathfrak{a}^*$ to denote the system of simple restricted roots corresponding to the choice of $\mathfrak{a}^+$. Then

$$\mathfrak{g}=\mathfrak{g}_0\oplus\bigoplus_{\alpha\in\Sigma}\mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha=\{X\in\mathfrak{g}:[H,X]=\alpha(H)X \text{ for all } H\in\mathfrak{a}\}.$$

Let Σ^+ (resp. Σ^-) denote the restricted roots which are non-negative (respectively non-positive) linear combinations of elements of Δ.

3.1. **Cartan and Jordan projections.** Let $\mathsf{K}<\mathsf{G}$ denote the maximal compact subgroup with Lie algebra $\mathfrak{k}$. Recall that every $g\in\mathsf{G}$ has a *Cartan decomposition*, i.e. $g=ke^{\kappa(g)}\ell$ for some $k,\ell\in\mathsf{K}$ and a unique $\kappa(g)\in\mathfrak{a}^+$ (the elements k,ℓ are not unique). The map $\kappa:\mathsf{G}\to\mathfrak{a}^+$ is called the *Cartan projection*. The *Jordan projection* $\lambda:\mathsf{G}\to\mathfrak{a}^+$ is defined as

$$\lambda(g):=\lim_{n\to+\infty}\frac{\kappa(g^n)}{n}.$$

We fix a representative $w_0\in\mathsf{K}$ of the longest Weyl element which is of order 2. Let $\mathrm{i}:=-\mathrm{Ad}_{w_0}:\mathfrak{a}\to\mathfrak{a}$ denote the *opposition involution*. This map has the property that

$$\kappa(g^{-1})=\mathrm{i}\kappa(g)\quad\text{for all}\quad g\in\mathsf{G}\,. \tag{5}$$

The adjoint $\mathrm{i}^*:\mathfrak{a}^*\to\mathfrak{a}^*$ of the opposite involution preserves the set of simple roots and for a subset $\theta\subset\Delta$, we define

$$\mathrm{i}^*\theta:=\{\mathrm{i}^*\alpha:\alpha\in\theta\}.$$

3.2. **Parabolic subgroups and flag manifolds.** Given a non-empty $\theta\subset\Delta$, the associated parabolic subgroup P_θ is the stabilizer under the adjoint action of the Lie algebra

$$\mathfrak{u}_\theta^+:=\bigoplus_{\alpha\in\Sigma_\theta^+}\mathfrak{g}_\alpha$$

where $\Sigma_\theta^+:=\Sigma^+\smallsetminus\mathrm{span}(\Delta\smallsetminus\theta)$. We also set $\mathsf{A}:=\exp\mathfrak{a}$ and $\mathsf{A}^+:=\exp\mathfrak{a}^+$, and denote by $\mathsf{N}<\mathsf{P}_\Delta$ the unipotent radical of P_Δ.

The *Furstenberg boundary* and general *θ-boundary* are the quotient spaces

$$\mathcal{F}_\Delta := \mathsf{G} / \mathsf{P}_\Delta \quad \text{and} \quad \mathcal{F}_\theta := \mathsf{G} / \mathsf{P}_\theta .$$

We also call $\mathcal{F}_\theta$ a *partial flag manifold.* Two elements $x \in \mathcal{F}_\theta$ and $y \in \mathcal{F}_{\mathrm{i}^*\theta}$ are *transverse* if there exists $g \in \mathsf{G}$ such that

$$x = g\,\mathsf{P}_\theta \quad \text{and} \quad y = gw_0\,\mathsf{P}_{\mathrm{i}^*\theta},$$

equivalently (x, y) is contained in the unique open G-orbit in $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}^*\theta}$.

3.3. **The special linear group.** In this section, we fix some notation when $\mathsf{G} = \mathsf{SL}(V)$ and V is a finite dimensional real vector space endowed with an inner product.

For a linear tranformation $T : V \to V$, we let

$$\sigma_1(T) \geq \cdots \geq \sigma_d(T)$$

denote the singular values of T with respect to the inner product and let $\|T\| := \sigma_1(T)$ denote the operator norm. We assume that the fixed maximal compact K of $\mathsf{SL}(V)$ coincides with the orthogonal group of the fixed inner product. Then we can fix a Cartan subspace $\mathfrak{a}$, a positive Weyl chamber $\mathfrak{a}^+$, and a labelling $\Delta = \{\alpha_1, \ldots, \alpha_{\dim V - 1}\}$ of the simple roots such that

$$\alpha_j(\kappa(g)) = \log \frac{\sigma_j(g)}{\sigma_{j+1}(g)}$$

for each $1 \leq j \leq \dim V - 1$. Then the associated fundamental weights satisfy

$$\omega_{\alpha_j}(\kappa(g)) = \log\big(\sigma_1(g) \cdots \sigma_j(g)\big).$$

For notation simplicity we let $\mathsf{P}_k := \mathsf{P}_{\{\alpha_k\}}$, $\mathcal{F}_k := \mathcal{F}_{\{\alpha_k\}}$, and $\Lambda_k(\Gamma) := \Lambda_{\alpha_k}(\Gamma)$.

When $V = \mathbb{R}^d$, we always use the standard inner product, the standard Cartan subspace

$$\mathfrak{a} = \{\mathrm{diag}(t_1, \ldots, t_d) : t_1 + \cdots + t_d = 0\},$$

and the standard positive Weyl chamber

$$\mathfrak{a}^+ = \{\mathrm{diag}(t_1, \ldots, t_d) \in \mathfrak{a} : t_1 \geq \cdots \geq t_d\}.$$

Then

$$\alpha_j(\mathrm{diag}(t_1, \ldots, t_d)) = t_j - t_{j+1} \quad \text{and} \quad \omega_{\alpha_j}(\mathrm{diag}(t_1, \ldots, t_d)) = t_1 + \cdots + t_j.$$

3.4. **Projection to the flag manifold.** For $g \in \mathsf{G}$ with $\min_{\alpha\in\theta} \alpha(\kappa(g)) > 0$, we define

$$U_\theta(g) := k\,\mathsf{P}_\theta \in \mathcal{F}_\theta$$

where g has Cartan decomposition $g = ka\ell \in \mathsf{K}\mathsf{A}^+\mathsf{K}$ (the condition on the roots implies that $U_\theta(g)$ is well-defined). By Equation (5),

$$\min_{\alpha\in\theta} \alpha(\kappa(g)) = \min_{\alpha\in\mathrm{i}^*\theta} \alpha(\kappa(g^{-1}))$$

and so $U_{\mathrm{i}^*\theta}(g^{-1})$ is also well-defined when $\min_{\alpha\in\theta} \alpha(\kappa(g)) > 0$.

These projection maps have the following dynamical behavior (for a proof see for instance [KLP17, Section 4] or [CZZ24, Proposition 2.3]).

Proposition 3.1. *If $\{g_n\} \subset \mathsf{G}$, $x^+ \in \mathcal{F}_\theta$, and $x^- \in \mathcal{F}_{\mathrm{i}^*\theta}$, then the following are equivalent:*

(1) $g_n x \to x^+$ for all $x \in \mathcal{F}_\theta$ transverse to x^- and the convergence is uniform on compact subsets.

(2) $\min_{\alpha\in\theta} \alpha(\kappa(g_n)) \to +\infty$, $U_\theta(g_n) \to x^+$, *and* $U_{\mathrm{i}^*\theta}(g_n^{-1}) \to x^-$.

3.5. **The partial Iwasawa cocycle.** The *Iwasawa cocycle* $B_\Delta^{IW} : \mathsf{G} \times \mathcal{F}_\Delta \to \mathfrak{a}$ is defined as follows: for $g \in \mathsf{G}$ and $x \in \mathcal{F}_\Delta$, $B_\Delta^{IW}(g,x) \in \mathfrak{a}$ is the unique element such that

$$gk \in \mathsf{K}(\exp B_\Delta^{IW}(g,x))\mathsf{N}$$

for $k \in \mathsf{K}$ such that $k\,\mathsf{P}_\Delta = x$ in $\mathcal{F}_\Delta$.

For general $\theta \subset \Delta$, let

$$\mathfrak{a}_\theta := \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \notin \theta\}.$$

For $\alpha \in \Delta$, let ω_α denote the (restricted) fundamental weight associated to α. Then $\{\omega_\alpha|_{\mathfrak{a}_\theta}\}_{\alpha\in\theta}$ is a basis for $\mathfrak{a}_\theta^*$ and so there exists a unique projection

$$\pi_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$$

satisfying

$$\omega_\alpha \pi_\theta(H) = \omega_\alpha(H) \tag{6}$$

for all $H \in \mathfrak{a}$ and $\alpha \in \theta$.

The *partial Iwasawa cocycle* $B_\theta^{IW} : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ is defined as

$$B_\theta^{IW}(g,x) := \pi_\theta B_\Delta^{IW}(g,\tilde{x}) \tag{7}$$

for any $\tilde{x} \in \mathcal{F}_\Delta$ that projects to $x \in \mathcal{F}_\theta$ under the canonical projection $\mathcal{F}_\Delta \to \mathcal{F}_\theta$. The above definition is independent of the choice of $\tilde{x}$ and defines a cocycle [Qui02, Lemma 6.1].

Recall from the introduction that the partial Iwasawa cocycle can be used to define a notion of Patterson–Sullivan measures on the partial flag manifold $\mathcal{F}_\theta$ (see Definition 1.1).

3.6. **Linear Representations.** Throughout the paper, for each $\alpha \in \Delta$ we fix an irreducible representation $\Phi_\alpha : \mathsf{G} \to \mathsf{SL}(V_\alpha)$ and a $\Phi_\alpha(\mathsf{K})$-invariant inner product on V_α with the following properties (using the notation in Section 3.3):

(R1) There exists $N_\alpha \in \mathbb{N}$ such that if $g \in \mathsf{G}$, then

$$\log\|\Phi_\alpha(g)\| = N_\alpha \omega_\alpha(\kappa(g)) \quad \text{and} \quad \log\frac{\sigma_1(\Phi_\alpha(g))}{\sigma_2(\Phi_\alpha(g))} = \alpha(\kappa(g)).$$

(R2) There exists a $\Phi_\alpha(\mathsf{A})$-invariant orthogonal splitting $V_\alpha = V_\alpha^+ \oplus V_\alpha^-$ such that $\dim V_\alpha^+ = 1$. Moreover, if $H \in \mathfrak{a}$ and $v \in V_\alpha^+$, then

$$\Phi_\alpha(e^H)v = e^{N_\alpha\omega_\alpha(H)}v.$$

(R3) There exist Φ_α-equivariant boundary maps $\zeta_\alpha : \mathcal{F}_\alpha \to \mathbb{P}(V_\alpha)$ and $\zeta_\alpha^* : \mathcal{F}_{\mathrm{i}^*\alpha} \to \mathrm{Gr}_{\dim V_\alpha - 1}(V_\alpha)$ such that:
   (a) $\zeta_\alpha(\mathsf{P}_\alpha) = V_\alpha^+$ and $\zeta_\alpha^*(w_0\,\mathsf{P}_{\mathrm{i}^*\alpha}) = V_\alpha^-$.
   (b) $x \in \mathcal{F}_\alpha$ and $y \in \mathcal{F}_{\mathrm{i}^*\alpha}$ are transverse if and only if $\zeta_\alpha(x)$ and $\zeta_\alpha^*(y)$ are transverse.

Remark 3.2. Such representations exist due to Tits [Tit71, Theorem 7.2]. Indeed, Tits proved the first claim in Property (R1). For a proof of the second assertion in Property (R1) and Property (R2), see for instance [Smi18, Lemma 2.13] and [BQ16, Sections 6.8, 6.9]. For a proof of Property (R3), see for instance [GGKW17, Section 3].

Remark 3.3. We abuse notation and when $\alpha \in \theta$, also often use ζ_α to also denote the map $\mathcal{F}_\theta \to \mathbb{P}(V_\alpha)$ obtained by precomposing $\zeta_\alpha : \mathcal{F}_\alpha \to \mathbb{P}(V_\alpha)$ with the natural projection $\mathcal{F}_\theta \to \mathcal{F}_\alpha$. Likewise, we also use $\zeta^*_{\mathrm{i}^*\alpha}$ to denote the analogous map defined on $\mathcal{F}_{\mathrm{i}^*\theta}$.

The following lemma relates the projections to the flag manifolds introduced in Section 3.4 to these representations.

Lemma 3.4 ([KZ26, Lemma 3.5]). *Fix $\alpha \in \Delta$ and assume $\{g_n\} \subset \mathsf{G}$ is such that $\alpha(\kappa(g_n)) \to +\infty$, $U_\alpha(g_n) \to x$, and $U_{\mathrm{i}^*\alpha}(g_n^{-1}) \to y$. Then any limit point of*

$$\frac{1}{\|\Phi_\alpha(g_n)\|}\Phi_\alpha(g_n) \quad \text{in } \mathrm{End}(V_\alpha)$$

has image $\zeta_\alpha(x)$ and kernel $\zeta^*_\alpha(y)$.

3.7. **Irreducible actions.** A subgroup $\mathsf{H} < \mathsf{SL}(V)$ is called *irreducible* if there is no non-trivial and proper subspace of V invariant under H, and is called *strongly irreducible* if any finite index subgroup of H is irreducible. We transfer these notions to G using the Φ_α's.

Definition 3.5. A subgroup $\Gamma < \mathsf{G}$ is $(\Phi_\alpha)_{\alpha\in\theta}$*-irreducible* if $\Phi_\alpha(\Gamma) < \mathsf{SL}(V_\alpha)$ is irreducible for all $\alpha \in \theta$, and *strongly* $(\Phi_\alpha)_{\alpha\in\theta}$*-irreducible* if any finite index subgroup of Γ is $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible.

Remark 3.6. Notice that a Zariski dense subgroup is strongly $(\Phi_\alpha)_{\alpha\in\Delta}$-irreducible.

We will use the following observation several times.

Lemma 3.7 ([KZ26, Lemma 3.8]). *Suppose $\Gamma < \mathsf{G}$ is strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible. If*

- *$\alpha_1, \dots, \alpha_m$ are (possibly non-distinct) elements of θ,*
- *$v_i \in V_{\alpha_i} \smallsetminus \{0\}$ for $i = 1, \dots, m$, and*
- *$W_i \subset V_{\alpha_i}$ is a proper linear subspace for $i = 1, \dots, m$,*

then there exists $\gamma \in \Gamma$ with

$$\Phi_{\alpha_i}(\gamma)v_i \notin W_i$$

for all $i = 1, \dots, m$.

3.8. **(Relatively) Anosov and transverse groups.** In the rest of this section we recall the definitions of (relatively) Anosov and transverse groups, and some results about Patterson–Sullivan measures for such groups.

Given a discrete subgroup $\Gamma < \mathsf{G}$, a point $x \in \mathcal{F}_\theta$ is a *limit point* of Γ if there exists an escaping sequence $\{\gamma_n\} \subset \Gamma$ with $\min_{\alpha\in\theta} \alpha(\kappa(\gamma_n)) \to +\infty$ and $U_\theta(\gamma_n) \to x$. The *limit set of* Γ, denoted by

$$\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta,$$

is the set of all limit points of Γ. Proposition 3.1 implies that the points in $\Lambda_\theta(\Gamma)$ are exactly the points $x^+ \in \mathcal{F}_\theta$ where there exists a sequence $\{\gamma_n\} \subset \Gamma$ and a non-empty open set $\mathcal{U} \subset \mathcal{F}_\theta$ such that $\gamma_n x \to x^+$ for all $x \in \mathcal{U}$, uniformly on compact subsets.

A discrete subgroup $\Gamma < \mathsf{G}$ is P_θ*-transverse* if $\min_{\alpha\in\theta} \alpha(\kappa(\gamma_n)) \to +\infty$ for any sequence $\{\gamma_n\} \subset \Gamma$ of distinct elements and any two distinct points in $\Lambda_{\theta\cup\mathrm{i}^*\theta}(\Gamma)$ are transverse.

Sometimes the definition of transverse group includes the assumption that θ is symmetric (i.e., $\theta = \mathrm{i}^*\theta$). However, as the next observation demonstrates, this results in no loss of generality.

Observation 3.8. $\Gamma < \mathsf{G}$ is P_θ-transverse if and only if Γ is $\mathsf{P}_{\theta\cup\mathrm{i}^*\theta}$-transverse. Moreover, in this case the the projection $\mathcal{F}_{\theta\cup\mathrm{i}^*\theta} \to \mathcal{F}_\theta$ induces a homeomorphism $\Lambda_{\theta\cup\mathrm{i}^*\theta}(\Gamma) \to \Lambda_\theta(\Gamma)$.

A P_θ-transverse group is called *non-elementary* if $\#\Lambda_\theta(\Gamma) \geq 3$, in which case the natural Γ-action on $\Lambda_\theta(\Gamma)$ is a minimal convergence action and $\#\Lambda_\theta(\Gamma) = +\infty$, see [KLP17, Theorem 4.16] or [CZZ26, Proposition 3.3].

A P_θ-transverse group is P_θ*-Anosov* if the action of Γ on $\Lambda_\theta(\Gamma)$ is a uniform convergence action, equivalently Γ is word hyperbolic as an abstract group and there exists an equivariant homeomorphism from the Gromov boundary to the limit set $\Lambda_\theta(\Gamma)$ [Bow98]. Likewise, a P_θ-transverse group is P_θ*-relatively Anosov* if the action of Γ on $\Lambda_\theta(\Gamma)$ is geometrically finite, equivalently if Γ has the structure of a relatively hyperbolic group and there exists an equivariant homeomorphism from the associated Bowditch boundary to the limit set $\Lambda_\theta(\Gamma)$ [Yam04].

Canary, Zhang, and the second author established the following existence and uniqueness results for Patterson–Sullivan measure supported on the limit set.

Theorem 3.9. *Suppose $\Gamma < \mathsf{G}$ is a non-elementary P_θ-transverse group, $\phi \in \mathfrak{a}_\theta^*$, and $\delta^\phi(\Gamma) < +\infty$.*

(1) [CZZ24, Proposition 4.2] *There exists a $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure supported on $\Lambda_\theta(\Gamma)$.*
(2) [CZZ24, Proposition 8.1] *If μ is a (Γ, ϕ, β)-Patterson–Sullivan measure supported on $\Lambda_\theta(\Gamma)$, then $\beta \geq \delta^\phi(\Gamma)$.*
(3) [CZZ24, Corollaries 12.1 and 12.2, and Proposition 9.1] *If*
$$\sum_{\gamma\in\Gamma} e^{-\delta^\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty,$$
then there exists a unique $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure μ supported on $\Lambda_\theta(\Gamma)$. Moreover, Γ acts ergodically on $(\Lambda_\theta(\Gamma), \mu)$ and μ is supported on the conical limit set in the sense of the convergence action of Γ on $\Lambda_\theta(\Gamma)$.

Remark 3.10. When $\Gamma < \mathsf{G}$ is Zariski dense, the uniqueness holds for measures supported on $\mathcal{F}_\theta$, as shown by the first author, Oh, and Wang [KOW25b].

4. Vector-valued horofunction compactifications and PS-measures

In this section, we recall the vector-valued horofunction compactifications of the symmetric space $X = \mathsf{G}/\mathsf{K}$ associated to G introduced in our other work [KZ26]. It turns out that they contain θ-boundaries, and we also consider Patterson–Sullivan measures there. Similar compactifications, but using Finsler metrics, appear in [KL18, HSWW17, LP23, Lem23].

Fix the basepoint $o := \mathsf{K} \in X$. The symmetric space distance is given by
$$\mathrm{d}_X(go, ho) = \left\|\kappa(g^{-1}h)\right\|$$
where $\|\cdot\|$ is some norm on $\mathfrak{a}$. For $x = go$, define the vector-valued horofunction $b_x : X \to \mathfrak{a}$ by
$$b_x(ho) = \kappa(h^{-1}g) - \kappa(g).$$

Then the maps $\{b_x : x \in X\}$ are uniformly Lipschitz [KZ26, Lemma 4.1].

For non-empty $\theta \subset \Delta$, let $\pi_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ be the projection satisfying Equation (6) and then let $\partial_\theta X$ be the set of functions $\xi : X \to \mathfrak{a}_\theta$ where there exists an escaping sequence $\{x_n\} \subset X$ with $\pi_\theta b_{x_n} \to \xi$ in the compact-open topology. The uniform Lipschitzness above, together with the separability of X, implies that $\partial_\theta X$ is compact in the compact-open topology. Further, G acts on $\partial_\theta X$ by

$$g \cdot \xi = \xi \circ g^{-1} - \xi(g^{-1}o).$$

The space $\partial_\theta X$ can be used to compactify X.

Proposition 4.1 ([KZ26, Proposition 4.2])**.** *The space $\overline{X}^\theta := X \sqcup \partial_\theta X$ has a topology which makes it a compactification of X, that is $\overline{X}^\theta$ is a compact metrizable space and the inclusion $X \hookrightarrow \overline{X}^\theta$ is a topological embedding with open dense image. Moreover with respect to this topology:*

1. *$\{x_n\} \subset X$ converges to $\xi \in \partial_\theta X$ if and only if $\mathrm{d}_X(o, x_n) \to +\infty$ and $\pi_\theta b_{x_n} \to \xi$ in the compact-open topology.*
2. *The G-action on $\overline{X}^\theta$ is continuous.*

Patterson–Sullivan measures on $\partial_\theta X$ can naturally be defined as follows.

Definition 4.2. Given a subgroup $\Gamma < \mathsf{G}$, $\theta \subset \Delta$, $\phi \in \mathfrak{a}_\theta^*$, and $\delta \geq 0$, a Borel probability measure μ on $\partial_\theta X$ is a *coarse* (Γ, ϕ, δ)*-Patterson–Sullivan measure* if there exists $C \geq 1$ such that for every $\gamma \in \Gamma$ the measures μ, $\gamma_*\mu$ are absolutely continuous and

$$C^{-1} e^{-\delta \phi \xi(\gamma o)} \leq \frac{d\gamma_* \mu}{d\mu}(\xi) \leq C e^{-\delta \phi \xi(\gamma o)} \quad \mu\text{-a.e.}$$

We call μ a (Γ, ϕ, δ)*-Patterson–Sullivan measure* if $C = 1$.

Using Patterson's original construction for Fuchsian groups, we proved the following existence result.

Proposition 4.3 ([KZ26, Proposition 4.5])**.** *If $\Gamma < \mathsf{G}$ is discrete, $\phi \in \mathfrak{a}_\theta^*$, and $\delta^\phi(\Gamma) < +\infty$, then there exists a $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure on $\partial_\theta X$.*

4.1. **Embeddings of partial flag manifolds.** The partial flag manifold $\mathcal{F}_\theta$ turns out to be naturally embedded into $\partial_\theta X$. Recall that $B_\theta^{IW} : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ denotes the partial Iwasawa cocyle. Quint [Qui02, Lemma 6.6] proved that

$$\lim_{n \to \infty} \pi_\theta \kappa(g^{-1} h_n) - \pi_\theta \kappa(h_n) = B_\theta^{IW}(g^{-1}, x) \quad \text{for all} \quad g \in \mathsf{G}$$

when $\min_{\alpha \in \theta} \alpha(\kappa(h_n)) \to +\infty$ and $U_\theta(h_n) \to x$. Using this fact, we showed that $\mathcal{F}_\theta$ embeds into $\partial_\theta X$.

Proposition 4.4 ([KZ26, Proposition 4.7])**.** *There exists a topological embedding $\iota : \mathcal{F}_\theta \to \partial_\theta X$ that satisfies*

$$\iota(x)(ho) = B_\theta^{IW}(h^{-1}, x) \tag{8}$$

for all $x \in \mathcal{F}_\theta$ and $h \in \mathsf{G}$. Moreover:

1. *If a sequence $\{g_n\} \subset \mathsf{G}$ satisfies $\min_{\alpha \in \theta} \alpha(\kappa(g_n)) \to +\infty$ and $U_\theta(g_n) \to x$, then*

$$g_n o \to \iota(x) \quad \text{in} \quad \overline{X}^\theta.$$

(2) If μ is a (coarse, resp.) (Γ, ϕ, δ)-Patterson–Sullivan measure on $\mathcal{F}_\theta$ in the sense of Definition 1.1, then $\iota_* \mu$ is a (coarse, resp.) (Γ, ϕ, δ)-Patterson–Sullivan measure on $\partial_\theta X$ in the sense of Definition 4.2.

4.2. **Shadows and contracting conical limit sets.** We now define shadows on $\partial_\theta X$ and use them to introduce the contracting conical limit set of a discrete subgroup.

In [KZ26, Lemma 4.6], we proved: if $\xi \in \partial_\theta X$ and $g \in \mathsf{G}$, then

$$\omega_\alpha \xi(g^{-1}o) \leq \omega_\alpha \kappa(g)$$

for all $\alpha \in \theta$. Given $g \in \mathsf{G}$, we then define shadows in $\partial_\theta X$ by considering the set of functionals ξ that are close to maximizing the expression $\omega_\alpha \xi(g^{-1}o)$ for all $\alpha \in \theta$.

More precisely, for $g \in \mathsf{G}$ and $R > 0$, the associated *shadow* is defined by

$$\mathcal{O}_R^\theta(g) := g \cdot \{\xi \in \partial_\theta X : \omega_\alpha \xi(g^{-1}o) > \omega_\alpha \kappa(g) - R \text{ for all } \alpha \in \theta\}.$$

In what follows we use π_θ to denote both the projection $\mathfrak{a} \to \mathfrak{a}_\theta$ satisfying Equation (6) and the map $\partial_\Delta X \to \partial_\theta X$ obtained by the postcomposition with this projection. Since $\omega_\alpha \xi = \omega_\alpha \pi_\theta \xi$ for all $\alpha \in \theta$ and $\xi \in \partial_\Delta X$, we have

$$\pi_\theta \, \mathcal{O}_R^\Delta(g) \subset \mathcal{O}_R^\theta(g).$$

We also use Proposition 4.4 to view $\mathcal{F}_\theta$ as a subset of $\partial_\theta X$. Then Equation (7) and Proposition 4.4 imply that $\pi_\theta|_{\mathcal{F}_\Delta}$ coincides with the natural projection $\mathcal{F}_\Delta \to \mathcal{F}_\theta$ given by $g \, \mathsf{P}_\Delta \to g \, \mathsf{P}_\theta$.

The boundary $\partial_\theta X$ has a natural cocycle $B_\theta : \mathsf{G} \times \partial_\theta X \to \mathfrak{a}$ defined by

$$B_\theta(g, x) := \xi(g^{-1}o).$$

Indeed one can see that $B_\theta(g_1 g_2, \xi) = B_\theta(g_1, g_2 \xi) + B_\theta(g_2, \xi)$ for all $g_1, g_2 \in \mathsf{G}$ and $\xi \in \partial_\theta X$.

In [KZ26], we verified that the definitions above give PS-systems on $\overline{X}^\theta$.

Theorem 4.5 ([KZ26, Theorem 6.1])**.** *Suppose $\theta \subset \Delta$ and $\phi \in \mathfrak{a}_\theta^*$. If $\Gamma < \mathsf{G}$ is strongly $(\Phi_\alpha)_{\alpha \in \theta}$-irreducible and μ is a coarse (Γ, ϕ, δ)-Patterson–Sullivan measure on $\partial_\theta X$, then $(\partial_\theta X, \Gamma, \phi \circ B_\theta, \mu)$ is a PS-system, with magnitude $\|\gamma\|_\phi := \phi(\kappa(\gamma))$ and the R-shadows $\mathcal{O}_R^\theta(\gamma)$ for each $\gamma \in \Gamma$. Moreover, (PS5) holds.*

For transverse groups, we showed that the system is well-behaved.

Theorem 4.6 ([KZ26, Theorem 6.2])**.** *Suppose $\theta \subset \Delta$, $\phi \in \mathfrak{a}_\theta^*$, and $\Gamma < \mathsf{G}$ is a strongly $(\Phi_\alpha)_{\alpha \in \theta}$-irreducible P_θ-transverse group. Let μ be a coarse (Γ, ϕ, δ)-Patterson–Sullivan measure on $\partial_\theta X$. Then the PS-system $(\partial_\theta X, \Gamma, \phi \circ B_\theta, \mu)$ in Theorem 4.5 is well-behaved with respect to the trivial hierarchy $\mathcal{H}(R) \equiv \Gamma$.*

Remark 4.7. See [KZ26, Section 5.2] for a comparison of shadows in $\partial_\theta X$ with the shadows in $\mathcal{F}_\theta$ defined in terms of positive Weyl chambers in the symmetric space.

4.3. **Properties of shadows.** We record some properties of shadows. For part (3), we fix any metric generating the topology on $\overline{X}^\theta$.

Lemma 4.8 ([KZ26, Lemmas 5.4, 5.5, 5.6])**.**

(1) For any $g \in \mathsf{G}$ and $R > 0$, there exists $R' = R'(g, R) > 0$ such that: if $h \in \mathsf{G}$, then

$$g \, \mathcal{O}_R^\theta(h) \subset \mathcal{O}_{R'}^\theta(gh).$$

(2) For $g \in \mathsf{G}$ with $\min_{\alpha\in\theta} \alpha(\kappa(g)) > 0$,

$$U_\theta(g) \in \mathcal{O}^\theta_R(g)$$

for all $R > 0$.

(3) For a sequence $\{g_n\} \subset \mathsf{G}$, if $\min_{\alpha\in\theta} \alpha(\kappa(g_n)) \to +\infty$, then

$$\operatorname{diam} \mathcal{O}^\theta_R(g_n) \to 0.$$

4.4. **Contracting conical limit sets.** Given a subgroup $\Gamma < \mathsf{G}$, we define its *conical limit set* in $\partial_\theta X$ by

$$\Lambda^{\mathrm{con}}_\theta(\Gamma) := \left\{ \xi \in \partial_\theta X : \exists\, R > 0, \text{ escaping } \{\gamma_n\} \subset \Gamma \text{ s.t. } \xi \in \bigcap_{n\geq 1} \mathcal{O}^\theta_R(\gamma_n) \right\}, \tag{9}$$

following the classical definition of conical limit sets in rank one settings.

When G is of higher rank, the intersection $\bigcap_{n\geq 1} \mathcal{O}^\theta_R(\gamma_n)$ may not be a singleton, even after intersecting with the partial flag manifold $\mathcal{F}_\theta$ and the conical limit set $\Lambda^{\mathrm{con}}_\theta(\Gamma)$, even after intersecting with $\mathcal{F}_\theta$, may not be a subset of the limit set $\Lambda_\theta(\Gamma)$, see [KZ26, Example 5.8]. Hence, in view of Lemma 4.8(3), we define the following smaller subset of conical limit set, which only involves shrinking shadows.

Definition 4.9. Given a subgroup $\Gamma < \mathsf{G}$, we call $\xi \in \partial_\theta X$ a *contracting conical limit point* of Γ if there exist $R > 0$ and a sequence $\{\gamma_n\} \subset \Gamma$ such that

$$\lim_{n\to+\infty} \min_{\alpha\in\theta} \alpha(\kappa(\gamma_n)) = +\infty \quad \text{and} \quad \xi \in \bigcap_{n\geq 1} \mathcal{O}^\theta_R(\gamma_n).$$

We denote by $\Lambda^{\mathrm{concon}}_\theta(\Gamma)$ the *contracting conical limit set* of Γ, which is defined as the set of all contracting conical limit points of Γ.

For general $\Gamma < \mathsf{G}$, the contracting conical limit set $\Lambda^{\mathrm{concon}}_\theta(\Gamma)$ is a Γ-invariant subset of the limit set $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$ introduced in Section 3.8 (this follows from Lemma 4.8).

For transverse groups we have the following.

Proposition 4.10 ([KZ26, Proposition 5.3, Theorem 6.3])**.** *If $\Gamma < \mathsf{G}$ is a non-elementary P_θ-transverse group, then:*

(1) $\Lambda^{\mathrm{concon}}_\theta(\Gamma) = \Lambda^{\mathrm{con}}_\theta(\Gamma)$ and $\Lambda^{\mathrm{con}}_\theta(\Gamma)$ is a subset of the limit set $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$ introduced in Section 3.8.

(2) $\Lambda^{\mathrm{con}}_\theta(\Gamma)$ coincides with the conical limit set in the convergence group sense (recall that Γ acts on $\Lambda_\theta(\Gamma)$ as a convergence group).

(3) If $\phi \in \mathfrak{a}^*_\theta$, $\delta^\phi(\Gamma) < +\infty$, and $\sum_{\gamma\in\Gamma} e^{-\delta^\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty$, then there exists a unique $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure μ on $\partial_\theta X$. Moreover,

$$\mu(\Lambda^{\mathrm{con}}_\theta(\Gamma)) = 1.$$

Part 2. Construction of measurable boundary maps

5. Continuity properties of boundary maps

In this section we prove the following technical continuity result for measurable maps whose domains are well-behaved PS-systems. Notice that the theorem does not assume that the map has any equivariance properties.

Theorem 5.1. *Suppose (M, Γ, σ, μ) is a well-behaved PS-system with respect to a hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$. Assume $F : M \to (Y, \mathrm{d}_Y)$ is a Borel measurable map into a separable metric space.*

For μ-a.e. $x_0 \in M$, if $x_0 \in \bigcap_{n\geq 1} \gamma\, \mathcal{O}_R(\gamma_n)$ for some $\gamma \in \Gamma$, escaping sequence $\{\gamma_n\}$ with $\gamma_n \in \mathcal{H}(n)$, and $R > 0$, then there exist a subsequence $\{\gamma_{n_j}\}$, $h_1, \dots, h_m \in \Gamma$, and a μ-null set $E \subset M$ such that for each $x \in M \smallsetminus E$ there exists some $1 \leq i \leq m$ where

$$\lim_{j \to +\infty} F(\gamma \gamma_{n_j} h_i x) = F(x_0).$$

Moreover, if

$$\mu(Z) = 0$$

whenever $S_n \to +\infty$ and $\left[M \smallsetminus \gamma_n^{-1}\, \mathcal{O}_{S_n}(\gamma_n)\right] \to Z$ with respect to the Hausdorff distance, then we can assume that $m = 1$ and $h_1 = \mathrm{id}$.

We use the following result from our earlier work.

Lemma 5.2 ([KZ25, Corollary 5.5])**.** *Suppose (M, Γ, σ, μ) is a well-behaved PS-system with respect to a hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$. Assume that $F : M \to (Y, \mathrm{d}_Y)$ is a Borel measurable map into a separable metric space.*

If $R > 0$ is sufficiently large, then for μ-a.e. $x \in M$ we have

$$0 = \lim_{n \to +\infty} \frac{1}{\mu(\gamma\, \mathcal{O}_R(\gamma_n))} \mu\left(\{y \in \gamma\, \mathcal{O}_R(\gamma_n) : \mathrm{d}_Y(F(x), F(y)) > \epsilon\}\right)$$

for all $\epsilon > 0$ whenever $x \in \bigcap_{n\geq 1} \gamma\, \mathcal{O}_R(\gamma_n)$ for some $\gamma \in \Gamma$ and escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R)$.

5.1. **Proof of Theorem 5.1.** Fix $R_j \nearrow +\infty$. After possibly increasing each $\{R_j\}$, by Lemma 5.2 for each $j \geq 1$ there exists a full μ-measure set $M_j \subset M$ such that: if $x_0 \in M_j \cap \bigcap_{n\geq 1} \gamma\, \mathcal{O}_{R_j}(\gamma_n)$ for some $\gamma \in \Gamma$ and escaping sequence $\{\gamma_n\} \subset \mathcal{H}(R_j)$, then

$$0 = \lim_{n \to +\infty} \frac{1}{\mu(\gamma\, \mathcal{O}_{R_j}(\gamma_n))} \mu\left(\left\{x \in \gamma\, \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}_Y(F(x), F(x_0)) > \epsilon\right\}\right)$$

for all $\epsilon > 0$.

Let

$$M' := \bigcap_{\gamma \in \Gamma, j \geq 1} \gamma M_j.$$

Then M' is a Γ-invariant set with full μ-measure.

Fix $x_0 \in M'$, $\gamma \in \Gamma$, and $\{\gamma_n\} \subset \Gamma$ escaping such that $\gamma_n \in \mathcal{H}(n)$ and

$$x_0 \in \bigcap_{n \geq 1} \gamma\, \mathcal{O}_R(\gamma_n)$$

for some $R > 0$. Passing to a subsequence and relabelling, we can suppose that $\gamma_n \in \mathcal{H}(R_n)$ for all $n \geq 1$.

We can also assume $R_j \geq R$ for all j and hence by Property (PS6)

$$x_0 \in \bigcap_{n \geq 1} \gamma\, \mathcal{O}_{R_j}(\gamma_n).$$

Since $j \mapsto \mathcal{H}(R_j)$ is non-increasing, we have $\{\gamma_n\}_{n \geq j} \subset \mathcal{H}(R_j)$. So

$$0 = \lim_{n \to +\infty} \frac{1}{\mu(\gamma\, \mathcal{O}_{R_j}(\gamma_n))} \mu\left(\left\{x \in \gamma\, \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}_Y(F(x), F(x_0)) > \epsilon\right\}\right)$$

for all $j \in \mathbb{N}$ and $\epsilon > 0$. By Property (PS1),

$$\lim_{n\to+\infty} \frac{1}{\mu(\mathcal{O}_{R_j}(\gamma_n))} \mu\left(\left\{x \in \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}_Y(F(\gamma x), F(x_0)) > \epsilon\right\}\right)$$
$$\asymp \lim_{n\to+\infty} \frac{1}{\gamma_*^{-1}\mu(\mathcal{O}_{R_j}(\gamma_n))} \gamma_*^{-1}\mu\left(\left\{x \in \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}_Y(F(\gamma x), F(x_0)) > \epsilon\right\}\right)$$
$$= \lim_{n\to+\infty} \frac{1}{\mu(\gamma\, \mathcal{O}_{R_j}(\gamma_n))} \mu\left(\left\{x \in \gamma\, \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}_Y(F(x), F(x_0)) > \epsilon\right\}\right) = 0$$

for all $j \in \mathbb{N}$ and $\epsilon > 0$.

By Property (PS2), there exists $C_j > 1$ such that

$$C_j e^{-\delta\|\gamma_n\|_\sigma} \le \frac{d{\gamma_n^{-1}}_*\mu}{d\mu} \le C_j e^{-\delta\|\gamma_n\|_\sigma} \quad \mu\text{-a.e. on } \gamma_n^{-1}\, \mathcal{O}_{R_j}(\gamma_n)$$

where $\delta \ge 0$ is the dimension of the PS-system. Hence

$$0 = \lim_{n\to+\infty} \frac{1}{(\gamma_n^{-1})_*\mu(\gamma_n^{-1}\, \mathcal{O}_{R_j}(\gamma_n))} (\gamma_n^{-1})_*\mu\left(\left\{x \in \gamma_n^{-1}\, \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}_Y(F(\gamma\gamma_n x), F(x_0)) > \epsilon\right\}\right)$$
$$\asymp \lim_{n\to+\infty} \frac{1}{\mu(\gamma_n^{-1}\, \mathcal{O}_{R_j}(\gamma_n))} \mu\left(\left\{x \in \gamma_n^{-1}\, \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}_Y(F(\gamma\gamma_n x), F(x_0)) > \epsilon\right\}\right)$$

for all $j \in \mathbb{N}$ and $\epsilon > 0$. Since

$$\mu(\gamma_n^{-1}\, \mathcal{O}_{R_j}(\gamma_n)) \le 1,$$

then

$$0 = \lim_{n\to+\infty} \mu\left(\left\{x \in \gamma_n^{-1}\, \mathcal{O}_{R_j}(\gamma_n) : \mathrm{d}_Y(F(\gamma\gamma_n x), F(x_0)) > \epsilon\right\}\right)$$

for all $j \in \mathbb{N}$ and $\epsilon > 0$. Then we can find $S_n = R_{j_n} \to +\infty$ and $\epsilon_n \searrow 0$ such that $j_n \le n$ and after passing to a subsequence of $\{\gamma_n\}$,

$$\sum_{n\ge1} \mu\left(\left\{x \in \gamma_n^{-1}\, \mathcal{O}_{S_n}(\gamma_n) : \mathrm{d}_Y(F(\gamma\gamma_n x), F(x_0)) > \epsilon_n\right\}\right) < +\infty.$$

Let

$$E_n := \left\{x \in \gamma_n^{-1}\, \mathcal{O}_{S_n}(\gamma_n) : \mathrm{d}_Y(F(\gamma\gamma_n x), F(x_0)) > \epsilon_n\right\}$$

and

$$E' := \bigcap_{N\ge1} \bigcup_{n\ge N} E_n,$$

which is μ-null by the Borel–Cantelli Lemma.

Passing to a subsequence we can suppose that $\left[M \smallsetminus \gamma_n^{-1}\, \mathcal{O}_{S_n}(\gamma_n)\right] \to Z$ with respect to the Hausdorff distance.

If $\mu(Z) = 0$, then we set

$$E := E' \cup Z$$

which is μ-null. Fix $x \in M \smallsetminus E$. Since $x \notin Z$ and Z is closed, there exists some $N_0 \in \mathbb{N}$ such that $x \in \bigcap_{n\ge N_0} \gamma_n^{-1}\, \mathcal{O}_{S_n}(\gamma_n)$. Since $x \notin E'$, there exists some $N_1 \in \mathbb{N}$ such that $x \notin \bigcup_{n\ge N_1} E_n$. Hence for $n \ge \max(N_0, N_1)$ we have

$$\mathrm{d}_Y(F(\gamma\gamma_n x), F(x_0)) \le \epsilon_n.$$

Since $\epsilon_n \to 0$, this implies that $F(\gamma\gamma_n x) \to F(x_0)$, and the "moreover" part follows.

In general, we set

$$E := \Gamma \cdot E'$$

which is μ-null by the quasi-invariance of μ. By Property (PS3), we can pick $h_1, \dots, h_m \in \Gamma$ such that

$$\bigcap_{i=1}^m h_i^{-1} Z = \emptyset.$$

Then fix $r > 0$ such that

$$\bigcap_{i=1}^m h_i^{-1} \mathcal{N}_r(Z) = \emptyset,$$

equivalently

$$\bigcup_{i=1}^m h_i^{-1} \left(M \smallsetminus \mathcal{N}_r(Z) \right) = M.$$

Notice that

$$M \smallsetminus \mathcal{N}_r(Z) \subset \bigcap_{n \geq N_0} \gamma_n^{-1} \mathcal{O}_{S_n}(\gamma_n)$$

for some $N_0 \in \mathbb{N}$.

Now fix $x \in M \smallsetminus E$. Then we can fix $1 \leq i \leq m$ such that $h_i x \in M \smallsetminus \mathcal{N}_r(Z)$. Since $E = \Gamma \cdot E'$, we have $h_i x \notin E'$. In particular, there exists some $N_1 \in \mathbb{N}$ such that

$$h_i x \notin \bigcup_{n \geq N_1} E_n.$$

Then for $n \geq \max(N_0, N_1)$ we have

$$\mathrm{d}_Y(F(\gamma \gamma_n h_i x), F(x_0)) \leq \epsilon_n.$$

So $F(\gamma\gamma_n h_i x) \to F(x_0)$. □

6. Boundary maps and almost everywhere contraction

In this section, we prove the existence of boundary maps, which is the main technical result of this paper. We first define two notions which we need to state our result.

Suppose Γ is a locally compact group acting on a measure space (M, μ) where μ is a Γ-quasi-invariant measure. Then the action of Γ on (M, μ) is *amenable* if there exists a sequence $\lambda_n : M \to \mathrm{Prob}(\Gamma)$ so that for μ-a.e. $x \in M$ and every $g \in \Gamma$,

$$\lim_{n \to +\infty} \| \lambda_n(gx) - g_* \lambda_n(x) \| = 0$$

where $\mathrm{Prob}(\cdot)$ is the space of Borel probability measures and $\|\cdot\|$ denotes the total variation.

A subgroup $H < \mathsf{G}$ is P_θ-*contracting* if there exists a sequence $\{h_n\} \subset H$ such that

$$\alpha(\kappa(h_n)) \to +\infty \quad \text{for all } \alpha \in \theta.$$

If H is Zariski dense, then H is P_θ-contracting, see [GM89, Theorem 3.6] or [Pra94].

Theorem 6.1. *Suppose (M, Γ, σ, μ) is a well-behaved Patterson–Sullivan system with respect to a hierarchy $\mathcal{H} = \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$,*

$$\mu(\Lambda^{\mathrm{con}}(\mathcal{H})) = 1,$$

and the Γ-action on (M, μ) is amenable.

If $\rho : \Gamma \to \mathsf{G}$ is a representation such that $\rho(\Gamma)$ is P_θ-contracting and strongly $(\Phi_\alpha)_{\alpha \in \theta}$-irreducible, then there exists a ρ-equivariant μ-a.e. defined measurable map $f : M \to \Lambda_\theta(\Gamma)$. Moreover:

(1) *For μ-a.e. $x \in M$, if $\gamma \in \Gamma$, $\{\gamma_n\} \subset \Gamma$ is escaping, $\gamma_n \in \mathcal{H}(R_n)$ for some $R_n \to +\infty$, and*
$$x \in \bigcap_{n \geq 1} \gamma\, \mathcal{O}_R(\gamma_n),$$
for some $R > 0$, then
$$\alpha(\kappa(\rho(\gamma\gamma_n))) \to +\infty \quad \textit{for all} \quad \alpha \in \theta$$
and
$$U_\theta(\rho(\gamma\gamma_n)) \to f(x).$$
(2) *If $\mathsf{m} : M \to \mathrm{Prob}(\mathcal{F}_\theta)$ is some ρ-equivariant μ-a.e. defined measurable map, then $\mathsf{m}(x) = \mathcal{D}_{f(x)}$ μ-a.e. In particular, if $f' : M \to \mathcal{F}_\theta$ is a ρ-equivariant μ-a.e. defined measurable map, then $f = f'$ μ-a.e.*
(3) *If the hierarchy $\mathcal{H}$ is trivial (i.e., $\mathcal{H}(R) \equiv \Gamma$), then for μ-a.e. $x \in M$, there exist $R > 0$, $\gamma \in \Gamma$, and an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that*
$$\lim_{n \to +\infty} \min_{\alpha \in \theta} \alpha(\kappa(\rho(\gamma_n))) = +\infty, \quad x \in \bigcap_{n \geq 1} \gamma\, \mathcal{O}_R(\gamma_n), \quad \textit{and} \quad f(x) \in \bigcap_{n \geq 1} \rho(\gamma)\, \mathcal{O}^\theta_R(\rho(\gamma_n)).$$
In particular,
$$f_*\mu(\Lambda_\theta^{\mathrm{concon}}(\Gamma)) = 1.$$

Remark 6.2. Recall that the conditions in (1) imply that $\rho(\gamma\gamma_n)o \to f(x)$ in $\overline{X}^\theta$ by Proposition 4.4.

In the rest of this section, we assume the hypotheses of Theorem 6.1.

Lemma 6.3. *There exists a ρ-equivariant μ-a.e. defined measurable map $\mathsf{m}_0 : M \to \mathrm{Prob}(\mathcal{F}_\theta)$. Moreover, m_0 maps into a single G-orbit.*

Proof. The existence of the map m_0 follows from [Zim84, Proposition 4.3.9]. By Theorem 2.4, the Γ-action on (M, μ) is ergodic. Then the fact that m_0 maps into a single G-orbit follows from [Zim84, Corollary 3.2.17, Proposition 2.1.11], together with the ergodicity of the Γ-action on (M, μ). $\square$

Since m_0 maps into a single G-orbit, we can fix a measure $\nu_0 \in \mathrm{Prob}(\mathcal{F}_\theta)$ and a measurable map $x \in M \mapsto g_x \in \mathsf{G}$ such that
$$\mathsf{m}_0(x) = (g_x)_*\nu_0$$
for μ-a.e. $x \in M$.

Let $d_0 \in \mathbb{Z}_{\geq 0}$ be the smallest non-negative integer where there exists an (real) algebraic variety of dimension d_0 in $\mathcal{F}_\theta$ with positive ν_0 measure. Then let $\mathcal{A}$ denote the set of all irreducible d_0-dimensional algebraic varieties with positive ν_0 measure. Notice that if $Z_1, Z_2 \in \mathcal{A}$ are distinct, then $\nu_0(Z_1 \cap Z_2) = 0$ by the minimality of d_0. Hence
$$\nu_0 = \sum_{Z \in \mathcal{A}} \nu_0(\cdot \cap Z) + \tilde{\nu}_0$$
where $\tilde{\nu}_0$ is a non-negative measure.

Now fix $\epsilon > 0$ sufficiently small such that the set
$$\mathcal{A}_\epsilon := \{Z \in \mathcal{A} : \nu_0(Z) > \epsilon\}$$

is non-empty (notice that it must be finite). Then define

$$\nu_1 := \frac{1}{\sum_{Z\in\mathcal{A}_\epsilon}\nu_0(Z)}\sum_{Z\in\mathcal{A}_\epsilon}\nu_0(\cdot\cap Z)$$

and define $\mathsf{m}_1 : M \to \mathrm{Prob}(\mathcal{F}_\theta)$ by

$$\mathsf{m}_1(x) = (g_x)_*\nu_1.$$

Then by construction the map m_1 is ρ-equivariant and for μ-a.e. $x \in M$, $\mathsf{m}_1(x)$ is supported on finitely many irreducible d_0-dimensional algebraic varieties.

For $\alpha \in \theta$, let $\rho_\alpha := \Phi_\alpha \circ \rho : \Gamma \to \mathsf{SL}(V_\alpha)$. In what follows, we view the boundary map ζ_α in Property (R3) as having domain $\mathcal{F}_\theta$.

Proposition 6.4. *There exists a Γ-invariant full μ-measure set $M' \subset M$ such that: if $x_0 \in M'$, $\{\gamma_n\} \subset \Gamma$ is escaping, $\gamma_n \in \mathcal{H}(R_n)$ for some $R_n \to +\infty$,*

$$x_0 \in \bigcap_{n\geq 1}\gamma\,\mathcal{O}_R(\gamma_n)$$

for some $R > 0$, and

$$\frac{\rho_\alpha(\gamma\gamma_n)}{\|\rho_\alpha(\gamma\gamma_n)\|} \to S_\alpha \in \mathrm{End}(V_\alpha)$$

for all $\alpha \in \theta$, then

(1) $\mathrm{rank}\, S_\alpha = 1$ *for all $\alpha \in \theta$,*
(2) $\mathsf{m}_1(x_0) = \mathcal{D}_\xi$ *where $\xi \in \mathcal{F}_\theta$ is the unique point with $\zeta_\alpha(\xi) = \mathrm{im}\, S_\alpha$ for all $\alpha \in \theta$.*

Delaying the proof of Proposition 6.4 for a moment, we complete the proof of Theorem 6.1. The proposition implies that $\mathsf{m}_1(x)$ is a Dirac mass when $x \in M' \cap \Lambda^{\mathrm{con}}(\mathcal{H})$ and so $f(x) = \mathrm{supp}\,\mathsf{m}_1(x)$ provides a ρ-equivariant μ-a.e. defined measurable map $f : M \to \mathcal{F}_\theta$.

Lemma 6.5. *f satisfies part (1) of the “moreover” part of Theorem 6.1.*

Proof. Suppose $x \in M'$ and x, γ, $\{\gamma_n\}$ satisfy the hypothesis of part (1). Proposition 6.4 implies that any limit point of

$$\frac{\rho_\alpha(\gamma\gamma_n)}{\|\rho_\alpha(\gamma\gamma_n)\|}$$

has rank one and image $\zeta_\alpha f(x)$. So by Property (R1) of the representations Φ_α, we have

$$\alpha(\kappa(\rho(\gamma\gamma_n))) \to +\infty \quad \text{for all} \quad \alpha \in \theta.$$

Then Lemma 3.4 implies that $U_\theta(\rho(\gamma\gamma_n)) \to f(x)$. □

We now prove the part (2) of the theorem. With the notation introduced at the start of the proof, $d_0 = 0$ and $\mathcal{A}_\epsilon$ is a point. Since $\epsilon > 0$ was an arbitrary small positive number, this implies that $\mathcal{A}$ is a point. Hence there exists $\lambda > 0$ such that

$$\mathsf{m}_0(x) = \lambda\,\mathcal{D}_{f(x)} + \tilde{\mathsf{m}}_0(x)$$

μ-a.e., where $\tilde{\mathsf{m}}_0(x)$ is a non-atomic non-negative measure.

Lemma 6.6. *If $\mathsf{m} : M \to \mathrm{Prob}(\mathcal{F}_\theta)$ is an ρ-equivariant μ-a.e. defined map, then $\mathsf{m}(x) = \mathcal{D}_{f(x)}$ μ-a.e.*

Proof. At the start of the argument, $\mathsf{m}_0 : M \to \mathrm{Prob}(\mathcal{F}_\theta)$ was an arbitrary ρ-equivariant μ-a.e. defined map. Hence repeating the proof above, there exist $\lambda' > 0$ and a μ-a.e. defined ρ-equivariant map $f' : M \to \mathcal{F}_\theta$ such that $\mathsf{m}(x) = \lambda' \mathcal{D}_{f'(x)} + \tilde{\mathsf{m}}(x)$, where $\tilde{\mathsf{m}}(x)$ is a non-atomic non-negative measure. Further, f' satisfies part (1) of the moreover part of the theorem. Since f also satisfies part (1) of the moreover part of the theorem, we must have $f' = f$ μ-a.e.

Then it suffices to show that $\tilde{\mathsf{m}}(x) \equiv 0$ for μ-a.e. $x \in M$. Suppose not. Then since $x \mapsto \tilde{\mathsf{m}}(x)(\mathcal{F}_\theta)$ is Γ-invariant and Γ acts ergodically on (M, μ), we have $\tilde{\mathsf{m}}(x)$ is a non-zero measure for μ-a.e. x. Then $\frac{\tilde{\mathsf{m}}}{\|\tilde{\mathsf{m}}\|} : M \to \mathrm{Prob}(\mathcal{F}_\theta)$ is an ρ-equivariant μ-a.e. defined map. So repeating the argument again, we see that $\tilde{\mathsf{m}}(x)$ has an atom for μ-a.e. x, which is a contradiction. $\square$

Now it remains to show the part (3). Suppose that $\mathcal{H}(R) \equiv \Gamma$. Since $\mathcal{F}_\theta \subset \partial_\theta X$ by Proposition 4.4, we can consider f as the map into $\partial_\theta X$. We then consider the *mixed shadow*

$$\mathcal{O}_R^f(\gamma) := \mathcal{O}_R(\gamma) \cap f^{-1}(\mathcal{O}_R^\theta(\rho(\gamma)))$$

for $\gamma \in \Gamma$ and $R > 0$. By the part (1) of Theorem 6.1, it suffices to show that for μ-a.e. $x \in M$, there exist $R > 0$, $\gamma \in \Gamma$, and an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that $x \in \gamma\, \mathcal{O}_R^f(\gamma_n)$ for all $n \in \mathbb{N}$. Then the "in particular" part follows from Lemma 4.8(1).

By Property (PS4), we can fix an enumeration $\Gamma = \{\gamma_n\}$ such that

$$\|\gamma_1\|_\sigma \le \|\gamma_2\|_\sigma \le \cdots .$$

By the Mixed Shadow Lemma in our earlier work [KZ25, Theorem 6.2] and the Shadow Lemma (Proposition 2.5), we can fix $R > 0$ so that for some $C > 1$, we have

$$\frac{1}{C}\mu\left(\mathcal{O}_R(\gamma)\right) \le \mu\left(\mathcal{O}_R^f(\gamma)\right) \le C\mu\left(\mathcal{O}_R(\gamma)\right) \quad \text{for all} \quad \gamma \in \Gamma.$$

By [KZ25, Proof of Theorem 4.1], the sequence of sets $A_n := \mathcal{O}_R(\gamma_n)$, $n \in \mathbb{N}$, satisfies the following Kochen–Stone Lemma, with $\Omega = M$ and $\nu = \mu$.

Lemma 6.7 (Kochen–Stone Lemma, [KS64])**.** *Let (Ω, ν) be a finite measure space. If $\{A_n\} \subset \Omega$ is a sequence of measurable sets where*

$$\sum_{n \ge 1} \nu(A_n) = +\infty \quad \textit{and} \quad \liminf_{N \to +\infty} \frac{\sum_{n,m=1}^N \nu(A_n \cap A_m)}{\left(\sum_{n=1}^N \nu(A_n)\right)^2} < +\infty,$$

then

$$\nu\left(\{x \in \Omega : x \textit{ is contained in infinitely many of } A_1, A_2, \dots\}\right) > 0.$$

The uniform estimates on shadows implies that the new sequence of sets $B_n := \mathcal{O}_R^f(\gamma_n)$, $n \in \mathbb{N}$, also satisfies this Kochen–Stone Lemma. Therefore, setting

$$E := \left\{ x \in M : \exists\ R > 0,\ \gamma \in \Gamma,\ \text{escaping } \{\gamma_n\} \subset \Gamma \text{ s.t. } x \in \bigcap_{n \ge 1} \gamma\, \mathcal{O}_R^f(\gamma_n) \right\},$$

we have $\mu(E) > 0$ and E is Γ-invariant. Since the Γ-action on (M, μ) is ergodic (Theorem 2.4), we have $\mu(E) = 1$. This completes the proof of (3).

Therefore, Theorem 6.1 follows, and it remains to prove Proposition 6.4.

6.1. **Proof of Proposition 6.4.** Let $M' \subset M$ be a full μ-measure set satisfying the conclusion of Theorem 5.1. Suppose $x_0 \in M'$, $\gamma \in \Gamma$, $\{\gamma_n\} \subset \Gamma$ is escaping, $\gamma_n \in \mathcal{H}(R_n)$ for some $R_n \to +\infty$,

$$x_0 \in \bigcap_{n \geq 1} \gamma \, \mathcal{O}_R(\gamma_n)$$

for some $R > 0$, and

$$\frac{\rho_\alpha(\gamma\gamma_n)}{\|\rho_\alpha(\gamma\gamma_n)\|} \to S_\alpha \in \mathrm{End}(V_\alpha)$$

for all $\alpha \in \theta$.

Using Theorem 5.1 and replacing $\{\gamma_n\}$ by a subsequence, there exists a μ-null set $E \subset M$ and $h_1, \dots, h_N \in \Gamma$ such that: if $y \in M \smallsetminus E$, then

$$\mathsf{m}_1(x_0) = \lim_{n \to +\infty} \mathsf{m}_1(\gamma\gamma_n h_i y)$$

for some $1 \leq i \leq N$.

Given $T \in \mathrm{End}(V_\alpha)$ and a measure ν on $\mathbb{P}(V_\alpha)$, we define the pushforward measure

$$T_*\nu := T_* \left(\nu|_{\mathbb{P}(V_\alpha) \smallsetminus \mathbb{P}(\ker T)} \right).$$

The following well known observation will allow us to study the limit points of $\frac{\rho_\alpha(\gamma\gamma_n)}{\|\rho_\alpha(\gamma\gamma_n)\|}$ in $\mathrm{End}(V_\alpha)$.

Observation 6.8. Suppose ν is a Borel probability measure on $\mathbb{P}(\mathbb{R}^d)$ and $T_n \to T$ in $\mathrm{End}(\mathbb{R}^d)$. If $\nu(\mathbb{P}(\ker T)) = 0$, then $(T_n)_*\nu \to T_*\nu$.

Since $\rho(\Gamma)$ is P_θ-contracting, there exists a sequence $\{g_m\} \subset \Gamma$ such that

$$\min_{\alpha \in \theta} \alpha(\kappa(\rho(g_m))) \to +\infty.$$

Passing to a subsequence we can suppose that

$$\frac{\rho_\alpha(g_m)}{\|\rho_\alpha(g_m)\|} \to T_\alpha \in \mathrm{End}(V_\alpha)$$

for all $\alpha \in \theta$. By Property (R1),

$$\frac{\sigma_1(\rho_\alpha(g_m))}{\sigma_2(\rho_\alpha(g_m))} \to +\infty$$

and hence T_α has rank one for each $\alpha \in \theta$.

Fix

$$y_0 \in M \smallsetminus \Gamma \cdot E$$

(this set has full μ-measure and hence is non-empty).

Lemma 6.9. *There exist $u_1, u_2 \in \Gamma$ such that*

$$S_\alpha \rho_\alpha(h_i u_1) T_\alpha \neq 0 \quad \textit{for all} \quad \alpha \in \theta, \ 1 \leq i \leq N$$

and

$$\rho(u_2) Y \not\subset \zeta_\alpha^{-1}\Big(\mathbb{P}(\ker S_\alpha \rho_\alpha(h_i u_1) T_\alpha) \Big) \quad \textit{for all} \quad \alpha \in \theta, \ Y \in g_{y_0} \mathcal{A}_\epsilon .$$

Proof. For each $\alpha \in \theta$ and $1 \leq i \leq N$, fix $v_\alpha \in \operatorname{im} T_\alpha$ non-zero. By Lemma 3.7, there exists $u_1 \in \Gamma$ such that

$$\rho_\alpha(u_1)v_\alpha \notin \rho_\alpha(h_i)^{-1}\ker S_\alpha$$

for all $\alpha \in \theta$ and $1 \leq i \leq N$. Then u_1 satisfies the first part of the lemma. Next for each $Y \in g_{y_0}\mathcal{A}_\epsilon$, fix a non-zero vector $w_Y \in V_\alpha$ with $[w_Y] \in \zeta_\alpha(Y)$. Then by Lemma 3.7, there exists $u_2 \in \Gamma$ such that

$$\rho_\alpha(u_2)w_Y \notin \ker S_\alpha \rho_\alpha(h_i u_1)T_\alpha$$

for all $\alpha \in \theta$ and $Y \in g_{y_0}\mathcal{A}_\epsilon$ (recall that $\mathcal{A}_\epsilon$ is finite). Then u_2 satisfies the second part of the lemma. □

For each $m \in \mathbb{N}$, since $u_1 g_m u_2 y_0 \notin E$, there exists some $1 \leq i_m \leq N$ such that

$$\lim_{n\to+\infty} \rho(\gamma\gamma_n h_{i_m} u_1 g_m u_2)_* \mathsf{m}_1(y_0) = \lim_{n\to+\infty} \mathsf{m}_1(\gamma\gamma_n h_{i_m} u_1 g_m u_2 y_0) = \mathsf{m}_1(x_0).$$

Replacing $\{g_m\}$ with a subsequence and relabelling the $\{h_i\}$ we can assume that $h_{i_m} = h_1$ for all m. Next fix $n_m \to +\infty$ such that

$$\lim_{m\to+\infty} \rho(\gamma\gamma_{n_m} h_1 u_1 g_m u_2)_* \mathsf{m}_1(y_0) = \mathsf{m}_1(x_0).$$

Notice that

$$\lim_{m\to+\infty} \frac{\rho_\alpha(\gamma\gamma_{n_m} h_1 u_1 g_m u_2)}{\|\rho_\alpha(\gamma\gamma_{n_m} h_1 u_1 g_m u_2)\|} = \frac{S_\alpha \rho_\alpha(h_1 u_1) T_\alpha \rho_\alpha(u_2)}{\|S_\alpha \rho_\alpha(h_1 u_1) T_\alpha \rho_\alpha(u_2)\|} =: T'_\alpha \in \operatorname{End}(V_\alpha),$$

since $S_\alpha \rho_\alpha(h_1 u_1) T_\alpha \neq 0$.

By the choice of u_2, if $Y \in g_{y_0}\mathcal{A}_\epsilon$ and $\alpha \in \theta$, then

$$Y \not\subset \zeta_\alpha^{-1}\left(\mathbb{P}(\ker T'_\alpha)\right),$$

and so

$$\dim\left(Y \cap \zeta_\alpha^{-1}\left(\mathbb{P}(\ker T'_\alpha)\right)\right) < d_0.$$

Hence by the definition of m_1,

$$(\zeta_\alpha)_* \mathsf{m}_1(y_0)\left(\mathbb{P}(\ker T'_\alpha)\right) = \mathsf{m}_1(y_0)\left(\zeta_\alpha^{-1}\left(\mathbb{P}(\ker T'_\alpha)\right)\right) = 0.$$

So Observation 6.8 implies that

$$\begin{aligned}(\zeta_\alpha)_* \mathsf{m}_1(x_0) &= \lim_{m\to+\infty} (\zeta_\alpha)_* \rho(\gamma\gamma_{n_m} h_1 u_1 g_m u_2)_* \mathsf{m}_1(y_0) \\ &= \lim_{m\to+\infty} \rho_\alpha(\gamma\gamma_{n_m} h_1 u_1 g_m h_2)_* (\zeta_\alpha)_* \mathsf{m}_1(y_0) \\ &= (T'_\alpha)_* (\zeta_\alpha)_* \mathsf{m}_1(y_0).\end{aligned}$$

Since $\operatorname{rank} T'_\alpha = \operatorname{rank} T_\alpha = 1$ we then see that

$$(\zeta_\alpha)_* \mathsf{m}_1(x_0) = \mathcal{D}_{\mathbb{P}(\operatorname{im} T'_\alpha)}.$$

Thus $\mathsf{m}_1(x_0) = \mathcal{D}_\xi$ where $\xi \in \mathcal{F}_\theta$ is the unique point with $\zeta_\alpha(\xi) = \mathbb{P}(\operatorname{im} T'_\alpha)$ for all $\alpha \in \theta$. Since $\operatorname{im} T'_\alpha \subset \operatorname{im} S_\alpha$, to finish the proof it suffices to show that $\operatorname{rank} S_\alpha = 1$.

Fix $\alpha \in \theta$. Since $\mathsf{m}_1(x_0)$ is a Dirac mass, the definition of m_1 implies that $\mathsf{m}_1(x)$ is a Dirac mass for μ-a.e. x. Then $f_\alpha(x) := \operatorname{supp}(\zeta_\alpha)_* \mathsf{m}_1(x)$ defines a ρ-equivariant μ-a.e. defined measurable map $f_\alpha : M \to \mathbb{P}(V_\alpha)$.

Suppose for a contradiction that $\operatorname{rank} S_\alpha > 1$. Fix $z_0 \in M \smallsetminus \Gamma \cdot E$ such that $f_\alpha(z_0)$ is defined.

Lemma 6.10. *There exists* $u_3 \in \Gamma$ *such that*

$$\rho_\alpha(h_i u_3) f_\alpha(z_0) \notin \mathbb{P}(\ker S_\alpha)$$

and

$$S_\alpha \rho_\alpha(h_i u_3) f_\alpha(z_0) \neq f_\alpha(x_0)$$

for all $1 \leq i \leq N$.

Proof. Let $f_\alpha(z_0) = [v]$ for some non-zero $v \in V_\alpha$. Notice that since S_α has rank at least two, $S_\alpha^{-1}(f_\alpha(x_0)) \subset V_\alpha$ is a proper linear subspace. So by Lemma 3.7, there exists $u_3 \in \Gamma$ such that

$$\rho_\alpha(u_3)v \notin \rho_\alpha(h_i)^{-1} \ker S_\alpha$$

and

$$\rho_\alpha(u_3)v \notin \rho_\alpha(h_i)^{-1} S_\alpha^{-1}(f_\alpha(x_0))$$

for all $1 \leq i \leq N$. Then u_3 satisfies the desired conclusion. □

Since $u_3 z_0 \notin E$, there exists $1 \leq i \leq N$ such that

$$\mathsf{m}_1(x_0) = \lim_{n \to +\infty} \mathsf{m}_1(\gamma\gamma_n h_i u_3 z_0) = \lim_{n \to +\infty} \rho(\gamma\gamma_n h_i u_3)_* \mathsf{m}_1(z_0).$$

Applying $(\zeta_\alpha)_*$, Observation 6.8 implies that

$$\begin{aligned}\mathcal{D}_{f_\alpha(x_0)} = (\zeta_\alpha)_* \mathsf{m}_1(x_0) &= \lim_{n \to +\infty} \rho_\alpha(\gamma\gamma_n h_i u_3)_* \mathcal{D}_{f_\alpha(z_0)} \\ &= \mathcal{D}_{S_\alpha \rho_\alpha(h_i u_3) f_\alpha(z_0)}.\end{aligned}$$

Since $S_\alpha \rho_\alpha(h_i u_3) f_\alpha(z_0) \neq f_\alpha(x_0)$, we have a contradiction. Hence $\operatorname{rank} S_\alpha = 1$, completing the proof of Proposition 6.4. □

7. Amenable actions of transverse groups

In this section we prove that transverse groups act amenably on their limit sets, and hence the amenable assumption in Theorem 6.1 is satisfied.

Theorem 7.1. *Suppose* $\Gamma < \mathsf{G}$ *is a non-elementary* P_θ*-transverse group. If* μ *is a* Γ*-quasi-invariant Borel probability measure on* $\Lambda_\theta(\Gamma)$*, then the* Γ*-action on* $(\Lambda_\theta(\Gamma), \mu)$ *is amenable.*

To prove Theorem 7.1 we first use a result from [CZZ24] that says a transverse group can be identified with a group acting nicely on a properly convex domain in some projective space. The Hilbert metric on this properly convex domain then has enough hyperbolic behavior to adapt an argument of Kaimanovich [Kai04].

7.1. Background on convex real projective geometry.

We recall some terminology from real projective geometry.

Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is *properly convex*, that is, it is a bounded convex subset of some affine chart of $\mathbb{P}(\mathbb{R}^d)$. The *automorphism group* of Ω is

$$\mathsf{Aut}(\Omega) = \{g \in \mathsf{PGL}(d, \mathbb{R}) : g\Omega = \Omega\}.$$

The *limit set* of a subgroup $H < \mathsf{Aut}(\Omega)$, denoted by $\Lambda_\Omega(H)$, is the set of points $x \in \partial\Omega$ where there exist $o \in \Omega$ and $\{h_n\} \subset H$ with $h_n o \to x$.

Given $x, y \in \overline{\Omega}$, we let $[x, y]$ denote the projective line segment contained in $\overline{\Omega}$ joining x to y. We further define $(x, y) := [x, y] \smallsetminus \{x, y\}$, $[x, y) := [x, y] \smallsetminus \{y\}$, and $(x, y] := [x, y] \smallsetminus \{x\}$.

The *Hilbert metric* d_Ω on Ω is defined as follows: for $p, q \in \Omega$ distinct let $a, b \in \partial\Omega$ be the unique points with $p, q \in (a, b)$ and with order a, p, q, b, then define

$$\mathrm{d}_\Omega(x, y) := \frac{1}{2} \log \frac{\|a - q\| \, \|b - p\|}{\|a - p\| \, \|b - q\|}$$

where $\|\cdot\|$ is any norm on any affine chart containing $\overline{\Omega}$. The Hilbert metric is a proper geodesic metric where $\mathsf{Aut}(\Omega)$ acts by isometries. Further, for $p, q \in \Omega$ the line segment $[p, q]$ can be parametrized to be a geodesic.

An element $W \in \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ is a *supporting hyperplane* at $x \in \partial\Omega$ if $x \in \mathbb{P}(W)$ and $\mathbb{P}(W) \cap \Omega = \emptyset$. Convexity of Ω implies that every boundary point is contained in at least one supporting hyperplane and we say that $\partial\Omega$ is $\mathcal{C}^1$*-smooth at* $x \in \partial\Omega$ if x is contained in exactly one supporting hyperplane.

We end this section by recording some useful geometric properties.

Observation 7.2 (see e.g. the proof of [Ben04, Lemma 3.4])**.** Suppose $x \in \partial\Omega$ is a $\mathcal{C}^1$-smooth point. For $p \in \Omega$, let $\ell_p : [0, +\infty) \to \Omega$ denote the unit speed parametrization of the geodesic ray $[p, x)$. If $p, q \in \Omega$, then there exists $T_{p,q} \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} \mathrm{d}_\Omega(\ell_p(t), \ell_q(t + T_{p,q})) = 0.$$

Moreover $T_{p,q}$ depends continuously on p, q and the convergence is uniform on compact subsets of Ω.

Given a discrete group $\Gamma < \mathsf{Aut}(\Omega)$, the *Hilbert metric critical exponent* is

$$\delta_\Omega(\Gamma) := \limsup_{R \to +\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma : \mathrm{d}_\Omega(\gamma o, o) \leq R\}$$

where $o \in \Omega$ is any fixed point. A result of Tholozan implies that this critical exponent has a uniform upper bound, which only depends on the dimension d.

Theorem 7.3 ([Tho17])**.** *If* $\Gamma < \mathsf{Aut}(\Omega)$ *is discrete, then* $\delta_\Omega(\Gamma) \leq d - 2$.

7.2. **Projectively visible groups.** In [CZZ24, CZZ26], Canary, Zhang, and the second author studied transverse groups by constructing actions on certain types of properly convex domains.

Definition 7.4. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is properly convex. A discrete subgroup $\Gamma < \mathsf{Aut}(\Omega)$ is *projectively visible* if

- $(x, y) \subset \Omega$ for every $x, y \in \Lambda_\Omega(\Gamma)$,
- every $x \in \Lambda_\Omega(\Gamma)$ is a $\mathcal{C}^1$-smooth point of $\partial\Omega$.

Under some mild conditions on G and P_θ, every transverse group can be identified with a projectively visible subgroup.

Theorem 7.5 ([CZZ24, Theorem 6.2])**.** *Suppose* G *has trivial center,* $\theta \subset \Delta$*, and* P_θ *contains no simple factors of* G*. If* $\Gamma < \mathsf{G}$ *is* P_θ*-transverse, then there exist* $d \in \mathbb{N}$*, a properly convex domain* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$*, a projectively visible subgroup* $\Gamma_0 < \mathsf{Aut}(\Omega)$*, an isomorphism* $\rho : \Gamma_0 \to \Gamma$*, and a* ρ*-equivariant homeomorphism* $\xi : \Lambda_\Omega(\Gamma_0) \to \Lambda_\theta(\Gamma)$*.*

7.3. **Proof of Theorem 7.1.** By replacing G with a quotient, it suffices to consider the case where G has trivial center and P_θ does not contain any simple factors of G, see [CZZ24, Section 2.4].

Then by Theorem 7.5 there exist $d \in \mathbb{N}$, a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, a projectively visible subgroup $\Gamma_0 < \mathsf{Aut}(\Omega)$, an isomorphism $\rho : \Gamma_0 \to \Gamma$, and a ρ-equivariant homeomorphism $\xi : \Lambda_\Omega(\Gamma_0) \to \Lambda_\theta(\Gamma)$.

Then it suffices to fix a Γ_0-quasi-invariant Borel probability measure μ on $\Lambda_\Omega(\Gamma_0)$ and show that Γ_0 acts amenably on $(\Lambda_\Omega(\Gamma_0), \mu)$. Using Observation 7.2, we can argue exactly as in [Kai04, Theorems 1.33 and 3.15].

Lemma 7.6 (compare to [Kai04, Theorem 1.33, Theorem 1.38])**.** *There exists a sequence of Γ_0-equivariant maps $\lambda_n : \Omega \times \Lambda_\Omega(\Gamma_0) \to \mathrm{Prob}(\Gamma_0)$ such that*

$$\lim_{n\to+\infty} \|\lambda_n(p,x) - \lambda_n(q,x)\| = 0$$

for any $p, q \in \Omega$ and $x \in \Lambda_\Omega(\Gamma_0)$. Moreover, with x fixed the convergence is uniform on any compact subset of Ω.

Proof. Fix $\delta > \delta_\Omega(\Gamma_0)$ and $o \in \Omega$. Then for $p \in \Omega$ consider the measures

$$\nu_p := \frac{1}{\sum_{\gamma\in\Gamma_0} e^{-\delta\, \mathrm{d}_\Omega(p,\gamma o)}} \sum_{\gamma\in\Gamma_0} e^{-\delta\, \mathrm{d}_\Omega(p,\gamma o)} \mathcal{D}_\gamma \in \mathrm{Prob}(\Gamma_0)$$

where $\mathcal{D}_\gamma$ is the Dirac mass at γ.

Claim: If $\mathrm{d}_\Omega(p,q) \le 1$, then

$$\|\nu_p - \nu_q\| \le 2\delta e^{2\delta}\, \mathrm{d}_\Omega(p,q). \tag{10}$$

Proof of Claim: For $p \in \Omega$, let $M_p := \sum_{\gamma\in\Gamma_0} e^{-\delta\, \mathrm{d}_\Omega(p,\gamma o)}$. Now fix $p, q \in \Omega$ with $\mathrm{d}_\Omega(p,q) \le 1$. Then for every $\gamma \in \Gamma$,

$$e^{-\delta\, \mathrm{d}_\Omega(p,\gamma o)} = e^{\epsilon_\gamma} e^{-\delta\, \mathrm{d}_\Omega(q,\gamma o)} \quad \text{where} \quad |\epsilon_\gamma| \le \delta\, \mathrm{d}_\Omega(p,q).$$

So $M_p = e^{\epsilon} M_q$ where $|\epsilon| \le \delta\, \mathrm{d}_\Omega(p,q)$. Then

$$\begin{aligned}
\|\nu_p - \nu_q\| &= \sum_{\gamma\in\Gamma} \left| \frac{1}{M_p} e^{-\delta\, \mathrm{d}_\Omega(p,\gamma o)} - \frac{1}{M_q} e^{-\delta\, \mathrm{d}_\Omega(q,\gamma o)} \right| \\
&= \frac{1}{M_q} \sum_{\gamma\in\Gamma} \left| e^{-\epsilon} e^{-\delta\, \mathrm{d}_\Omega(p,\gamma o)} - e^{-\delta\, \mathrm{d}_\Omega(q,\gamma o)} \right| \\
&= \frac{1}{M_q} \sum_{\gamma\in\Gamma} e^{-\delta\, \mathrm{d}_\Omega(q,\gamma o)} \left| e^{-\epsilon+\epsilon_\gamma} - 1 \right| \le \sup_{\gamma\in\Gamma} \left| e^{-\epsilon+\epsilon_\gamma} - 1 \right|.
\end{aligned}$$

Now $|-\epsilon + \epsilon_\gamma| \le 2\delta\, \mathrm{d}_\Omega(p,q)$, so by the mean value theorem

$$\|\nu_p - \nu_q\| \le e^{2\delta\, \mathrm{d}_\Omega(p,q)} 2\delta\, \mathrm{d}_\Omega(p,q) \le 2\delta e^{2\delta}\, \mathrm{d}_\Omega(p,q).$$

◀

Next for $p \in \Omega$ and $x \in \Lambda_\Omega(\Gamma_0)$, let $\ell_{px} : [0,+\infty) \to \Omega$ be the unit speed parametrization of the geodesic ray $[p,x)$. Then define

$$\lambda_n(p,x) := \frac{1}{n} \int_0^n \nu_{\ell_{px}(t)} dt.$$

Observation 7.2 and Equation (10) imply that these measures have the desired properties. □

Next define $\lambda_n : \Lambda_\Omega(\Gamma_0) \to \mathrm{Prob}(\Gamma_0)$ by

$$\lambda_n(x) := \lambda_n(o, x).$$

Then

$$\lim_{n\to+\infty} \|\lambda_n(\gamma x) - \gamma_* \lambda_n(x)\| = \lim_{n\to+\infty} \|\lambda_n(o, \gamma x) - \lambda_n(\gamma o, \gamma x)\| = 0$$

for every $x \in \Lambda_\Omega(\Gamma_0)$ and every $\gamma \in \Gamma_0$. □

Part 3. **Applications**

8. Lifting Patterson–Sullivan measures

For the rest of the section, suppose $\Gamma < \mathsf{G}$ is a P_θ-transverse group, $\phi \in \mathfrak{a}_\theta^*$, $\delta := \delta^\phi(\Gamma) < +\infty$, and

$$\sum_{\gamma\in\Gamma} e^{-\delta\phi(\kappa(\gamma))} = +\infty.$$

Let μ denote the unique (Γ, ϕ, δ)-Patterson–Sullivan measure on $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta \subset \partial_\theta X$, see Theorem 3.9. This section is devoted to the proof of the following.

Theorem 8.1. *With the notations above, suppose $\Theta \supset \theta$ and Γ is P_Θ-contracting and strongly $(\Phi_\alpha)_{\alpha\in\Theta}$-irreducible. Then the following holds:*

1. *There exists a unique μ-a.e defined injective Γ-equivariant measurable map*
$$f : \Lambda_\theta(\Gamma) \to \mathcal{F}_\Theta .$$
2. *The pushforward $f_*\mu$ is the unique (Γ, ϕ, δ)-Patterson–Sullivan measure on $\partial_\Theta X$.*
3. $(f_*\mu)(\Lambda_\Theta^{\mathrm{concon}}(\Gamma)) = 1$. *Moreover, for μ-a.e. $x \in \Lambda_\theta(\Gamma)$, there exist $R > 0$ and an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that*
$$\lim_{n\to+\infty} \min_{\alpha\in\Theta} \alpha(\kappa(\gamma_n)) = +\infty, \quad x \in \bigcap_{n\geq 1} \mathcal{O}_R^\theta(\gamma_n), \quad \textit{and} \quad f(x) \in \bigcap_{n\geq 1} \mathcal{O}_R^\Theta(\gamma_n).$$
4. *If μ' is a (Γ, ϕ, β)-Patterson–Sullivan measure on $\partial_\Theta X$, then $\beta \geq \delta$.*

By Theorem 4.6, $(\partial_\theta X, \Gamma, \phi \circ B_\theta, \mu)$ is a well-behaved PS-system with respect to the trivial hierarchy $\mathcal{H}(R) \equiv \Gamma$. Hence, together with Theorem 2.3 and [Ben97], we obtain singularity between Patterson–Sullivan measures.

Corollary 8.2. *Suppose further that Γ is Zariski dense in G. If $\psi \in \mathfrak{a}_\Theta^*$, $\beta \geq 0$, and μ_ψ is a (Γ, ψ, β)-Patterson–Sullivan measure on $\partial_\Theta X$, then*

$$f_*\mu \text{ and } \mu_\psi \text{ are non-singular} \iff f_*\mu = \mu_\psi \iff \delta \cdot \phi = \beta \cdot \psi.$$

Proof. If $\delta \cdot \phi = \beta \cdot \psi$, then it follows from the uniqueness in Theorem 8.1(2) that $f_*\mu = \mu_\psi$. Clearly, if $f_*\mu = \mu_\psi$, then they are non-singular. So it suffices to show that non-singularity implies that the functionals coincide up to an appropriate scaling.

Suppose $f_*\mu$ and μ_ψ are non-singular. Then by Theorem 2.3 we have

$$\sup_{\gamma\in\Gamma} |\delta \cdot \phi(\kappa(\gamma)) - \beta \cdot \psi(\kappa(\gamma))| < +\infty.$$

By definition, the Jordan projections then satisfy

$$\delta \cdot \phi(\lambda(\gamma)) = \beta \cdot \psi(\lambda(\gamma))$$

for all $\gamma \in \Gamma$. Since Γ is Zariski dense, this implies $\delta \cdot \phi = \beta \cdot \psi$ by [Ben97]. □

The rest of this section is devoted to the proof of Theorem 8.1.

8.1. **Part (1).** Theorem 7.1 implies that the Γ-action on $(\Lambda_\theta(\Gamma), \mu)$ is amenable and Theorem 3.9 (and Proposition 4.10) implies that μ is supported on the conical limit set. So we can apply Theorem 6.1 to the inclusion $\Gamma \hookrightarrow \mathsf{G}$. As a result, we have a unique measurable Γ-equivariant μ-a.e. defined map

$$f : \Lambda_\theta(\Gamma) \to \mathcal{F}_\Theta .$$

To finish the proof of part (1) we only need to show that f is injective on a set of full μ-measure. Let $\pi : \mathcal{F}_\Theta \to \mathcal{F}_\theta$ be the natural projection. Then πf is a Γ-equivariant μ-a.e. defined map. However, Γ is P_θ-contracting and strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible, so by Theorem 6.1 the identity is the unique measurable Γ-equivariant μ-a.e. defined map $\Lambda_\theta(\Gamma) \to \mathcal{F}_\theta$. Thus we must have

$$\pi f = \mathrm{id}_{\mathcal{F}_\theta} \quad \mu\text{-a.e.} \tag{11}$$

Hence f is injective on a set of full μ-measure.

8.2. **Parts (2) and (4).** Since $\phi \in \mathfrak{a}_\theta^* = \mathrm{span}\{\omega_\alpha : \alpha \in \theta\}$, Equation (6) implies that

$$\phi B_\Theta^{IW}(g, x) = \phi B_\theta^{IW}(g, \pi(x))$$

for all $x \in \mathcal{F}_\Theta$ and $g \in \mathsf{G}$. Hence Equation (11) implies that $f_*\mu$ is a (Γ, ϕ, δ)-Patterson–Sullivan measure on $\mathcal{F}_\Theta \subset \partial_\Theta X$.

The uniqueness in part (2) and the estimate in part (4) will be a consequence of the following general result (which we also use in the proof of Theorem 10.1 below).

Proposition 8.3. *Suppose that μ' is a Radon measure on $\partial_\Theta X$ such that for each $R > 0$ sufficiently large there exists $C_0 = C_0(R) > 0$ so that*

$$(f_*\mu)(\mathcal{O}_R^\Theta(\gamma)) \leq C_0 \mu'(\mathcal{O}_R^\Theta(\gamma)) \quad \textit{for all} \quad \gamma \in \Gamma.$$

Then

$$f_*\mu \ll \mu'.$$

Proof. Since $f_*\mu$ is supported on $\mathcal{F}_\Theta$, it suffices to fix a Borel subset $E \subset \mathcal{F}_\Theta$ with $\mu'(E) = 0$ and then show that $f_*\mu(E) = 0$.

For each $R > 0$, let $E_R \subset f^{-1}(E)$ be the subset satisfying:

- for each $x \in E_R$, there exists an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that

$$x \in \bigcap_{n\geq 1} \mathcal{O}_R^\theta(\gamma_n)$$

  and
- Theorem 6.1(1) holds for all $x \in E_R$.

Then Proposition 4.10 implies that $(f_*\mu)(E) = \mu\left(\bigcup_{R>0} E_R\right)$. Since $\{E_R\}_{R>0}$ is an increasing family of sets, it suffices to fix $R > 0$ sufficiently large and show that $\mu(E_R) = 0$.

Fix $\epsilon > 0$ and let $\mathcal{U} \subset \partial_\Theta X$ be an open neighborhood of E such that

$$\mu'(\mathcal{U}) \leq \epsilon.$$

Claim: If $x \in E_R$, then there exists $\gamma_x \in \Gamma$ so that $x \in \mathcal{O}_R^\theta(\gamma_x)$ and $\mathcal{O}_R^\Theta(\gamma_x) \subset \mathcal{U}$.

Proof of Claim: By the definition of E_R and Theorem 6.1(1), there exists an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that

$$x \in \bigcap_{n \geq 1} \mathcal{O}_R^\theta(\gamma_n), \quad \lim_{n \to +\infty} \min_{\alpha \in \Theta} \alpha(\kappa(\gamma_n)) = +\infty, \quad \text{and} \quad f(x) = \lim_{n \to +\infty} U_\Theta(\gamma_n) \in \mathcal{F}_\Theta .$$

By Lemma 4.8(2), $U_\Theta(\gamma_n) \in \mathcal{O}_R^\Theta(\gamma_n)$ for all large $n \geq 1$. Moreover, $\operatorname{diam} \mathcal{O}_R^\Theta(\gamma_n) \to 0$ as $n \to +\infty$ by Lemma 4.8(3). Since $\mathcal{U} \subset \partial_\Theta X$ is an open neighborhood of $f(x)$, it follows that there exists $\gamma_x \in \Gamma$ such that

$$x \in \mathcal{O}_R^\theta(\gamma_x) \quad \text{and} \quad \mathcal{O}_R^\Theta(\gamma_x) \subset \mathcal{U} .$$

◀

Hence we have a countable subset $I := \{\gamma_x : x \in E_R\} \subset \Gamma$ so that

$$E_R \subset \bigcup_{\gamma \in I} \mathcal{O}_R^\theta(\gamma) \quad \text{and} \quad \bigcup_{\gamma \in I} \mathcal{O}_R^\Theta(\gamma) \subset \mathcal{U} .$$

By Lemma 2.6, there exist $J \subset I$ and $R' \geq R$ such that the shadows $\left\{\mathcal{O}_R^\theta(\gamma) : \gamma \in J\right\}$ are pairwise disjoint and

$$\bigcup_{\gamma \in I} \mathcal{O}_R^\theta(\gamma) \subset \bigcup_{\gamma \in J} \mathcal{O}_{R'}^\theta(\gamma).$$

We then have

$$\mu(E_R) \leq \mu\left(\bigcup_{\gamma \in I} \mathcal{O}_R^\theta(\gamma)\right) \leq \sum_{\gamma \in J} \mu(\mathcal{O}_{R'}^\theta(\gamma)).$$

By Theorem 4.5, $(\partial_\theta X, \Gamma, \phi \circ B_\theta, \mu)$ and $(\partial_\Theta X, \Gamma, \phi \circ B_\Theta, f_* \mu)$ are PS-systems. So, after possibly increasing R, the Shadow Lemma (Proposition 2.5) implies that there exists $C_1 = C_1(R) > 0$ so that

$$\mu(\mathcal{O}_{R'}^\theta(\gamma)) \leq C_1 \mu(\mathcal{O}_R^\theta(\gamma))$$

and

$$\mu(\mathcal{O}_R^\theta(\gamma)) \leq C_1 (f_*\mu)(\mathcal{O}_R^\Theta(\gamma))$$

for all $\gamma \in \Gamma$. This implies that

$$\mu(E_R) \leq C_1 \sum_{\gamma \in J} \mu(\mathcal{O}_R^\theta(\gamma)) \leq C_1^2 \sum_{\gamma \in J} (f_*\mu)(\mathcal{O}_R^\Theta(\gamma)) \leq C_0 C_1^2 \sum_{\gamma \in J} \mu'(\mathcal{O}_R^\Theta(\gamma)).$$

Let $\pi_\theta : \partial_\Theta X \to \partial_\theta X$ denote the map obtained by postcomposing with the canonical projection $\pi_\theta : \mathfrak{a}_\Delta \to \mathfrak{a}_\theta$ restricted to $\mathfrak{a}_\Theta$. Equation (6) implies that $\pi_\theta \, \mathcal{O}_R^\Theta(\gamma) \subset \mathcal{O}_R^\theta(\gamma)$. Then, since the shadows $\left\{\mathcal{O}_R^\theta(\gamma) : \gamma \in J\right\}$ are pairwise disjoint, the shadows $\left\{\mathcal{O}_R^\Theta(\gamma) : \gamma \in J\right\}$ are pairwise disjoint as well. Therefore,

$$\mu(E_R) \leq C_0 C_1^2 \mu'\left(\bigcup_{\gamma \in J} \mathcal{O}_R^\Theta(\gamma)\right) \leq C_0 C_1^2 \mu'(\mathcal{U}) \leq C_0 C_1^2 \epsilon.$$

Since $\epsilon > 0$ is arbitrary and C_0, C_1 are independent of ϵ, we have

$$\mu(E_R) = 0.$$

This finishes the proof. □

Now we deduce the uniqueness in part (2) and the estimate in part (4) using Proposition 8.3.

Lemma 8.4. *If $\beta \leq \delta$ and μ' is a (Γ, ϕ, β)-Patterson–Sullivan measure on $\partial_\Theta X$, then $\beta = \delta$ and $\mu' = \mu$.*

Proof. Since both $(\partial_\Theta X, \Gamma, \phi \circ B_\Theta, f_*\mu)$ and $(\partial_\Theta X, \Gamma, \phi \circ B_\Theta, \mu')$ are PS-systems by Theorem 4.5, it follows from the Shadow Lemma (Proposition 2.5) that for all large enough $R > 0$, there exists $C_0 > 0$ such that

$$(f_*\mu)(\mathcal{O}_R^\Theta(\gamma)) \leq C_0 \mu'(\mathcal{O}_R^\Theta(\gamma)) \quad \text{for all} \quad \gamma \in \Gamma.$$

Hence, by Proposition 8.3, we have $f_*\mu \ll \mu'$.

Fix a full μ-measure Γ-invariant set $Y \subset \Lambda_\theta(\Gamma)$ where f is defined. We claim that $\mu'(f(Y)) = 1$. If not, then

$$\mu'' := \frac{1}{\mu'(\partial_\Theta X \smallsetminus f(Y))} \mu'|_{\partial_\Theta X \smallsetminus f(Y)}$$

is a (Γ, ϕ, β)-Patterson–Sullivan measure. Then arguing as above, we have $f_*\mu \ll \mu''$, which is impossible. Hence $\mu'(f(Y)) = 1$.

Recall that $\pi f = \mathrm{id}_{\mathcal{F}_\theta}$, see Equation (11). So $\pi_*\mu'$ is a (Γ, ϕ, β)-Patterson–Sullivan measure on $\Lambda_\theta(\Gamma)$. Then $\beta = \delta$ and $\mu = f_*^{-1}\mu'$ by [CZZ24] (see Theorem 3.9). □

8.3. **Part (3).** This follows from the part (3) of Theorem 6.1 and Lemma 4.8(1), as the associated hierarchy $\mathcal{H}$ is trivial (Theorem 4.6). □

9. Ergodic dichotomy for BMS-measures on homogeneous spaces

In this section, we prove the ergodic dichotomy for the diagonal action on homogeneous spaces. Let $\Gamma < \mathsf{G}$ be a P_θ-transverse group which is P_Δ-contracting and strongly $(\Phi_\alpha)_{\alpha \in \Delta}$-irreducible. In this section, we apply the machinery developed in earlier sections to study ergodic theory on the homogeneous space $\Gamma \backslash \mathsf{G}$ or $\Gamma \backslash \mathsf{G} / \mathsf{M}$.

First recall the Hopf parametrization $\mathsf{G}/\mathsf{M} = \mathcal{F}_\Delta^{(2)} \times \mathfrak{a}$ given by

$$g\mathsf{M} = \left(g\,\mathsf{P}_\Delta, gw_0\,\mathsf{P}_\Delta, B_\Delta^{IW}(g, \mathsf{P}_\Delta)\right)$$

where $\mathcal{F}_\Delta^{(2)} \subset \mathcal{F}_\Delta \times \mathcal{F}_\Delta$ is the subset of transverse pairs. Then right multiplication A on G/M corresponds to translation on the $\mathfrak{a}$-component of $\mathcal{F}_\Delta^{(2)} \times \mathfrak{a}$.

Fix $\phi \in \mathfrak{a}_\theta^*$ and $\delta \geq 0$, and suppose that there exist (Γ, ϕ, δ) and $(\Gamma, \mathrm{i}^*\phi, \delta)$-PS measures μ_ϕ and $\mu_{\mathrm{i}^*\phi}$ on $\Lambda_\Delta(\Gamma) \subset \mathcal{F}_\Delta$ respectively. While they may not exist or they may not be unique in general, we assume the existence and make a choice.

We first define a Γ-invariant Radon measure ν_ϕ on $\mathcal{F}_\Delta^{(2)}$ by

$$d\nu_\phi(\xi, \eta) := e^{\delta\phi\,\mathcal{G}_\Delta(\xi,\eta)} d\mu_\phi(\xi) d\mu_{\mathrm{i}^*\phi}(\eta)$$

where $\mathcal{G}_\Delta$ is the $\mathfrak{a}$-*valued Gromov product*, i.e.

$$\mathcal{G}_\Delta(\xi, \eta) := -\left(B_\Delta^{IW}(g^{-1}, \xi) + \mathrm{i}B(g^{-1}, \eta)\right) \tag{12}$$

for $g \in \mathsf{G}$ with $(g\,\mathsf{P}_\Delta, gw_0\,\mathsf{P}_\Delta) = (\xi, \eta)$. By the irreducibility of Γ and the quasi-invariance of μ_ϕ under the Γ-action, it follows from Fubini's theorem that

$$\nu_\phi(\mathcal{F}_\Delta^{(2)}) > 0. \tag{13}$$

See [KOW25b, Proposition 10.2] for instance.

Then the measure $\nu_\phi \otimes \mathrm{Leb}_{\mathfrak{a}}$ on $\mathsf{G}/\mathsf{M} = \mathcal{F}_\Delta^{(2)} \times \mathfrak{a}$ is Γ-invariant, and hence it induces the A-invariant Radon measure

$$\mathsf{m}_\phi \quad \text{on} \quad \Gamma \backslash \mathsf{G}/\mathsf{M}$$

which we call the *Bowen–Margulis–Sullivan measure* associated to $(\mu_\phi, \mu_{\phi \circ \mathrm{i}})$.

Combining the ergodicity results for the diagonal action of Γ on $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}^*\theta}$ established in [CZZ24] with our lifting theorem, we obtain the ergodic dichotomy for ν_ϕ and m_ϕ, which generalizes the classical Hopf–Tsuji–Sullivan dichotomy.

Theorem 9.1. *The following dichotomy holds.*

(1) *If* $\sum_{\gamma \in \Gamma} e^{-\delta \phi(\kappa(\gamma))} = +\infty$, *then* $\delta = \delta^\phi(\Gamma)$ *and* $(\mu_\phi, \mu_{\mathrm{i}^*\phi})$ *is a unique pair of PS-measures, and we have*
   - $\mu_\phi(\Lambda_\Delta^{\mathrm{concon}}(\Gamma)) = \mu_{\mathrm{i}^*\phi}(\Lambda_\Delta^{\mathrm{concon}}(\Gamma)) = 1$,
   - *the diagonal* Γ*-action on* $(\mathcal{F}_\Delta^{(2)}, \nu_\phi)$ *is ergodic, and*
   - *the* A*-action on* $(\Gamma \backslash \mathsf{G}/\mathsf{M}, \mathsf{m}_\phi)$ *is ergodic.*

(2) *If* $\sum_{\gamma \in \Gamma} e^{-\delta \phi(\kappa(\gamma))} < +\infty$, *then we have*
   - $\mu_\phi(\Lambda_\Delta^{\mathrm{con}}(\Gamma)) = \mu_{\mathrm{i}^*\phi}(\Lambda_\Delta^{\mathrm{con}}(\Gamma)) = 0$,
   - *the diagonal* Γ*-action on* $(\mathcal{F}_\Delta^{(2)}, \nu_\phi)$ *is not ergodic, and*
   - *the* A*-action on* $(\Gamma \backslash \mathsf{G}/\mathsf{M}, \mathsf{m}_\phi)$ *is not ergodic.*

Proof. Suppose first that $\sum_{\gamma \in \Gamma} e^{-\delta \phi(\kappa(\gamma))} = +\infty$. The existence of $(\mu_\phi, \mu_{\mathrm{i}^*\phi})$ implies $\delta \geq \delta^\phi(\Gamma)$ by Theorem 8.1(4), and hence $\delta = \delta^\phi(\Gamma)$ due to the divergence of the series.

Without loss of generality, we may assume $\theta = \mathrm{i}^*\theta$ (Observation 3.8). Now by Theorem 8.1, we have $\mu_\phi = f_*\mu$ and $\mu_{\mathrm{i}^*\phi} = \bar{f}_*\bar{\mu}$ where

- μ is the (Γ, ϕ, δ)-Patterson–Sullivan measure on $\Lambda_\theta(\Gamma)$ and $f : \Lambda_\theta(\Gamma) \to \mathcal{F}_\Delta$ is the μ-a.e. defined injective Γ-equivariant map,
- $\bar{\mu}$ is the $(\Gamma, \mathrm{i}^*\phi, \delta)$-Patterson–Sullivan measure $\Lambda_\theta(\Gamma)$ and $\bar{f} : \Lambda_\theta(\Gamma) \to \mathcal{F}_\Delta$ is the $\bar{\mu}$-a.e. defined injective Γ-equivariant map.

Theorem 8.1 also implies that

$$\mu_\phi(\Lambda_\Delta^{\mathrm{concon}}(\Gamma)) = \mu_{\mathrm{i}^*\phi}(\Lambda_\Delta^{\mathrm{concon}}(\Gamma)) = 1.$$

As proved in [CZZ24, Corollary 12.1], the diagonal Γ-action on $(\Lambda_\theta(\Gamma)^2, \mu \otimes \bar{\mu})$ is ergodic. Hence, the Γ-action on $(\mathcal{F}_\Delta^2, \mu_\phi \otimes \mu_{\mathrm{i}^*\phi})$ is ergodic as well. Since $\mathcal{F}_\Delta^{(2)} \subset \mathcal{F}_\Delta^2$ is Γ-invariant, the Γ-action on $(\mathcal{F}_\Delta^{(2)}, \nu_\phi)$ is ergodic as well. It then follows from the definition of m_ϕ that the A-action on $(\Gamma \backslash \mathsf{G}/\mathsf{M}, \mathsf{m}_\phi)$ is also ergodic.

Next suppose that $\sum_{\gamma \in \Gamma} e^{-\delta \phi(\kappa(\gamma))} < +\infty$. The Shadow Lemma (Proposition 2.5) implies that $\mu_\phi(\Lambda_\Delta^{\mathrm{con}}(\Gamma)) = \mu_{\mathrm{i}^*\phi}(\Lambda_\Delta^{\mathrm{con}}(\Gamma)) = 0$. It now suffices to show that the Γ-action on $(\mathcal{F}_\Delta^{(2)}, \nu_\phi)$ is not ergodic, which also implies the non-ergodicity of the A-action on $(\Gamma \backslash \mathsf{G}/\mathsf{M}, \mathsf{m}_\phi)$.

Suppose to the contrary that the Γ-action on $(\mathcal{F}_\Delta^{(2)}, \nu_\phi)$ is ergodic. Then passing through the projection $\pi : \mathcal{F}_\Delta^{(2)} \to \mathcal{F}_\theta^{(2)}$, the Γ-action on $(\mathcal{F}_\theta^{(2)}, \pi_*\nu_\phi)$ is ergodic, where $\mathcal{F}_\theta^{(2)}$ is the set of transverse pairs in $\mathcal{F}_\theta^2$. On the other hand, since $\pi_*\nu_\phi$ is supported on $\Lambda_\theta(\Gamma) \times \Lambda_\theta(\Gamma)$, such an ergodicity implies $\sum_{\gamma \in \Gamma} e^{-\delta \phi(\kappa(\gamma))} = +\infty$ by [CZZ24, Corollary 12.1], contradiction. ∎

10. Strict convexity of critical exponent

This section is devoted to the proof of our main theorem on the strict convexity of the entropy. We combine the strategy of [BCZZ24, Proposition 14.5] with the machinery developed in this paper.

Theorem 10.1. *Suppose $\Gamma < \mathsf{G}$ is a P_θ-transverse group, $\Theta \supset \theta$, and $\Phi_\alpha(\Gamma) < \mathsf{SL}(V_\alpha)$ has semisimple Zariski closure for each $\alpha \in \Theta$. Assume:*

- *$\phi \in \mathfrak{a}_\theta^*$ satisfies $\delta^\phi(\Gamma) < +\infty$ and $\sum_{\gamma\in\Gamma} e^{-\delta^\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty$.*
- *$\phi_1, \phi_2 \in \mathfrak{a}_\Theta^*$ satisfy $\delta^{\phi_1}(\Gamma) = \delta^{\phi_2}(\Gamma) = 1$.*
- *There exist $t \in (0,1)$ and $C \geq 0$ such that*
$$\phi(\kappa(\gamma)) \geq t\phi_1(\kappa(\gamma)) + (1-t)\phi_2(\kappa(\gamma)) - C \quad \text{for all } \gamma \in \Gamma.$$

Then $\delta^\phi(\Gamma) \leq 1$ and equality holds if and only if
$$\sup_{\gamma\in\Gamma} |\phi(\kappa(\gamma)) - \phi_i(\kappa(\gamma))| < +\infty \quad \text{for } i = 1, 2.$$
In particular, if $\delta^\phi(\Gamma) = 1$, then
$$\phi(\lambda(\gamma)) = \phi_1(\lambda(\gamma)) = \phi_2(\lambda(\gamma)) \quad \text{for all } \gamma \in \Gamma.$$

Delaying the proof of Theorem 10.1, we first deduce an analogous statement for general linear combinations.

Corollary 10.2. *Suppose $\Gamma < \mathsf{G}$ is a P_θ-transverse group, $\Theta \supset \theta$, and $\Phi_\alpha(\Gamma) < \mathsf{SL}(V_\alpha)$ has semisimple Zariski closure for each $\alpha \in \Theta$. Assume:*

- *$\phi \in \mathfrak{a}_\theta^*$ satisfies $\delta^\phi(\Gamma) < +\infty$ and $\sum_{\gamma\in\Gamma} e^{-\delta^\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty$.*
- *$\phi_1, \phi_2 \in \mathfrak{a}_\Theta^*$ satisfy $\delta^{\phi_1}(\Gamma), \delta^{\phi_2}(\Gamma) < +\infty$ and*
$$\phi \geq c_1\phi_1 + c_2\phi_2 \quad \text{on} \quad \mathfrak{a}^+$$
for some $c_1, c_2 > 0$.

Then
$$\delta^\phi(\Gamma) \leq \frac{1}{\frac{c_1}{\delta^{\phi_1}(\Gamma)} + \frac{c_2}{\delta^{\phi_2}(\Gamma)}}$$
and equality holds if and only if
$$\sup_{\gamma\in\Gamma} \left|\delta^\phi(\Gamma)\phi(\kappa(\gamma)) - \delta^{\phi_i}(\Gamma)\phi_i(\kappa(\gamma))\right| < +\infty \quad \text{for} \quad i = 1, 2.$$

Proof. Let
$$\psi := \frac{1}{\frac{c_1}{\delta^{\phi_1}(\Gamma)} + \frac{c_2}{\delta^{\phi_2}(\Gamma)}} \cdot \phi \in \mathfrak{a}_\theta^*.$$
Then we have
$$\delta^\psi(\Gamma) = \left(\frac{c_1}{\delta^{\phi_1}(\Gamma)} + \frac{c_2}{\delta^{\phi_2}(\Gamma)}\right)\delta^\phi(\Gamma) \quad \text{and} \quad \sum_{\gamma\in\Gamma} e^{-\delta^\psi(\Gamma)\psi(\kappa(\gamma))} = +\infty.$$
In addition, setting $\psi_i := \delta^{\phi_i}(\Gamma) \cdot \phi_i$, we have $\delta^{\psi_i}(\Gamma) = 1$ for $i = 1, 2$, and moreover
$$\psi \geq t\psi_1 + (1-t)\psi_2$$
where $t := \frac{\frac{c_1}{\delta^{\phi_1}(\Gamma)}}{\frac{c_1}{\delta^{\phi_1}(\Gamma)} + \frac{c_2}{\delta^{\phi_2}(\Gamma)}} \in (0,1)$. Then applying Theorem 10.1 to ψ, ψ_1, and ψ_2 finishes the proof. □

10.1. **Proof of Theorem 10.1 in a special case.** We first prove the theorem with an extra assumption.

Extra Assumption: Γ is P_Θ-contracting and strongly $(\Phi_\alpha)_{\alpha\in\Theta}$-irreducible.

Notice that, if $s > 0$, then Hölder inequality implies that

$$\begin{aligned}\sum_{\gamma\in\Gamma} e^{-s\phi(\kappa(\gamma))} &\leq e^{Cs}\sum_{\gamma\in\Gamma} e^{-st\phi_1(\kappa(\gamma))}e^{-s(1-t)\phi_2(\kappa(\gamma))}\\ &\leq e^{Cs}\left(\sum_{\gamma\in\Gamma} e^{-s\phi_1(\kappa(\gamma))}\right)^t\left(\sum_{\gamma\in\Gamma} e^{-s\phi_2(\kappa(\gamma))}\right)^{1-t}.\end{aligned}$$

So Equation (1) implies that

$$\delta^\phi(\Gamma) \leq 1.$$

In addition, it is straightforward that $\sup_{\gamma\in\Gamma}|\phi(\kappa(\gamma)) - \phi_i(\kappa(\gamma))| < +\infty$ for $i = 1, 2$ implies $\delta^\phi(\Gamma) = 1$, and therefore the theorem reduces to the following.

Claim: If $\delta^\phi(\Gamma) = 1$, then $\sup_{\gamma\in\Gamma}|\phi(\kappa(\gamma)) - \phi_i(\kappa(\gamma))| < +\infty$ for $i = 1, 2$.

Assume $\delta^\phi(\Gamma) = 1$. Let μ be the unique $(\Gamma, \phi, 1)$-Patterson–Sullivan measure on $\Lambda_\theta(\Gamma)$ (see Theorem 3.9) and let $f : \Lambda_\theta(\Gamma) \to \mathcal{F}_\Theta$ denote the map in Theorem 8.1. Then $f_*\mu$ is a $(\Gamma, \phi, 1)$-Patterson–Sullivan measure on $\mathcal{F}_\Theta \subset \partial_\Theta X$. Next for $i = 1, 2$ let μ_i be a $(\Gamma, \phi_i, 1)$-Patterson–Sullivan measure on $\partial_\Theta X$ (see Proposition 4.3).

By Theorems 4.5 and 8.1, the measures $f_*\mu$, μ_1, and μ_2 are each part of a Patterson–Sullivan system on $\partial_\Theta X$. In particular, these measures satisfy the Shadow Lemma (Proposition 2.5). Then we have that for any large $R > 0$, there exists $C_0 = C_0(R) > 1$ such that for all $\gamma \in \Gamma$ and $i = 1, 2$,

$$(f_*\mu)(\mathcal{O}_R^\Theta(\gamma)) \leq C_0 e^{-\phi(\kappa(\gamma))} \quad \text{and} \quad \mu_i(\mathcal{O}_R^\Theta(\gamma)) \geq C_0^{-1}e^{-\phi_i(\kappa(\gamma))}.$$

Hence, it follows from $\phi(\kappa(\gamma)) \geq t\phi_1(\kappa(\gamma)) + (1-t)\phi_2(\kappa(\gamma)) - C$ that for all $\gamma \in \Gamma$,

$$\begin{aligned}(f_*\mu)(\mathcal{O}_R^\Theta(\gamma)) &\leq C_0^2 e^C \mu_1(\mathcal{O}_R^\Theta(\gamma))^t \mu_2(\mathcal{O}_R^\Theta(\gamma))^{1-t}\\ &\leq C_0^2 e^C\left(t\mu_1(\mathcal{O}_R^\Theta(\gamma)) + (1-t)\mu_2(\mathcal{O}_R^\Theta(\gamma))\right)\\ &\leq C_0^2 e^C(\mu_1+\mu_2)(\mathcal{O}_R^\Theta(\gamma))\end{aligned}$$

where the weighted arithmetic-mean geometric-mean inequality is used in the second inequality.

By Proposition 8.3, we then have

$$f_*\mu \ll \mu_1 + \mu_2.$$

In particular, at least one of μ_1 or μ_2 is non-singular to $f_*\mu$. After relabeling, we can suppose that μ_1 is non-singular to $f_*\mu$. Then by Theorem 2.3, we have

$$\sup_{\gamma\in\Gamma}|\phi(\kappa(\gamma)) - \phi_1(\kappa(\gamma))| < +\infty.$$

This implies that there there exists $C' > 0$ such that

$$\phi(\kappa(\gamma)) \geq \phi_2(\kappa(\gamma)) - C'$$

for all $\gamma \in \Gamma$. Then using the Shadow Lemma and Proposition 8.3 as above, we have

$$f_*\mu \ll \mu_2.$$

Therefore by Theorem 2.3,

$$\sup_{\gamma\in\Gamma} |\phi(\kappa(\gamma)) - \phi_2(\kappa(\gamma))| < +\infty.$$

This completes the proof. □

10.2. **Proof of Theorem 10.1 in general.** We now prove the theorem in full generality. The idea is to replace G with a different group where Γ is contracting and strongly irreducible.

We will freely use the notation in Section 3.3. For each $\alpha \in \Theta$, let $\mathsf{H}_\alpha < \mathsf{SL}(V_\alpha)$ denote the Zariski closure of $\Phi_\alpha(\Gamma) < \mathsf{SL}(V_\alpha)$. Replacing Γ with a finite index subgroup, we can assume that each H_α is connected.

Lemma 10.3. *For each $\alpha \in \Theta$ there exists an irreducible representation $\rho_\alpha : \mathsf{H}_\alpha \to \mathsf{SL}(W_\alpha)$ such that*

(1) There exist $C_\alpha > 0$ and $r_\alpha \in \mathbb{N}$ where

$$|\omega_1(\kappa(\rho_\alpha(h))) - r_\alpha \omega_1(\kappa(h))| \leq C_\alpha$$

for all $h \in \mathsf{H}_\alpha$.

(2) $\rho_\alpha(\mathsf{H}_\alpha)$ is P_{α_1}-contracting.

When $\alpha \in \theta$, we can assume that $r_\alpha = 1$ and $W_\alpha < V_\alpha$ is an irreducible factor of the H_α-action on V_α.

Remark 10.4. In the lemma, we assume that W_α is endowed with some inner product (the properties do not depend on the choice).

Proof. Fix $\alpha \in \Theta$, let $d = \dim V_\alpha$, and identify $\mathsf{SL}(V_\alpha)$ with $\mathsf{SL}(d, \mathbb{R})$ using an orthogonal basis of V_α. Conjugating H_α (this is what introduces the additive error), we can assume that the Cartan subgroup A_α of H_α consists of positive diagonal matrices and the maximal compact subgroup of H_α is a subgroup of $\mathsf{SO}(d)$.

Fix $r \in \mathbb{N}$ minimally so that $\alpha_r(\kappa(a)) > 0$ for some $a \in \mathsf{A}_\alpha$. Consider the standard representation $\tau : \mathsf{SL}(d, \mathbb{R}) \to \mathsf{SL}(\wedge^r \mathbb{R}^d)$, and identify $\mathsf{SL}(\wedge^r \mathbb{R}^d)$ with $\mathsf{SL}(\binom{d}{r}, \mathbb{R})$. Notice that if $a \in \mathsf{A}_\alpha$, then the r largest entries along the diagonal in a are equal to $e^{\omega_1(\kappa(a))}$. Hence $\omega_1(\kappa(\tau(a))) = r\omega_1(\kappa(a))$. Thus $\omega_1(\kappa(\tau(h))) = r\omega_1(\kappa(h))$ for all $h \in \mathsf{H}_\alpha$. Further, by the choice of r, there exists $a \in \mathsf{A}_\alpha$ so that

$$\alpha_1(\kappa(\tau(a))) = \alpha_r(\kappa(a)) > 0$$

and hence

$$\alpha_1(\kappa(\tau(a^n))) \to +\infty.$$

Finally, since H_α is semisimple, we can decompose $\wedge^r \mathbb{R}^d$ into $\tau|_{\mathsf{H}_\alpha}$-irreducible factors and then pick the factor $W_\alpha \subset \wedge^r \mathbb{R}^d$ where the associated representation $\rho : \mathsf{H}_\alpha \to \mathsf{SL}(W_\alpha)$ satisfies

$$\omega_1(\kappa(\rho(h))) = \omega_1(\kappa(\tau(h)))$$

for all $h \in \mathsf{H}_\alpha$. Then $\rho_\alpha := \rho$ and $r_\alpha := r$ have the desired properties.

Notice that when $\alpha \in \theta$, then $r = 1$ by the divergence property of transverse groups, from which the last claim follows. □

Let $\bar{\mathsf{G}} := \prod_{\alpha\in\Theta} \mathsf{SL}(W_\alpha)$ and $\rho := (\rho_\alpha \circ \Phi_\alpha|_\Gamma)_{\alpha\in\Theta} : \Gamma \to \bar{\mathsf{G}}$ using the representations in Lemma 10.3, and let

$$\bar{\Gamma} := \rho(\Gamma).$$

We can assume that the simple roots of $\bar{\mathsf{G}}$ are $\cup_{\alpha\in\Theta}\{\beta_j^\alpha\}_{j=1}^{d_\alpha-1}$ where $\beta_1^\alpha,\dots,\beta_{d_\alpha-1}^\alpha$ are the standard simple roots of $\mathsf{SL}(W_\alpha)=\mathsf{SL}(d_\alpha,\mathbb{R})$ described in Section 3.3. Notice that we can choose the irreducible representations $\Phi_{\beta_1^\alpha}$ for $\bar{\mathsf{G}}$ to just be projection onto the associated factor. Then let

$$\bar{\Theta}:=\{\beta_1^\alpha\}_{\alpha\in\Theta}\quad\text{and}\quad\bar{\theta}:=\{\beta_1^\alpha\}_{\alpha\in\theta}.$$

Lemma 10.5. *The group $\bar{\Gamma}$ is $\mathsf{P}_{\bar{\Theta}}$-contracting, strongly $(\Phi_\beta)_{\beta\in\bar{\Theta}}$-irreducible, and $\bar{\Gamma}$ is $\mathsf{P}_{\bar{\theta}}$-transverse.*

Proof. First notice that the strong $(\Phi_\beta)_{\beta\in\bar{\Theta}}$-irreducibility follows from the irreducibility of ρ_α and that H_α is the Zariski closure of $\Phi_\alpha(\Gamma)$. Moreover, since H_α is the identity component of the Zariski closure of $\Phi_\alpha(\Gamma)$, it follows from Lemma 10.3(2) and [BQ16, Lemma 6.23] that for each $\beta\in\bar{\Theta}$, there exists a sequence $\{\gamma_n\}\subset\Gamma$ satisfying

$$\beta(\kappa(\rho(\gamma_n)))\to+\infty.$$

Now by [BQ16, Lemma 6.25], the sequence $\{\gamma_n\}\subset\Gamma$ can be chosen independent of the choice of β, i.e., there exists a sequence $\{\gamma_n\}\subset\Gamma$ such that for each $\beta\in\bar{\Theta}$,

$$\beta(\kappa(\rho(\gamma_n)))\to+\infty.$$

Hence, $\bar{\Gamma}$ is $\mathsf{P}_{\bar{\Theta}}$-contracting.

It remains to prove that $\bar{\Gamma}$ is $\mathsf{P}_{\bar{\theta}}$-transverse. For $\alpha\in\theta$, note that W_α can be chosen as an irreducible factor of the H_α-action on V_α and $r_\alpha=1$ in Lemma 10.3. We then have

$$\beta(\kappa(\rho(\gamma_n)))\to+\infty$$

for all $\beta\in\bar{\theta}$ and an escaping sequence $\{\gamma_n\}\subset\Gamma$, since Γ is P_θ-transverse. In addition, we also have that $\bar{\mathsf{G}}/\mathsf{P}_{\bar{\theta}}=\prod_{\alpha\in\theta}\mathbb{P}(W_\alpha)$, which is a subspace of $\prod_{\alpha\in\theta}\mathbb{P}(V_\alpha)$. Since the limit set of $(\Phi_\alpha)_{\alpha\in\theta}(\Gamma)<\prod_{\alpha\in\theta}\mathsf{SL}(V_\alpha)$ is transverse by Property (R3), this implies that the limit set of $\bar{\Gamma}$ in $\bar{\mathsf{G}}/\mathsf{P}_{\bar{\theta}}$ is transverse as well. □

Since $\phi\in\mathfrak{a}_\theta^*=\operatorname{span}\{\omega_\alpha\}_{\alpha\in\theta}$, we can write $\phi=\sum_{\alpha\in\theta}c_\alpha\omega_\alpha$. Let

$$\bar{\phi}:=\sum_{\alpha\in\theta}\frac{c_\alpha}{r_\alpha}\omega_{\beta_1^\alpha}\in\mathfrak{a}_{\bar{\theta}}^*.$$

Then there exists $C>0$ such that

$$\left|\phi(\kappa(\gamma))-\bar{\phi}(\kappa(\rho(\gamma)))\right|\le C\quad\text{for all }\gamma\in\Gamma.$$

Likewise, we can define $\bar{\phi}_1,\bar{\phi}_2\in\mathfrak{a}_{\bar{\Theta}}^*$ corresponding to ϕ_1,ϕ_2, respectively. Then after increasing $C>0$ we can assume that

$$\left|\phi_i(\kappa(\gamma))-\bar{\phi}_i(\kappa(\rho(\gamma)))\right|\le C\quad\text{for all }\gamma\in\Gamma.$$

Finally, applying the special case of the theorem to $\bar{\Gamma}$ and $\bar{\phi},\bar{\phi}_1,\bar{\phi}_2$ finishes the proof. □

11. Lipschitz limit sets

In this section, we prove the following entropy rigidity result for Anosov groups with Lipschitz limit sets.

Theorem 11.1. *Suppose $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is a P_1-Anosov group acting strongly irreducibly on $\mathbb{R}^d$ and on $\wedge^{p+1}\mathbb{R}^d$ whose limit set $\Lambda_1(\Gamma)$ is a Lipschitz p-manifold, for some $p \le d-2$. Then*

$$\delta^{\phi_{\mathrm{H}}}(\Gamma) = p,$$

if and only if Γ is conjugate to a uniform lattice in $\mathsf{SO}(d-1,1)$ and $p = d-2$.

We will freely use the notation in Section 3.3. In addition, notice that the Jordan projection $\lambda : \mathsf{SL}(d,\mathbb{R}) \to \mathfrak{a}^+$ is given by

$$\lambda(g) = \mathrm{diag}(\log\lambda_1(g), \dots, \log\lambda_d(g))$$

where

$$\lambda_1(g) \ge \cdots \ge \lambda_d(g)$$

are the absolute values of the eigenvalues of g. Also, we can choose the representation Φ_{α_j} to be the standard irreducible representation $\mathsf{SL}(d,\mathbb{R}) \to \mathsf{SL}(\wedge^j \mathbb{R}^d)$.

In terms of the fundamental weights, the Hilbert functional $\phi_{\mathrm{H}} \in \mathfrak{a}^*$ satisfies

$$\phi_{\mathrm{H}} = \frac{1}{2}(\omega_1 + \omega_{d-1}).$$

Let $\phi_p := (p+1)\omega_1 - \omega_{p+1}$ and $\bar\phi_p := (p+1)\omega_{d-1} - \omega_{d-p-1}$. Equation (5) implies that

$$\bar\phi_p(\kappa(g)) = \phi_p(\kappa(g^{-1}))$$

for all $g \in \mathsf{SL}(d,\mathbb{R})$, so we have

$$\delta^{\bar\phi_p}(\Gamma) = \delta^{\phi_p}(\Gamma)$$

for every subgroup $\Gamma < \mathsf{SL}(d,\mathbb{R})$.

We use the following result of Pozzetti–Sambarino–Wienhard.

Theorem 11.2 ([PSW23, Theorem A])**.** *Suppose $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is a P_1-Anosov group acting strongly irreducibly on $\mathbb{R}^d$ and on $\wedge^{p+1}\mathbb{R}^d$ for some $p \le d-2$. If $\Lambda_1(\Gamma)$ is a Lipschitz p-manifold, then*

$$\delta^{\phi_p}(\Gamma) = \delta^{\bar\phi_p}(\Gamma) = 1.$$

Remark 11.3. Theorem A in [PSW23] does not include the assumption that Γ acts strongly irreducibly on $\wedge^{p+1}\mathbb{R}^d$, however the proof of [PSW23, Lemma 6.8] uses [Lab06, Proposition 10.3] which appears to be false. Assuming this irreducibility allows one to avoid this citation in the proof.

The functions ϕ_{H}, ϕ_p, and $\bar\phi_p$ have the following relation.

Lemma 11.4. *If $X = \mathrm{diag}(t_1, \dots, t_d) \in \mathfrak{a}^+$, then*

$$p\phi_{\mathrm{H}}(X) \ge \frac{1}{2}(\phi_p + \bar\phi_p)(X)$$

with equality if and only if $t_2 = \cdots = t_{d-1}$.

Proof. Note

$$\begin{aligned}\frac{1}{2}(\phi_p + \bar\phi_p)(X) &= \frac{p+1}{2}(t_1 - t_d) - \frac{1}{2}(t_1 + \cdots + t_{p+1} - t_{d-p} - \cdots - t_d)\\ &= \frac{p}{2}(t_1 - t_d) - \frac{1}{2}(t_2 - t_{d-1}) - \frac{1}{2}(t_3 - t_{d-2}) - \cdots - \frac{1}{2}(t_r - t_{d-r+1})\end{aligned}$$

where $r := \min\{\lfloor d/2 \rfloor, p+1\}$. So

$$p\phi_{\mathrm{H}}(X) \geq \frac{1}{2}(\phi_p + \bar{\phi}_p)(X)$$

with equality if and only if $t_2 = t_{d-1}$, $t_3 = t_{d-2}$,..., and $t_r = t_{d-r+1}$. Since $t_2 \geq \cdots \geq t_{d-1}$ the result follows. □

11.1. **Proof of Theorem 11.1.** Suppose $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is a P_1-Anosov group such that $\Lambda_1(\Gamma)$ is a Lipschitz p-manifold and Γ acts strongly irreducibly on $\mathbb{R}^d$ and on $\wedge^{p+1}\mathbb{R}^d$. By [BCLS15, Corollary 2.20], Γ has semisimple Zariski closure, and hence each $\Phi_\alpha(\Gamma)$ has semisimple Zariski closure. This implies that Γ satisfies the first line of Theorem 10.1 with

$$\theta = \{\alpha_1, \alpha_{d-1}\} \quad \text{and} \quad \Theta = \{\alpha_1, \alpha_{p+1}, \alpha_{d-p-1}, \alpha_{d-1}\}.$$

Further, $\phi_{\mathrm{H}} \in \mathfrak{a}_\theta^*$ and $\phi_p, \bar{\phi}_p \in \mathfrak{a}_\Theta^*$.

Theorem 11.2 implies that

$$\delta^{\phi_p}(\Gamma) = \delta^{\bar{\phi}_p}(\Gamma) = 1.$$

Then Theorem 10.1 and Lemma 11.4 imply that

$$\delta^{\phi_{\mathrm{H}}}(\Gamma) = p\delta^{p\phi_{\mathrm{H}}}(\Gamma) \leq p \cdot 1 = p$$

with equality if and only if

$$\phi_p(\lambda(\gamma)) = \bar{\phi}_p(\lambda(\gamma)) = p\phi_{\mathrm{H}}(\lambda(\gamma)) \tag{14}$$

for all $\gamma \in \Gamma$.

The backward direction of the theorem is clear, so suppose for the rest of the section that $\delta^{\phi_{\mathrm{H}}}(\Gamma) = p$. Then Lemma 11.4 and Equation (14) imply that

$$\lambda_2(\gamma) = \cdots = \lambda_{d-1}(\gamma) \tag{15}$$

for all $\gamma \in \Gamma$.

Lemma 11.5. $\lambda_2(\gamma) = \cdots = \lambda_{d-1}(\gamma) = 1$ *for all* $\gamma \in \Gamma$.

Proof. Fix $\gamma \in \Gamma$. Then Equations (14) and (15) imply that

$$p \log \lambda_1(\gamma) - p \log \lambda_2(\gamma) = \frac{p}{2}(\log \lambda_1(\gamma) - \log \lambda_d(\gamma)).$$

So

$$\log \lambda_2(\gamma) = \frac{1}{2}(\log \lambda_1(\gamma) + \log \lambda_d(\gamma)).$$

On the other hand

$$\log \lambda_1(\gamma) + (d-2)\log \lambda_2(\gamma) + \log \lambda_d(\gamma) = 0.$$

So

$$\frac{d}{2}\left(\log \lambda_1(\gamma) + \log \lambda_d(\gamma)\right) = 0,$$

which implies that $\lambda_d(\gamma) = \lambda_1(\gamma)^{-1}$ and hence

$$\lambda_2(\gamma) = \cdots = \lambda_{d-1}(\gamma) = 1.$$ □

To finish the proof we use the following observation, which is a consequence of a theorem of Benoist and the basic theory of irreducible representations of semisimple Lie groups.

Observation 11.6. If $d \geq 3$, $G < \mathsf{SL}(d,\mathbb{R})$ is a strongly irreducible P_1-contracting subgroup, and

$$\lambda_2(g) = \cdots = \lambda_{d-1}(g) = 1$$

for all $g \in G$, then G is conjugate to a Zariski dense subgroup of $\mathsf{SO}_0(d-1,1)$ or $\mathsf{SO}(d-1,1)$.

Delaying the proof of the observation, we finish the proof of the theorem. Observation 11.6 implies that Γ is conjugate into $\mathsf{SO}(d-1,1)$. Next consider the Klein-Beltrami model $\mathbb{H}^{d-1} \subset \mathbb{P}(\mathbb{R}^d)$ of real hyperbolic $(d-1)$-space. Then $\mathsf{SO}(d-1,1) \to \mathsf{Isom}(\mathbb{H}^d)$ is a finite cover and the limit set $\Lambda_1(\Gamma) \subset \mathbb{P}(\mathbb{R}^d)$ coincides with the hyperbolic limit set of the image of Γ in $\mathsf{Isom}(\mathbb{H}^d)$. Then, since $\Lambda_1(\Gamma)$ is a Lipschitz p-manifold, [Kap09, Theorem 1.3] implies that Γ preserves and acts cocompactly on a totally geodesic copy of $\mathbb{H}^{p+1}$ inside $\mathbb{H}^{d-1}$. Since Γ is strongly irreducible, we must have $\mathbb{H}^{p+1} = \mathbb{H}^{d-1}$. Hence $p = d-2$ and Γ is a uniform lattice in $\mathsf{SO}(d-1,1)$. □

11.2. **Proof of Observation 11.6.** Let G' be the image of G in $\mathsf{PSL}(d,\mathbb{R})$, let H denote the Zariski closure of G' in $\mathsf{PSL}(d,\mathbb{R})$, and let $\mathsf{H}^0 < \mathsf{H}$ denote the connected component of the identity. By [BCLS15, Lemma 2.18], H^0 is a connected semisimple Lie group with trivial center and no compact factors. By a theorem of Benoist [Ben97],

$$\lambda_2(h) = \cdots = \lambda_{d-1}(h) = 1$$

for all $h \in \mathsf{H}^0$. Thus H^0 is a rank one non-compact simple group. Let X be the symmetric space associated to H^0 and let $\rho : \mathsf{H}^0 \to \mathsf{Isom}(X)$ be the induced map. Since H^0 has trivial center, ρ induces an isomorphism between H^0 and $\mathsf{Isom}_0(X)$, the connected component of the identity in $\mathsf{Isom}(X)$. Further, X is a negatively curved symmetric space, the geodesic boundary has an $\mathsf{Isom}(X)$-invariant smooth structure, and there exists a ρ^{-1}-equivariant smooth embedding $\xi : \partial X \hookrightarrow \mathbb{P}(\mathbb{R}^d)$ of the boundary of X (for details about the construction of ξ, see for instance [ZZ24b, Section 4]).

Lemma 11.7. *$X = \mathbb{H}^m$ is the real hyperbolic m-space, $m = \dim X$.*

Proof. Suppose $\gamma \in \mathsf{Isom}(X)$ is loxodromic, i.e. γ has no fixed points in X and has two fixed points $x^{\pm}$ in ∂X. Then the eigenvalue condition implies that all eigenvalues of the derivative $d(\gamma)_{x^\pm} : T_{x^\pm}\partial X \to T_{x^\pm}\partial X$ have the same modulus. From the description of the negatively curved symmetric spaces in [Mos73, Chapther 19], this is only possible if X is a real hyperbolic space. □

Now we can identify $\mathsf{Isom}(X)$ with $\mathsf{PO}(m,1)$ and view ρ^{-1} as an irreducible representation of $\mathsf{PO}_0(m,1)$, the connected component of the identity in $\mathsf{PO}(m,1)$. It then follows from the eigenvalue condition and the theory of highest weights (see for instance [ZZ24a, Lemma 10.4]) that $m = d-1$ and $\mathsf{H}^0 = \rho^{-1}(\mathsf{PO}_0(d-1,1))$ is conjugate to $\mathsf{PO}_0(d-1,1)$. So, after conjugating, we can assume that $\mathsf{H}^0 = \mathsf{PO}_0(d-1,1)$.

Next let $\widehat{\mathsf{H}}$ be the normalizer of $\mathsf{PO}_0(d-1,1)$ in $\mathsf{PGL}(d,\mathbb{R})$ and let $\tau : \widehat{\mathsf{H}} \to \mathsf{Aut}(\mathsf{PO}_0(d-1,1))$ be the map induced by conjugation. By Schur's lemma, τ is injective. Further, $\tau|_{\mathsf{PO}(d-1,1)}$ is onto. Hence $\mathsf{H} < \widehat{\mathsf{H}} = \mathsf{PO}(d-1,1)$. This implies that G is a Zariski dense subgroup of $\mathsf{SO}_0(d-1,1)$ or $\mathsf{SO}(d-1,1)$. □

12. Representations of mapping class groups and boundary maps

In this section, we apply our theory to representations of mapping class groups. Let S be a connected orientable surface of finite type with negative Euler characteristic. Its mapping class group $\mathsf{Mod}(S)$ is the group of isotopy classes of orientation-preserving homoeomorphisms on S.

Let $\mathcal{PML}$ denote the projective space of measured laminations on S, on which the mapping class group $\mathsf{Mod}(S)$ naturally acts. The space $\mathcal{PML}$ is homeomorphic to a sphere and admits a Lebesgue measure class. We show that representations of $\mathsf{Mod}(S)$ transfer this measure structure on $\mathcal{PML}$ to partial flag manifolds.

Theorem 12.1. *Suppose $\rho : \mathsf{Mod}(S) \to \mathsf{G}$ is a representation such that $\rho(\mathsf{Mod}(S))$ is P_θ-contracting and strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible. Then there exists a unique* Leb-*a.e. defined ρ-equivariant map*

$$f : \mathcal{PML} \to \mathcal{F}_\theta .$$

Moreover, $f(x) \in \Lambda_\theta^{\mathrm{concon}}(\rho(\mathsf{Mod}(S)))$ for Leb-*a.e. $x \in \mathcal{PML}$.*

Proof. We now deduce Theorem 12.1 from Theorem 6.1 and the universal amenability of the $\mathsf{Mod}(S)$-action on the boundary of the curve graph $\mathcal{C}(S)$ of S, shown by Hamenstädt [Ham09a]. The curve graph $\mathcal{C}(S)$ is the simplicial graph whose vertices are isotopy classes of essential simple closed curves on S, and two of them are connected by an edge if they have disjoint representatives. There is a natural action of $\mathsf{Mod}(S)$ on $\mathcal{C}(S)$ by isometries, and Masur and Minsky showed that $\mathcal{C}(S)$ is Gromov hyperblic [MM99]. In particular, the $\mathsf{Mod}(S)$-action on $\mathcal{C}(S)$ continuously extends to the Gromov boundary $\partial\,\mathcal{C}(S)$, and Hamenstädt proved that the $\mathsf{Mod}(S)$-action on $\partial\,\mathcal{C}(S)$ is amenable with resepect to any quasi-invariant Borel measure.

We now relate the above discussion to $\mathcal{PML}$. Klarreich [Kla22] and Hamenstädt [Ham06] characterized $\partial\,\mathcal{C}(S)$ as the space of (unmeasured) filling geodesic laminations on S. Hence, denoting by $\mathcal{FML} \subset \mathcal{PML}$ the projective space of filling measured laminations on S, there exists a measure-forgetting map

$$\pi : \mathcal{FML} \to \partial\,\mathcal{C}(S)$$

which is continuous [Ham09b, Lemma 3.12]. In particular, on the space $\mathcal{UE} \subset \mathcal{PML}$ of uniquely ergodic measured laminations, whose elements have unique transverse measures up to scaling, the map π is injective.

By Masur [Mas82] and Veech [Vee82], $\mathcal{UE}$ is Leb-conull. Therefore, the universal amenability of the $\mathsf{Mod}(S)$-action on $\partial\,\mathcal{C}(S)$ implies that the amenabiilty of the $\mathsf{Mod}(S)$-action on $(\mathcal{UE}, \mathrm{Leb})$.

In addition, $\mathcal{UE} \subset \mathcal{PML}$ is also embedded in the Gardiner–Masur boundary $\partial_{GM}\,\mathcal{T}(S)$ [GM91, Miy13] of the Teichmüller space $\mathcal{T}(S)$ of S. In our earlier work [KZ25, Theorem 10.1], we showed that the $\mathsf{Mod}(S)$-action on $\partial_{GM}\,\mathcal{T}(S)$ equipped with the pushforward of Leb under the embedding $\mathcal{UE} \hookrightarrow \partial_{GM}\,\mathcal{T}(S)$ is a well-behaved Patterson–Sullivan system whose conical limit set is conull. Therefore, we can apply Theorem 6.1 and then Theorem 12.1 follows, except for the "moreover" part.

To show the "moreover" part, we first note that the hierarchy involved in the associated well-behaved PS-system is not trivial (see [KZ25, Section 10] for explicit description of shadows and hierarchy), and hence the part (3) of Theorem 6.1 does not apply directly. Nevertheless, it was shown in [Cou24, Section 5.3] that its shadows induced from the hierarchy satisfy the Kochen–Stone Lemma (Lemma

6.7). Hence, in the proof of the part (3) of Theorem 6.1, one can replace shadows with mixed shadows and deduce the desired property. □

13. Gromov hyperbolic spaces with exponentially bounded geometry

In this section, we apply our framework to a Gromov hyperbolic space when it has exponentially bounded geometry.

Definition 13.1. A proper geodesic Gromov hyperbolic space (X, d_X) is said to have *exponentially bounded geometry* if there exist $a, r > 0$ such that for any $x \in X$ and $R > 0$, the radius R-ball $B_X(x, R) \subset X$ centered at x can contain at most a^R number of pairwise disjoint balls of radius r.

In the rest of this section, we fix a proper geodesic Gromov hyperbolic space (X, d_X) with exponentially bounded geometry, and a discrete subgroup $\Gamma < \mathsf{Isom}(X)$ of isometries, i.e., the Γ-action on X is proper. In this case, the critical exponent $\delta_X(\Gamma)$ of the Poincaré series

$$s \mapsto \sum_{\gamma \in \Gamma} e^{-s\, \mathrm{d}_X(o, \gamma o)}$$

is finite [Kai04, Proposition 1.29], for any fixed basepoint $o \in X$.

We denote by ∂X the Gromov boundary of X and by $\Lambda(\Gamma) \subset \partial X$ the limit set of Γ, the set of accumulation points of $\Gamma o \subset X$. We further assume that $\#\Lambda(\Gamma) \geq 3$, that is, Γ is non-elementary. We then have

$$0 < \delta_X(\Gamma) < +\infty,$$

and that the Γ-action on $\Lambda(\Gamma) \subset \partial X$ is a minimal convergence action.

The *Busemann coarse-cocycle* $\beta : \Gamma \times \Lambda(\Gamma) \to \mathbb{R}$ is defined by

$$\beta(\gamma, x) := \limsup_{p \to x} \mathrm{d}_X(\gamma^{-1} o, p) - \mathrm{d}_X(o, p).$$

In [Coo93], Coornaert constructed a coarse $(\Gamma, \beta, \delta_X(\Gamma))$-Patterson–Sullivan measure in the sense of Equation (3).

As an application of our theory, we also show that representations of Γ admit measurable boundary maps.

Theorem 13.2. *Suppose (X, d_X) and Γ are as above, $\sum_{\gamma \in \Gamma} e^{-\delta_X(\Gamma)\, \mathrm{d}_X(o, \gamma o)} = +\infty$, and μ is a coarse $(\Gamma, \beta, \delta_X(\Gamma))$-PS measure. If $\rho : \Gamma \to \mathsf{G}$ is a representation such that $\rho(\Gamma)$ is P_θ-contracting and strongly $(\Phi_\alpha)_{\alpha \in \theta}$-irreducible, then there exists a unique μ-a.e. defined ρ-equivariant map*

$$f : \Lambda(\Gamma) \to \mathcal{F}_\theta .$$

Moreover, $f(x) \in \Lambda_\theta^{\mathrm{concon}}(\rho(\Gamma))$ for μ-a.e. $x \in \Lambda(\Gamma)$.

Proof. It is easy to verify that this coarse PS-measure gives rise to a well-behaved Patterson–Sullivan system with respect to the trivial hierarchy $\mathcal{H}(R) \equiv \Gamma$, see [KZ25, Section 8].

It is also a straightforward application of the Borel–Cantelli Lemma to verify that the assumption $\sum_{\gamma \in \Gamma} e^{-\delta_X(\Gamma)\, \mathrm{d}_X(o, \gamma o)} = +\infty$ implies that

$$\mu(\Lambda^{\mathrm{con}}(\mathcal{H})) = 1,$$

see for instance [BCZZ24, Proposition 7.1]. Moreover, Kaimanovich showed that the Γ-action on $\Lambda(\Gamma)$ is amenable with respect to any Γ-quasi-invariant Borel measure [Kai04, Theorem 3.15]. Therefore, Theorem 6.1 applies.

□

14. Proving everything claimed in the introduction

In this last section, we explain why all of the the statements in the introduction are true. We first note that Zariski dense subgroups of G are strongly $(\Phi_\alpha)_{\alpha\in\Delta}$-irreducible (Remark 3.6) and P_Δ-contracting (see Section 6).

If $\Gamma < \mathsf{G}$ is a non-elementary P_θ-Anosov group, then Γ is P_θ-transverse. Moreover in this case, $\Lambda_\theta(\Gamma) = \Lambda_\theta^{\rm con} = \Lambda_\theta^{\rm concon}(\Gamma)$. Hence, for any $\phi \in \mathfrak{a}_\theta^*$ with $\delta^\phi(\Gamma) < +\infty$, the existence of a $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure on $\Lambda_\theta(\Gamma)$ and the Shadow Lemma imply $\sum_{\gamma\in\Gamma} e^{-\delta^\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty$ [Qui02, Sam24].

- Theorem 1.2 is a special case of Theorem 6.1.
- Theorem 1.4 follows from Theorem 8.1 and Proposition 4.4.
- Theorem 1.7 is a special case of Corollary 8.2.
- Theorem 1.8 is a special case of Theorem 9.1, together with Equation (13).
- Theorem 1.11 follows from Theorem 10.1.
- Theorem 1.12 is a special case of Corollary 10.2.
- Theorem 1.14 is proved as Theorem 11.1.

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