VECTOR-VALUED HOROFUNCTION BOUNDARIES AND PATTERSON–SULLIVAN MEASURES

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ABSTRACT. In higher rank, there is a well-studied theory of Patterson–Sullivan measures supported on partial flag manifolds. However, establishing the existence and uniqueness of such measures is a difficult question. In this paper, we develop a theory for Patterson–Sullivan measures supported on certain vector-valued horofunction boundaries of the associated symmetric space, where existence is straightforward. We also introduce a notion of shadows for this compactification and establish a shadow lemma. For transverse groups, we prove uniqueness and ergodicity results.

CONTENTS

1. INTRODUCTION

Throughout this paper G will be a semisimple Lie group with finite center and no compact factors. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ of the Lie algebra, a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$, and a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Then let $\Delta \subset \mathfrak{a}^*$ denote the system of simple restricted roots corresponding to the choice of $\mathfrak{a}^+$.

Given a subset $\theta \subset \Delta$, let $\mathsf{P}_\theta < \mathsf{G}$ denote the associated parabolic subgroup and let $\mathcal{F}_\theta := \mathsf{G} / \mathsf{P}_\theta$ denote the associated partial flag manifold. There is a natural vector-valued cocycle $B_\theta^{IW} : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ called the *(partial) Iwasawa cocycle*, where $\mathfrak{a}_\theta \subset \mathfrak{a}$ is the *partial Cartan subspace*. This cocycle can be used to define Patterson–Sullivan measures as follows.

Definition 1.1. Given a subgroup $\Gamma < \mathsf{G}$, $\theta \subset \Delta$ non-empty, a functional $\phi \in \mathfrak{a}_\theta^*$, and $\delta \geq 0$, a Borel probability measure μ on $\mathcal{F}_\theta$ is a (Γ, ϕ, δ)-*Patterson–Sullivan*

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measure if for every $\gamma \in \Gamma$ the measures μ, $\gamma_*\mu$ are absolutely continuous and

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-\delta\phi B_\theta^{IW}(\gamma^{-1},x)} \quad \mu\text{-a.e.}$$

When G has rank one, the above definition (with an appropriate choice of functional) coincides with the classical Patterson–Sullivan measures introduced by Patterson [Pat76] and Sullivan [Sul79]. In higher rank, the above definition is due to Quint [Qui02].

For $\alpha \in \Delta$, let ω_α denote the associated fundamental weight. The dual space of the partial Cartan subspace $\mathfrak{a}_\theta \subset \mathfrak{a}$ can be identified with $\operatorname{span}\{\omega_\alpha\}_{\alpha\in\theta}$. Then given $\phi \in \mathfrak{a}_\theta^* = \operatorname{span}\{\omega_\alpha\}_{\alpha\in\theta}$, the *$\phi$-critical exponent* of a discrete subgroup $\Gamma < \mathsf{G}$ is the exponential growth rate

$$\delta^\phi(\Gamma) := \limsup_{T\to+\infty} \frac{1}{T}\log \#\{\gamma \in \Gamma : \phi(\kappa(\gamma)) \leq T\} \in [0,+\infty]. \tag{1}$$

When G has rank one, $\Delta = \{\alpha\}$ is a singleton and $\delta^{\omega_\alpha}(\Gamma)$ coincides with the classical symmetric space critical exponent. Further, there always exists a Patterson–Sullivan measure with dimension $\delta^{\omega_\alpha}(\Gamma)$. On the other hand, in higher rank, existence is much more subtle and for Zariski dense discrete subgroups the most general criteria for existence is due to Quint [Qui02] and involves a not very easy condition to check on the growth indicator function.

In this paper, we observe that the symmetric space X associated to G admits a vector-valued compactification $\overline{X}^\theta$ and there is a natural notion of Patterson–Sullivan measures on the boundary $\partial_\theta X$ of this compactification. Similar compactifications, but using Finsler metrics, appear in [KL18, HSWW17, LP23, Lem23].

We further show that the partial flag manifold $\mathcal{F}_\theta$ naturally embeds into $\partial_\theta X$ and under this embedding Patterson–Sullivan measures on $\mathcal{F}_\theta$ are sent to Patterson–Sullivan measures on $\partial_\theta X$. The boundary $\partial_\theta X$ is larger than $\mathcal{F}_\theta$, but has the advantage that existence of Patterson–Sullivan measures is straightforward to establish.

We further show that when Γ is sufficiently irreducible, these Patterson–Sullivan measures are part of a "Patterson–Sullivan system," an abstract notion introduced in our earlier work [KZ25]. When Γ is transverse, we show that these Patterson–Sullivan measures are part of a "well-behaved Patterson–Sullivan system."

One of the motivations for this work appears in a companion paper [KZ26], where we combine the theory developed here with our earlier work in [KZ25] to establish new strict convexity results for variations of critical exponent and a new entropy rigidity result.

1.1. **Compactifications.** We now state the results of this paper more precisely. Let $\mathsf{K} < \mathsf{G}$ denote the maximal compact subgroup with Lie algebra $\mathfrak{k}$. Every element $g \in \mathsf{G}$ can be written as $g = ke^{\kappa(g)}\ell$ for some $k, \ell \in \mathsf{K}$ and a unique $\kappa(g) \in \mathfrak{a}^+$. Then map $\kappa : \mathsf{G} \to \mathfrak{a}^+$ is called the *Cartan projection.*

Let $X := \mathsf{G}/\mathsf{K}$ denote the symmetric space associated to G and fix the basepoint $o := \mathsf{K} \in X$. For $x = go \in X$, define $b_x : X \to \mathfrak{a}$ by

$$b_x(ho) = \kappa(h^{-1}g) - \kappa(g).$$

We will verify that the set $\{b_x : x \in X\}$ is relatively compact in the space of continuous maps $X \to \mathfrak{a}$ and hence we can compactify X by taking the closure in this space, which we denote by $\overline{X}^\Delta$. We also let $\partial_\Delta X := \overline{X}^\Delta \smallsetminus X$.

More generally, given $\theta \subset \Delta$ non-empty there is a natural projection $\pi_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ we can use it to define a θ-boundary $\partial_\theta X := \{\pi_\theta \circ \xi : \xi \in \partial_\Delta X\}$. We then show that $\partial_\theta X$ naturally compactifies X.

Proposition 1.2 (see Proposition 4.2 below). *The space* $\overline{X}^\theta := X \sqcup \partial_\theta X$ *has a topology which makes it a compactification of* X*, that is* $\overline{X}^\theta$ *is a compact metrizable space and the inclusion* $X \hookrightarrow \overline{X}^\theta$ *is a topological embedding with open dense image.*

1.2. **Patterson–Sullivan measures.** There is a well-known definition of Patterson–Sullivan measures on the horofunction boundary of a metric space, see e.g. [LW10]. In the case of a vector-valued horofunction boundary, a choice of functional leads to a natural notion of Patterson–Sullivan measure in this compactification.

Definition 1.3. Given a subgroup $\Gamma < \mathsf{G}$, $\theta \subset \Delta$ non-empty, a functional $\phi \in \mathfrak{a}_\theta^*$, and $\delta \geq 0$, a Borel probability measure μ on $\partial_\theta X$ is a (Γ, ϕ, δ)-*Patterson–Sullivan measure* if for every $\gamma \in \Gamma$ the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$\frac{d\gamma_*\mu}{d\mu}(\xi) = e^{-\delta\phi\xi(\gamma o)} \quad \mu\text{-a.e.}$$

Using the horofunction-like definition of these compactifications, it is possible to use Patterson's original construction to build Patterson–Sullivan measures.

Proposition 1.4 (see Proposition 4.5 below). *Suppose* $\Gamma < \mathsf{G}$ *is discrete. If* $\phi \in \mathfrak{a}_\theta^*$ *and* $\delta^\phi(\Gamma) < +\infty$*, then there exists a* $(\Gamma, \phi, \delta^\phi(\Gamma))$*-Patterson–Sullivan measure on* $\partial_\theta X$*.*

We further show that Patterson–Sullivan measures on $\mathcal{F}_\theta$ are a special case of those on $\partial_\theta X$.

Proposition 1.5 (see Proposition 4.7 below). *There is a topological embedding* $\iota : \mathcal{F}_\theta \hookrightarrow \partial_\theta X$ *which satisfies*

$$\iota(x)(go) = B_\theta^{IW}(g^{-1}, x)$$

for all $x \in \mathcal{F}_\theta$ *and all* $g \in \mathsf{G}$*. Hence the pushforward of any Patterson–Sullivan measure on* $\mathcal{F}_\theta$ *(in the sense of Definition 1.1) is a Patterson–Sullivan measure on* $\partial_\theta X$ *(in the sense of Definition 1.3).*

We further show that these measures are parts of "Patterson–Sullivan systems" (see Section 2), which were introduced in our previous work [KZ25]. This allows us to use the theory of such systems developed there and in particular implies that these Patterson–Sullivan measures satisfy a version of the shadow lemma.

In a companion paper [KZ26], we use the fact that these measures are parts of "Patterson–Sullivan systems" to apply a version of Tukia's measurable boundary rigidity theorem for such systems (established in [KZ25]).

1.3. **Transverse groups.** Building on work in [CZZ24], we further establish uniqueness and ergodicity results for the class of transverse groups. Fix a non-empty $\theta \subset \Delta$. For simplicity in the introduction, we assume that θ is invariant under the opposition involution (see Equation (3)), and we only consider Zariski dense subgroups. Our results in fact hold for any $\theta \subset \Delta$ and under weaker irreducibility assumptions than Zariski density (see Theorem 6.3 below).

For a Zariski dense subgroup $\Gamma < \mathsf{G}$, there exists a unique closed Γ-minimal set $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$ called the *limit set* of Γ [Ben97]. Then the group Γ is P_θ*-transverse* if

- for any escaping $\{\gamma_n\} \subset \Gamma$ and $\alpha \in \theta$,
$$\alpha(\kappa(\gamma_n)) \to +\infty$$
and
- any distinct $x, y \in \Lambda_\theta(\Gamma)$ are transverse, i.e., the diagonal G-orbit $\mathsf{G} \cdot (x, y) \subset \mathcal{F}_\theta \times \mathcal{F}_\theta$ is open.

The notion of transverse groups is a higher rank generalization of rank one discrete subgroups, and all Anosov and relatively Anosov groups are transverse groups. Transverse groups are sometimes called regular antipodal groups.

An important property of a P_θ-transverse group $\Gamma < \mathsf{G}$ is that the natural Γ-action on the limit set $\Lambda_\theta(\Gamma)$ is a convergence action, see [KLP17, Theorem 4.16] or [CZZ26, Proposition 3.3]. Hence, the notion of *conical limit set* $\Lambda_\theta^{\mathrm{con}}(\Gamma) \subset \Lambda_\theta(\Gamma)$ is naturally defined.

In [CZZ24], Canary, Zhang, and the second author established a higher rank generalization of the classical Hopf–Tsuji–Sullivan dichotomy for transverse groups which implies uniqueness of Patterson–Sullivan measure when the associated Poincaré series diverges. Furthermore, they also showed that such a unique measure is supported on the conical limit set. For Zariski dense groups, uniqueness was extended to measures on $\mathcal{F}_\theta$ by the first author, Oh, and Wang [KOW25b].

Using this previous work and our realization of $\partial_\theta X$ as a part of Patterson–Sullivan system, we further extend the uniqueness of Patterson–Sullivan measure to the boundary $\partial_\theta X$.

Theorem 1.6 (see Theorem 6.3 below)**.** *Suppose $\theta \subset \Delta$ and $\Gamma < \mathsf{G}$ is a Zariski dense P_θ-transverse group. If $\phi \in \mathfrak{a}_\theta^*$, $\delta^\phi(\Gamma) < +\infty$, and*
$$\sum_{\gamma \in \Gamma} e^{-\delta^\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty,$$
then there is a unique $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure μ on $\partial_\theta X$, the Γ-action on $(\partial_\theta X, \mu)$ is ergodic, and
$$\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 1$$
(in particular, μ is supported on $\mathcal{F}_\theta$).

1.4. **Outline of Paper.** The first two sections of the paper are expository. In Section 2 we recall the definition and some properties of abstract Patterson–Sullivan systems, which were introduced in our earlier work [KZ25]. In Section 3, we fix the notation involving semisimple Lie groups that we will use throughout the paper.

In Section 4, we precisely define the compactifications $\overline{X}^\theta$ and establish some basic properties. In Section 5, we define shadows in these compactifications and use them to introduce the contracting conical limit set of a discrete subgroup. In Section 6, we show that Patterson–Sullivan measures on these compactifications satisfy the axioms of abstract Patterson–Sullivan systems and prove Theorem 1.6.

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2. Patterson–Sullivan systems

In this section, we recall the definition and some properties of abstract Patterson–Sullivan systems, which were introduced in our earlier work [KZ25]. The main idea in this previous work was to identify the key features of a group action on a probability space that allows one to extend the theory of Patterson–Sullivan measures. We note that a different framework for abstract Patterson–Sullivan-like measures was given in [BCZZ24b].

Given a compact metric space M, a subgroup $\Gamma < \mathsf{Homeo}(M)$, and $\kappa \geq 0$, a function $\sigma : \Gamma \times M \to \mathbb{R}$ is called a *κ-coarse-cocycle* if

$$|\sigma(\gamma_1\gamma_2, x) - (\sigma(\gamma_1, \gamma_2 x) + \sigma(\gamma_2, x))| \leq \kappa$$

for any $\gamma_1, \gamma_2 \in \Gamma$ and $x \in M$. Given such a coarse-cocycle and $\delta \geq 0$, a Borel probability measure μ on M is called *coarse* (Γ, σ, δ)*-Patterson–Sullivan measure* if there exists $C \geq 1$ such that for any $\gamma \in \Gamma$ the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$C^{-1} e^{-\delta\sigma(\gamma^{-1}, x)} \leq \frac{d\gamma_*\mu}{d\mu}(x) \leq C e^{-\delta\sigma(\gamma^{-1}, x)} \quad \text{for } \mu\text{-a.e. } x \in M. \tag{2}$$

When $C = 1$ and hence equality holds in Equation (2), we call μ a (σ, δ)*-Patterson–Sullivan measure*.

Now we recall the definition of Patterson–Sullivan systems.

Definition 2.1. A *Patterson–Sullivan-system (PS-system) of dimension* $\delta \geq 0$ consists of

- a coarse-cocycle $\sigma : \Gamma \times M \to \mathbb{R}$,
- a coarse (σ, δ)-Patterson–Sullivan measure (PS-measure) μ,
- for each $\gamma \in \Gamma$, a number $\|\gamma\|_\sigma \in \mathbb{R}$ called the *σ-magnitude of* γ, and
- for each $\gamma \in \Gamma$ and $R > 0$, a non-empty open set $\mathcal{O}_R(\gamma) \subset M$ called the *R-shadow of* γ

such that:

(PS1) For any $\gamma \in \Gamma$, there exists $c = c(\gamma) > 0$ such that $|\sigma(\gamma, x)| \leq c(\gamma)$ for μ-a.e. $x \in M$.

(PS2) For every $R > 0$ there is a constant $C = C(R) > 0$ such that

$$\|\gamma\|_\sigma - C \leq \sigma(\gamma, x) \leq \|\gamma\|_\sigma + C$$

for all $\gamma \in \Gamma$ and μ-a.e. $x \in \gamma^{-1}\mathcal{O}_R(\gamma)$.

(PS3) If $\{\gamma_n\} \subset \Gamma$, $R_n \to +\infty$, $Z \subset M$ is compact, and $[M \smallsetminus \gamma_n^{-1}\mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance, then for any $x \in Z$, there exists $g \in \Gamma$ such that

$$gx \notin Z.$$

We call the PS-system *well-behaved* with respect to a collection

$$\mathcal{H} := \{\mathcal{H}(R) \subset \Gamma : R \geq 0\}$$

of non-increasing subsets of Γ if the following additional properties hold:

(PS4) Γ is countable and for any $T > 0$, the set $\{\gamma \in \mathcal{H}(0) : \|\gamma\|_\sigma \leq T\}$ is finite.

(PS5) If $\{\gamma_n\} \subset \Gamma$, $R_n \to +\infty$, $Z \subset M$ is compact, and $[M \smallsetminus \gamma_n^{-1} \mathcal{O}_{R_n}(\gamma_n)] \to Z$ with respect to the Hausdorff distance, then for any $h_1, \dots, h_m \in \Gamma$ and $x \in Z$, there exists $g \in \Gamma$ such that
$$gx \notin \bigcup_{i=1}^{m} h_i Z.$$
(PS6) If $R_1 \leq R_2$ and $\gamma \in \mathcal{H}(0)$, then $\mathcal{O}_{R_1}(\gamma) \subset \mathcal{O}_{R_2}(\gamma)$.
(PS7) For any $R > 0$ there exist $C > 0$ and $R' > 0$ such that: if $\alpha, \beta \in \mathcal{H}(R)$, $\|\alpha\|_\sigma \leq \|\beta\|_\sigma$, and $\mathcal{O}_R(\alpha) \cap \mathcal{O}_R(\beta) \neq \emptyset$, then
$$\mathcal{O}_R(\beta) \subset \mathcal{O}_{R'}(\alpha)$$
and
$$\left| \|\beta\|_\sigma - (\|\alpha\|_\sigma + \left\|\alpha^{-1}\beta\right\|_\sigma) \right| \leq C.$$
(PS8) For every $R > 0$, there exists a set $M' \subset M$ of full μ-measure such that
$$\lim_{n \to +\infty} \operatorname{diam} \mathcal{O}_R(\gamma_n) = 0$$
whenever $\{\gamma_n\} \subset \mathcal{H}(R)$ is an escaping sequence and
$$x \in M' \cap \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n).$$

We call the collection $\mathcal{H}$ the *hierarchy* of the Patterson–Sullivan system.

The axioms above are meant to identify the key features of shadows in hyperbolic geometry. The notion of hierarchy is designed to include examples where Γ acts on a metric space by isometries with "contracting" elements, but not every element is "contracting." For instance, for the mapping class group action on the Teichmüller space, pseudo-Anosov mapping classes are precisely "contracting" elements by Minsky's Contraction Theorem [Min96]. In this case, the space M on which Patterson–Sullivan measures are defined is the Gardiner–Masur boundary of the Teichmüller space. See [KZ25, Section 10].

PS-systems always satisfy a natural analogue of the Shadow Lemma.

Proposition 2.2 (Shadow Lemma, [KZ25, Proposition 3.1])**.** *Let (M, Γ, σ, μ) be a PS-system of dimension $\delta \geq 0$. For any $R > 0$ sufficiently large there exists $C = C(R) > 1$ such that*
$$\frac{1}{C} e^{-\delta \|\gamma\|_\sigma} \leq \mu(\mathcal{O}_R(\gamma)) \leq C e^{-\delta \|\gamma\|_\sigma}$$
for all $\gamma \in \Gamma$.

Recall that in hyperbolic geometry, $x \in \partial \mathbb{H}^n$ is a *conical limit point* of a discrete subgroup $\Gamma < \mathsf{Isom}(\mathbb{H}^n)$ if there exist $R > 0$ and an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that x is contained in the R-shadow of each γ_n. Further, the set of conical limit point has full measure if and only if the Poincaré series diverges, i.e.
$$\sum_{\gamma \in \Gamma} e^{-\delta_{\mathbb{H}^n}(\Gamma) \, \mathrm{d}(o, \gamma o)} = +\infty$$
by the classical Hopf–Tsuji–Sullivan dichotomy [Tsu59, Hop71, Sul79, AS84, Rob03].

In a similar fashion, for well-behaved PS-systems with respect to the trivial hierarchy, divergence of the "Poincaré series" at the critical exponent implies that the set of "conical limit points" has positive measure.

Theorem 2.3 ([KZ25, Theorem 4.1]). *Let (M, Γ, σ, μ) be a PS-system of dimension $\delta \geq 0$. If (M, Γ, σ, μ) is well-behaved with respect to the trivial hierarchy $\mathcal{H}(R) \equiv \Gamma$ and*

$$\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\sigma} = +\infty,$$

then the set

$$E := \left\{ x \in M : \exists \{\gamma_n\} \subset \Gamma \text{ escaping and } R > 0 \text{ such that } x \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n) \right\}$$

has positive μ-measure.

Remark 2.4. In many particular examples, one can prove that if $g \in \Gamma$ and $R > 0$, then there exists $R' > 0$ such that

$$g\, \mathcal{O}_R(\gamma) \subset \mathcal{O}_{R'}(g\gamma)$$

for all $\gamma \in \Gamma$. In this case, the set E is Theorem 2.3 is Γ-invariant and further has full μ-measure. Indeed, if $\mu(E) < 1$, then the Γ-invariance implies that $\mu' := \frac{1}{\mu(M \smallsetminus E)} \mu(\cdot \cap (M \smallsetminus E))$ is also a PS-measure, but then applying Theorem 2.3 to this measure shows that $\mu'(E) > 0$ which is impossible.

3. Notations for semisimple Lie groups

In this section we fix the notation involving semisimple Lie groups that we will use throughout the paper. Of particular importance for our arguments are the linear representations fixed in Section 3.4.

Recall from the introduction that G is a semisimple Lie group with finite center and no compact factors, $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ is a fixed Cartan decomposition of the Lie algebra, $\mathfrak{a} \subset \mathfrak{p}$ is a fixed Cartan subspace, and $\mathfrak{a}^+ \subset \mathfrak{a}$ is a fixed positive Weyl chamber. We use $\Sigma \subset \mathfrak{a}^*$ to denote the set of restricted roots and use $\Delta \subset \mathfrak{a}^*$ to denote the system of simple restricted roots corresponding to the choice of $\mathfrak{a}^+$. Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H) X \text{ for all } H \in \mathfrak{a}\}.$$

Let Σ^+ (resp. Σ^-) denote the restricted roots which are non-negative (respectively non-positive) linear combinations of elements of Δ.

Recall that $\kappa : \mathsf{G} \to \mathfrak{a}^+$ denotes the Cartan projection. We fix a representative $w_0 \in \mathsf{K}$ of the longest Weyl element which is of order 2. Let $\mathrm{i} := -\mathrm{Ad}_{w_0} : \mathfrak{a} \to \mathfrak{a}$ denote the *opposition involution.* This map has the property that

$$\kappa(g^{-1}) = \mathrm{i}\kappa(g) \quad \text{for all} \quad g \in \mathsf{G}. \tag{3}$$

The adjoint i^* of the opposite involution preserves the set of simple roots and for a subset $\theta \subset \Delta$, we define

$$\mathrm{i}^*\theta := \{\mathrm{i}^*\alpha : \alpha \in \theta\}.$$

3.1. **Parabolic subgroups and flag manifolds.** Given a non-empty $\theta \subset \Delta$, the associated parabolic subgroup P_θ is the stabilizer under the adjoint action of the Lie algebra

$$\mathfrak{u}_\theta^+ := \bigoplus_{\alpha \in \Sigma_\theta^+} \mathfrak{g}_\alpha$$

where $\Sigma_\theta^+ := \Sigma^+ \smallsetminus \operatorname{span}(\Delta \smallsetminus \theta)$. We also set $\mathsf{A} := \exp \mathfrak{a}$ and $\mathsf{A}^+ := \exp \mathfrak{a}^+$, and denote by $\mathsf{N} < \mathsf{P}_\Delta$ the unipotent radical of P_Δ.

The *Furstenberg boundary* and general *θ-boundary* are the quotient spaces

$$\mathcal{F}_\Delta := \mathsf{G} / \mathsf{P}_\Delta \quad \text{and} \quad \mathcal{F}_\theta := \mathsf{G} / \mathsf{P}_\theta .$$

Two elements $x \in \mathcal{F}_\theta$ and $y \in \mathcal{F}_{\mathrm{i}^*\theta}$ are *transverse* if there exists $g \in \mathsf{G}$ such that

$$x = g \, \mathsf{P}_\theta \quad \text{and} \quad y = g w_0 \, \mathsf{P}_{\mathrm{i}^*\theta},$$

equivalently (x, y) is contained in the unique open G-orbit in $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}^*\theta}$.

3.2. **Projection to the flag manifold.** For $g \in \mathsf{G}$ with $\min_{\alpha \in \theta} \alpha(\kappa(g)) > 0$, we define

$$U_\theta(g) := k \, \mathsf{P}_\theta \in \mathcal{F}_\theta$$

where g has Cartan decomposition $g = k a \ell \in \mathsf{K}\mathsf{A}^+\mathsf{K}$ (the condition on the roots implies that $U_\theta(g)$ is well-defined).

Observation 3.1. If $g \in \mathsf{G}$ has Cartan decomposition $g = k a \ell \in \mathsf{K}\mathsf{A}^+\mathsf{K}$ and $\min_{\alpha \in \theta} \alpha(\kappa(g)) > 0$, then $\min_{\alpha \in \mathrm{i}^*\theta} \alpha(\kappa(g^{-1})) > 0$ and

$$U_{\mathrm{i}^*\theta}(g^{-1}) = \ell^{-1} w_0 \, \mathsf{P}_{\mathrm{i}^*\theta} .$$

Proof. Notice that g^{-1} has Cartan decomposition $g^{-1} = \ell^{-1} w_0 (w_0 a w_0) w_0 k^{-1}$. □

These projection maps have the following dynamical behavior (for a proof see for instance [KLP17, Section 4] or [CZZ24, Proposition 2.3]).

Proposition 3.2. *If $\{g_n\} \subset \mathsf{G}$, $x^+ \in \mathcal{F}_\theta$, and $x^- \in \mathcal{F}_{\mathrm{i}^*\theta}$, then the following are equivalent:*

1. *$g_n x \to x^+$ for all $x \in \mathcal{F}_\theta$ transverse to x^- and the convergence is uniform on compact subsets.*
2. *$\min_{\alpha \in \theta} \alpha(\kappa(g_n)) \to +\infty$, $U_\theta(g_n) \to x^+$, and $U_{\mathrm{i}^*\theta}(g_n^{-1}) \to x^-$.*

3.3. **The partial Iwasawa cocycle.** The *Iwasawa cocycle* $B_\Delta^{IW} : \mathsf{G} \times \mathcal{F}_\Delta \to \mathfrak{a}$ is defined as follows: for $g \in \mathsf{G}$ and $x \in \mathcal{F}_\Delta$, $B_\Delta^{IW}(g, x) \in \mathfrak{a}$ is the unique element such that

$$gk \in \mathsf{K}(\exp B_\Delta^{IW}(g, x))\mathsf{N}$$

for $k \in \mathsf{K}$ such that $k \, \mathsf{P}_\Delta = x$ in $\mathcal{F}_\Delta$.

For general $\theta \subset \Delta$, let

$$\mathfrak{a}_\theta := \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \notin \theta\}.$$

For $\alpha \in \Delta$, let ω_α denote the (restricted) fundamental weight associated to α. Then $\{\omega_\alpha|_{\mathfrak{a}_\theta}\}_{\alpha \in \theta}$ is a basis for $\mathfrak{a}_\theta^*$ and so there exists a unique projection

$$\pi_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$$

satisfying

$$\omega_\alpha \pi_\theta(H) = \omega_\alpha(H) \tag{4}$$

for all $H \in \mathfrak{a}$ and $\alpha \in \theta$.

The *partial Iwasawa cocycle* $B^{IW}_\theta : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ is defined as

$$B^{IW}_\theta(g, x) := \pi_\theta B^{IW}_\Delta(g, \tilde{x}) \tag{5}$$

for any $\tilde{x} \in \mathcal{F}_\Delta$ that projects to $x \in \mathcal{F}_\theta$ under the canonical projection $\mathcal{F}_\Delta \to \mathcal{F}_\theta$. The above definition is independent of the choice of $\tilde{x}$ and defines a cocycle [Qui02, Lemma 6.1].

Recall from the introduction that the partial Iwasawa cocycle can be used to define a notion of Patterson–Sullivan measures on the partial flag manifold $\mathcal{F}_\theta$ (see Definition 1.1).

3.4. **Linear Representations.** Given a d-dimensional vector space V endowed with an inner product and $g \in \mathsf{SL}(V)$, we let

$$\sigma_1(g) \geq \cdots \geq \sigma_d(g)$$

denote the singular values of g with respect to the inner product and let $\|g\| = \sigma_1(g)$ denote the operator norm.

Throughout the paper, for each $\alpha \in \Delta$ we fix an irreducible representation $\Phi_\alpha : \mathsf{G} \to \mathsf{SL}(V_\alpha)$ and a $\Phi_\alpha(\mathsf{K})$-invariant inner product on V_α with the following properties:

(R1) There exists $N_\alpha \in \mathbb{N}$ such that if $g \in \mathsf{G}$, then

$$\log \|\Phi_\alpha(g)\| = N_\alpha \omega_\alpha(\kappa(g)) \quad \text{and} \quad \log \frac{\sigma_1(\Phi_\alpha(g))}{\sigma_2(\Phi_\alpha(g))} = \alpha(\kappa(g)).$$

(R2) There exists a $\Phi_\alpha(\mathsf{A})$-invariant orthogonal splitting $V_\alpha = V_\alpha^+ \oplus V_\alpha^-$ such that $\dim V_\alpha^+ = 1$. Moreover, if $H \in \mathfrak{a}$ and $v \in V_\alpha^+$, then

$$\Phi_\alpha(e^H)v = e^{N_\alpha \omega_\alpha(H)} v.$$

(R3) There exist Φ_α-equivariant boundary maps $\zeta_\alpha : \mathcal{F}_\alpha \to \mathbb{P}(V_\alpha)$ and $\zeta_\alpha^* : \mathcal{F}_{\mathrm{i}^*\alpha} \to \mathrm{Gr}_{\dim V_\alpha - 1}(V_\alpha)$ such that:
  (a) $\zeta_\alpha(\mathsf{P}_\alpha) = V_\alpha^+$ and $\zeta_\alpha^*(w_0 \mathsf{P}_{\mathrm{i}^*\alpha}) = V_\alpha^-$.
  (b) $x \in \mathcal{F}_\alpha$ and $y \in \mathcal{F}_{\mathrm{i}^*\alpha}$ are transverse if and only if $\zeta_\alpha(x)$ and $\zeta_\alpha^*(y)$ are transverse.

Remark 3.3. Such representations exist due to Tits [Tit71, Theorem 7.2]. Indeed, Tits proved the first claim in Property (R1). For a proof of the second assertion in Property (R1) and Property (R2), see for instance [Smi18, Lemma 2.13] and [BQ16, Sections 6.8, 6.9]. For a proof of Property (R3), see for instance [GGKW17, Section 3].

Remark 3.4. We abuse notation and when $\alpha \in \theta$, also often use ζ_α to also denote the map $\mathcal{F}_\theta \to \mathbb{P}(V_\alpha)$ obtained by precomposing $\zeta_\alpha : \mathcal{F}_\alpha \to \mathbb{P}(V_\alpha)$ with the natural projection $\mathcal{F}_\theta \to \mathcal{F}_\alpha$. Likewise, we also use $\zeta^*_{\mathrm{i}^*\alpha}$ to denote the analogous map defined on $\mathcal{F}_{\mathrm{i}^*\theta}$.

The following lemma relates the projections to the flag manifolds introduced in Section 3.2 to these representations.

Lemma 3.5. *Fix $\alpha \in \Delta$ and assume $\{g_n\} \subset \mathsf{G}$ is such that $\alpha(\kappa(g_n)) \to +\infty$, $U_\alpha(g_n) \to x$, and $U_{\mathrm{i}^*\alpha}(g_n^{-1}) \to y$. Then any limit point of*

$$\frac{1}{\|\Phi_\alpha(g_n)\|} \Phi_\alpha(g_n) \quad \textit{in } \mathrm{End}(V_\alpha)$$

has image $\zeta_\alpha(x)$ and kernel $\zeta_\alpha^*(y)$.

Proof. Suppose that $\frac{1}{\|\Phi_\alpha(g_n)\|}\Phi_\alpha(g_n) \to T$. Fix a Cartan decomposition $g_n = k_n a_n \ell_n \in \mathsf{K}\mathsf{A}^+\mathsf{K}$. Passing to subsequences we can suppose that $k_n \to k$, $\ell_n \to \ell$, and $\frac{1}{\|\Phi_\alpha(a_n)\|}\Phi_\alpha(a_n) \to S$. Then $x = k\,\mathsf{P}_\alpha$ and $T = \Phi_\alpha(k)S\Phi_\alpha(\ell)$. Further, Observation 3.1 implies that $y = \ell^{-1}w_0\,\mathsf{P}_{\mathrm{i}^*\alpha}$.

By Properties (R1) and (R2), S has image V_α^+ and kernel V_α^-. Then by Property (R3), T has image

$$\Phi_\alpha(k)V_\alpha^+ = \zeta_\alpha(k\,\mathsf{P}_\alpha) = \zeta_\alpha(x)$$

and kernel

$$\Phi_\alpha(\ell)^{-1}V_\alpha^- = \zeta_\alpha^*(\ell^{-1}w_0\,\mathsf{P}_{\mathrm{i}^*\alpha}) = \zeta_\alpha^*(y).$$

□

3.5. **Irreducible actions.** A subgroup $\mathsf{H} < \mathsf{SL}(V)$ is called *irreducible* if there is no non-trivial and proper subspace of V invariant under H, and is called *strongly irreducible* if any finite index subgroup of H is irreducible. We transfer these notions to G using the Φ_α's.

Definition 3.6. A subgroup $\Gamma < \mathsf{G}$ is *$(\Phi_\alpha)_{\alpha\in\theta}$-irreducible* if $\Phi_\alpha(\Gamma) < \mathsf{SL}(V_\alpha)$ is irreducible for all $\alpha \in \theta$, and *strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible* if any finite index subgroup of Γ is $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible.

Remark 3.7. Notice that a Zariski dense subgroup is strongly $(\Phi_\alpha)_{\alpha\in\Delta}$-irreducible.

We will use the following observation several times.

Lemma 3.8. *Suppose $\Gamma < \mathsf{G}$ is strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible. If*

- *$\alpha_1, \dots, \alpha_m$ are (possibly non-distinct) elements of θ,*
- *$v_i \in V_{\alpha_i} \smallsetminus \{0\}$ for $i = 1, \dots, m$, and*
- *$W_i \subset V_{\alpha_i}$ is a proper linear subspace for $i = 1, \dots, m$,*

then there exists $\gamma \in \Gamma$ with

$$\Phi_{\alpha_i}(\gamma)v_i \notin W_i$$

for $i = 1, \dots, m$.

Proof. Let $\mathsf{H} < \mathsf{G}$ be the Zariski closure of Γ in G and let $\mathsf{H}^0 < \mathsf{H}$ be the identity component. Since Γ is strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible, H^0 is $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible as well. Then for each $1 \le i \le m$, the set

$$\mathcal{O}_i := \{h \in \mathsf{H}^0 : \Phi_{\alpha_i}(h)v_i \notin W_i\}$$

is non-empty and Zariski open. Since $\Gamma \cap \mathsf{H}^0$ is Zariski dense in H^0, there exists

$$\gamma \in \Gamma \cap \bigcap_{i=1}^m \mathcal{O}_i,$$

which satisfies the desired properties. □

3.6. **Limit sets and transverse groups.** In this section we recall the definition a transverse group.

Given a discrete subgroup $\Gamma < \mathsf{G}$, a point $x \in \mathcal{F}_\theta$ is a *limit point* of Γ if there exists an escaping sequence $\{\gamma_n\} \subset \Gamma$ with $\min_{\alpha\in\theta}\alpha(\kappa(\gamma_n)) \to +\infty$ and $U_\theta(\gamma_n) \to x$. The *limit set of* Γ, denoted by

$$\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta,$$

is the set of all limit points of Γ. Proposition 3.2 implies that the points in $\Lambda_\theta(\Gamma)$ are exactly the points $x^+ \in \mathcal{F}_\theta$ where there exists a sequence $\{\gamma_n\} \subset \Gamma$ and a non-empty open set $\mathcal{U} \subset \mathcal{F}_\theta$ such that $\gamma_n x \to x^+$ for all $x \in \mathcal{U}$, uniformly on compact subsets.

A discrete subgroup $\Gamma < \mathsf{G}$ is P_θ-*transverse* if $\min_{\alpha\in\theta} \alpha(\kappa(\gamma_n)) \to +\infty$ for any sequence $\{\gamma_n\} \subset \Gamma$ of distinct elements and any two distinct points in $\Lambda_{\theta\cup \mathrm{i}^*\theta}(\Gamma)$ are transverse.

Sometimes the definition of transverse group includes the assumption that θ is symmetric (i.e., $\theta = \mathrm{i}^*\theta$). However, as the next observation demonstrates, this results in no loss of generality.

Observation 3.9. $\Gamma < \mathsf{G}$ is P_θ-transverse if and only if Γ is $\mathsf{P}_{\theta\cup\mathrm{i}^*\theta}$-transverse. Moreover, in this case the the projection $\mathcal{F}_{\theta\cup\mathrm{i}^*\theta} \to \mathcal{F}_\theta$ induces a homeomorphism $\Lambda_{\theta\cup\mathrm{i}^*\theta}(\Gamma) \to \Lambda_\theta(\Gamma)$.

A P_θ-transverse group is called *non-elementary* if $\#\Lambda_\theta(\Gamma) \geq 3$, in which case the natural Γ-action on $\Lambda_\theta(\Gamma)$ is a minimal convergence action and $\#\Lambda_\theta(\Gamma) = +\infty$, see [KLP17, Theorem 4.16] or [CZZ26, Proposition 3.3].

Two well-studied classes of transverse groups are Anosov groups and relatively Anosov groups. A P_θ-transverse group is P_θ-*Anosov* if the Γ-action on $\Lambda_\theta(\Gamma)$ is a uniform convergence action, equivalently Γ is word hyperbolic as an abstract group and there exists an equivariant homeomorphism from the Gromov boundary to the limit set $\Lambda_\theta(\Gamma)$ [Bow98]. Likewise, a P_θ-transverse group is P_θ-*relatively Anosov* if the Γ-action on $\Lambda_\theta(\Gamma)$ is geometrically finite, equivalently if Γ has the structure of a relatively hyperbolic group and there exists an equivariant homeomorphism from the associated Bowditch boundary to the limit set $\Lambda_\theta(\Gamma)$ [Yam04].

4. Compactifications and Patterson–Sullivan measures

In this section, we introduce vector-valued horofunction compactifications of the symmetric space $X = \mathsf{G}/\mathsf{K}$ associated to G, using Cartan projections. It turns out that they contain partial flag manifolds, and we also consider Patterson–Sullivan measures there.

Fix the basepoint $o := \mathsf{K} \in X$. The symmetric space distance is given by

$$\mathrm{d}_X(go, ho) = \left\|\kappa(g^{-1}h)\right\|$$

where $\|\cdot\|$ is some norm on $\mathfrak{a}$. For $x = go$, define $b_x : X \to \mathfrak{a}$ by

$$b_x(ho) = \kappa(h^{-1}g) - \kappa(g).$$

Lemma 4.1. *The maps $\{b_x : x \in X\}$ are uniformly Lipschitz.*

Proof. Fix $x = go \in X$. Let $h_1 o, h_2 o \in X$. Then for each $\alpha \in \Delta$, it follows from Property (R1) that

$$\begin{aligned} N_\alpha \omega_\alpha \left(b_x(h_1 o) - b_x(h_2 o)\right) &= \log \left\|\Phi_\alpha(h_1^{-1}g)\right\| - \log\left\|\Phi_\alpha(h_2^{-1}g)\right\| \\ &\leq \log\left\|\Phi_\alpha(h_1^{-1}h_2)\right\| = N_\alpha\omega_\alpha(\kappa(h_1^{-1}h_2)). \end{aligned}$$

And similarly,

$$\omega_\alpha\left(b_x(h_2 o) - b_x(h_1 o)\right) \leq \omega_\alpha(\kappa(h_2^{-1}h_1)).$$

This implies

$$\sum_{\alpha\in\Delta} \left|\omega_\alpha\left(b_x(h_2 o) - b_x(h_1 o)\right)\right| \leq 2\sum_{\alpha\in\Delta}\left|\omega_\alpha(\kappa({h_1}^{-1}h_2))\right|.$$

Since $\sum_{\alpha\in\Delta} |\omega_\alpha(\cdot)|$ is a norm on $\mathfrak{a}$ and

$$\mathrm{d}_X(h_1 o, h_2 o) = \left\|\kappa(h_1^{-1}h_2)\right\|,$$

this finishes the proof. □

For non-empty $\theta \subset \Delta$, let $\pi_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ be the projection satisfying Equation (4) and then let $\partial_\theta X$ be the set of functions $\xi : X \to \mathfrak{a}_\theta$ where there exists an escaping sequence $\{x_n\} \subset X$ with $\pi_\theta b_{x_n} \to \xi$ in the compact-open topology. Lemma 4.1, together with the separability of X, implies that $\partial_\theta X$ is compact in the compact-open topology. Further, G acts on $\partial_\theta X$ by

$$g \cdot \xi = \xi \circ g^{-1} - \xi(g^{-1}o).$$

The next proposition shows that $\partial_\theta X$ can be used to compactify X.

Proposition 4.2. *The space $\overline{X}^\theta := X \sqcup \partial_\theta X$ has a topology which makes it a compactification of X, that is $\overline{X}^\theta$ is a compact metrizable space and the inclusion $X \hookrightarrow \overline{X}^\theta$ is a topological embedding with open dense image. Moreover with respect to this topology:*

1. *$\{x_n\} \subset X$ converges to $\xi \in \partial_\theta X$ if and only if $\mathrm{d}_X(o, x_n) \to +\infty$ and $\pi_\theta b_{x_n} \to \xi$ in the compact-open topology.*
2. *The G-action on $\overline{X}^\theta$ is continuous.*
3. *The function $B_\theta : \mathsf{G} \times \overline{X}^\theta \to \mathfrak{a}$ defined by*

$$B_\theta(g, x) = \begin{cases} \pi_\theta b_x(g^{-1}o) & \text{if } x \in X \\ x(g^{-1}o) & \text{if } x \in \partial_\theta X \end{cases}$$

is continuous.

Remark 4.3. Notice that the function B_θ is a linear cocycle:

$$B_\theta(g_1 g_2, x) = B_\theta(g_1, g_2 x) + B_\theta(g_2, x)$$

for all $g_1, g_2 \in \mathsf{G}$ and $x \in \overline{X}^\theta$.

Proof. For two Lipschitz functions $\xi_1, \xi_2 : \mathsf{G} \to \mathfrak{a}_\theta$ define

$$\mathrm{d}_0(\xi_1, \xi_2) := \sum_{n\geq 1} \frac{1}{2^n} \max_{x\in B_X(o,n)} \|\xi_1(x) - \xi_2(x)\|$$

where $B_X(o,n) \subset X$ is the n-ball centered at o. Also define $h : X \to (0, +\infty)$ by $h(x) := \frac{1}{1+\mathrm{d}_X(o,x)}$. Then define a metric d on $\overline{X}^\theta$ by

$$\begin{aligned} \mathrm{d}(x,y) &:= \min\{\mathrm{d}_X(x,y), h(x)+h(y)\} + \mathrm{d}_0(b_x, b_y) \quad \text{if} \quad x, y \in X, \\ \mathrm{d}(x,\xi) &:= h(x) + \mathrm{d}_0(b_x, \xi) \quad \text{if} \quad x \in X,\ \xi \in \partial_\theta X, \\ \mathrm{d}(\xi_1,\xi_2) &:= \mathrm{d}_0(\xi_1, \xi_2) \quad \text{if} \quad \xi_1, \xi_2 \in \partial_\theta X. \end{aligned}$$

Following [Man89, Section 3], together with the fact that $|h(x) - h(y)| \leq \mathrm{d}_X(x,y)$ for $x, y \in X$, one can check that $\mathrm{d}(\cdot,\cdot)$ is indeed a metric on $\overline{X}^\theta$, and the topology induced by this metric has all the desired properties. □

Patterson–Sullivan measures on $\partial_\theta X$ can naturally be defined as follows.

Definition 4.4. Given a subgroup $\Gamma < \mathsf{G}$, $\theta \subset \Delta$ non-empty, a functional $\phi \in \mathfrak{a}_\theta^*$, and $\delta \geq 0$, a Borel probability measure μ on $\partial_\theta X$ is a *coarse* (Γ, ϕ, δ)*-Patterson–Sullivan measure* if there exists $C \geq 1$ such that for every $\gamma \in \Gamma$ the measures μ, $\gamma_*\mu$ are absolutely continuous and

$$C^{-1}e^{-\delta\phi\xi(\gamma o)} \leq \frac{d\gamma_*\mu}{d\mu}(\xi) \leq Ce^{-\delta\phi\xi(\gamma o)} \quad \mu\text{-a.e.}$$

We call μ a (Γ, ϕ, δ)*-Patterson–Sullivan measure* if $C = 1$.

Using Patterson's original construction for Fuchsian groups, we can prove the following existence result.

Proposition 4.5. *If $\Gamma < \mathsf{G}$ is discrete, $\phi \in \mathfrak{a}_\theta^*$, and $\delta^\phi(\Gamma) < +\infty$, then there exists a $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure on $\partial_\theta X$.*

Proof. For $\gamma \in \Gamma$ fixed, the function $f_\gamma := \phi B_\theta(\gamma, \cdot) : \overline{X}^\theta \to \mathbb{R}$ is continuous. Hence we can follow Patterson's original argument for constructing Patterson–Sullivan measures for Fuchsian groups. In particular, by [Pat76, Lemma 3.1], there exists a continuous non-decreasing function $h : \mathbb{R} \to \mathbb{R}^+$ such that:

1. The series $\sum_{\gamma\in\Gamma} h(\phi(\kappa(\gamma)))e^{-s\phi(\kappa(\gamma))}$ diverges at $s = \delta^\phi(\Gamma)$ and converges for $s > \delta^\phi(\Gamma)$.
2. For any $\epsilon > 0$ there exists $t_0 > 0$ such that if $t > t_0$ and $s > 1$, then $h(st) \leq s^\epsilon h(t)$.

(In the case when $\sum_{\gamma\in\Gamma} e^{-\delta^\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty$, we can take $h \equiv 1$.)

For $s > \delta^\phi(\Gamma)$, consider the probability measure

$$\mu_s := \frac{1}{\sum_{\gamma\in\Gamma} h(\phi(\kappa(\gamma)))e^{-s\phi(\kappa(\gamma))}} \sum_{\gamma\in\Gamma} h(\phi(\kappa(\gamma)))e^{-s\phi(\kappa(\gamma))}\,\mathcal{D}_{\gamma o}$$

on $\overline{X}^\theta$, where $\mathcal{D}_{\gamma o}$ is the Dirac mass at γo. Next, fix $s_n \searrow \delta^\phi(\Gamma)$ such that μ_{s_n} converges to a probability measure μ in the weak-$*$ topology. Then using the continuity of $B_\theta(\gamma^{-1}, \cdot)$ on $\overline{X}^\theta$ it is straightforward to show that μ is a $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure. ☐

Next we provide a formula for elements in $\partial_\theta X$ in terms of the linear representations introduced in Section 3.4.

Lemma 4.6. *If $g_n o \to \xi$ in $\overline{X}^\theta$ and $\frac{\Phi_\alpha(g_n)}{\|\Phi_\alpha(g_n)\|} \to T_\alpha$ in $\mathrm{End}(V_\alpha)$ for all $\alpha \in \theta$, then*

$$N_\alpha\omega_\alpha\xi(ho) = \log\left\|\Phi_\alpha(h^{-1})T_\alpha\right\|$$

for all $h \in \mathsf{G}$ and all $\alpha \in \theta$. In particular,

$$\omega_\alpha\xi(h^{-1}o) \leq \omega_\alpha\kappa(h)$$

for all $h \in \mathsf{G}$ and all $\alpha \in \theta$.

Proof. Let $g \in \mathsf{G}$ and recall that $b_{go}(ho) = \kappa(h^{-1}g) - \kappa(g)$. For each $\alpha \in \theta$, Property (R1) implies that

$$N_\alpha\omega_\alpha\kappa(h^{-1}g) = \log\left\|\Phi_\alpha(h^{-1}g)\right\| \quad \text{and} \quad N_\alpha\omega_\alpha\kappa(g) = \log\|\Phi_\alpha(g)\|.$$

Hence,

$$N_\alpha\omega_\alpha b_{go}(ho) = \log\left\|\Phi_\alpha(h^{-1})\frac{\Phi_\alpha(g)}{\|\Phi_\alpha(g)\|}\right\|.$$

The first claim then follows. For the "in particular" part, since $\|T_\alpha\| = 1$,

$$\omega_\alpha \kappa(h) = \frac{1}{N_\alpha} \log \|\Phi_\alpha(h)\| \geq \frac{1}{N_\alpha} \log \|\Phi_\alpha(h) T_\alpha\| = \omega_\alpha \xi(h^{-1} o). \qquad \square$$

Recall that $B_\theta^{IW} : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ denotes the partial Iwasawa cocyle. Using this cocycle, we show that $\mathcal{F}_\theta$ embeds into $\partial_\theta X$.

Proposition 4.7. *There is a topological embedding $\iota : \mathcal{F}_\theta \to \partial_\theta X$ that satisfies*

$$\iota(x)(ho) = B_\theta^{IW}(h^{-1}, x) \tag{6}$$

for all $x \in \mathcal{F}_\theta$ and $h \in \mathsf{G}$.

Moreover:

1. *If a sequence $\{g_n\} \subset \mathsf{G}$ satisfies $\min_{\alpha \in \theta} \alpha(\kappa(g_n)) \to +\infty$ and $U_\theta(g_n) \to x$, then*
$$g_n o \to \iota(x) \quad \text{in } \overline{X}^\theta.$$
2. *For $\alpha \in \theta$, if $x \in \mathcal{F}_\theta$, $\alpha \in \theta$, and $v_\alpha \in V_\alpha$ is a unit vector with $\zeta_\alpha(x) = [v_\alpha]$, then*
$$N_\alpha \omega_\alpha \iota(x)(ho) = \log \left\| \Phi_\alpha(h^{-1}) v_\alpha \right\|.$$
3. *If μ is a (coarse, resp.) (Γ, ϕ, δ)-Patterson–Sullivan measure on $\mathcal{F}_\theta$ in the sense of Definition 1.1, then $\iota_* \mu$ is a (coarse, resp.) (Γ, ϕ, δ)-Patterson–Sullivan measure on $\partial_\theta X$ in the sense of Definition 4.4.*

Proof. By [Qui02, Lemma 6.6], if a sequence $\{g_n\} \subset \mathsf{G}$ satisfies $\min_{\alpha \in \theta} \alpha(\kappa(g_n)) \to +\infty$ and $U_\theta(g_n) \to x$, then

$$\lim_{n \to +\infty} \pi_\theta b_{g_n o}(ho) = \lim_{n \to +\infty} \pi_\theta \kappa(h^{-1} g_n) - \pi_\theta \kappa(g_n) = B_\theta^{IW}(h^{-1}, x) = \iota(x)(ho)$$

for all $h \in \mathsf{G}$. This shows that Equation (6) defines a continuous map $\iota : \mathcal{F}_\theta \to \partial_\theta X$ and also establishes the first "moreover" part. Notice that the second "moreover" part is a consequence of the first, Lemma 3.5, and Lemma 4.6. The third "moreover" part is an immediate consequence of the definition of ι.

To finish the proof we need to show that ι is a topological embedding. Since ι is continuous and $\mathcal{F}_\theta$ is compact, it suffices to show that ι is injective. Suppose that $\iota(x) = \iota(y)$. Let $x = k_1 \mathsf{P}_\theta$ and $y = k_2 \mathsf{P}_\theta$ where $k_1, k_2 \in \mathsf{K}$.

Fix $\alpha \in \theta$ and fix a unit vector u_α in V_α^+. Property (R3) implies that $\zeta_\alpha(x) = [\Phi_\alpha(k_1) u_\alpha]$ and $\zeta_\alpha(y) = [\Phi_\alpha(k_2) u_\alpha]$. Thus by part (2) of the moreover part of this proposition,

$$\left\| \Phi_\alpha(h^{-1}) \Phi_\alpha(k_1) u_\alpha \right\| = e^{N_\alpha \omega_\alpha \iota(x)(ho)} = e^{N_\alpha \omega_\alpha \iota(y)(ho)} = \left\| \Phi_\alpha(h^{-1}) \Phi_\alpha(k_2) u_\alpha \right\|$$

for all $h \in \mathsf{G}$. Fix $H \in \mathfrak{a}^+$ with $\min_{\alpha \in \theta} \alpha(H) > 0$. Then the above equation with $h := k_2 e^{-H}$ implies that

$$\left\| \Phi_\alpha(e^H) \Phi_\alpha(k_2^{-1} k_1) u_\alpha \right\| = \left\| \Phi_\alpha(e^H) u_\alpha \right\|.$$

By Properties (R1) and (R2), the map $v \in V_\alpha \smallsetminus \{0\} \mapsto \frac{\left\| \Phi_\alpha(e^H) v \right\|}{\|v\|}$ is maximized only on $V_\alpha^+ \smallsetminus \{0\}$. So we must have $\Phi_\alpha(k_2^{-1} k_1) u_\alpha = \pm u_\alpha$. Then

$$\zeta_\alpha(x) = [\Phi_\alpha(k_1) u_\alpha] = [\Phi_\alpha(k_2) u_\alpha] = \zeta_\alpha(y).$$

Since this holds for all $\alpha \in \theta$, we have $x = y$. $\square$

5. Shadows and contracting conical limit sets

In this section, we define shadows on $\partial_\theta X$ and use them to introduce the contracting conical limit set of a discrete subgroup.

Recall, from Lemma 4.6, that if $\xi \in \partial_\theta X$ and $g \in \mathsf{G}$, then

$$\omega_\alpha \xi(g^{-1}o) \leq \omega_\alpha \kappa(g)$$

for all $\alpha \in \theta$. Given $g \in \mathsf{G}$, we introduce shadows in $\partial_\theta X$ by considering the set of functionals ξ that are close to maximizing the expression $\omega_\alpha \xi(g^{-1}o)$ for all $\alpha \in \theta$.

More precisely, for $g \in \mathsf{G}$ and $R > 0$, define the associated *shadow* by

$$\mathcal{O}_R^\theta(g) := g \cdot \{\xi \in \partial_\theta X : \omega_\alpha \xi(g^{-1}o) > \omega_\alpha \kappa(g) - R \text{ for all } \alpha \in \theta\}.$$

In what follows we use π_θ to denote both the projection $\mathfrak{a} \to \mathfrak{a}_\theta$ satisfying Equation (4) and the map $\partial_\Delta X \to \partial_\theta X$ obtained by the postcomposition with this projection. Since $\omega_\alpha \xi = \omega_\alpha \pi_\theta \xi$ for all $\alpha \in \theta$ and $\xi \in \partial_\Delta X$, we have

$$\pi_\theta \, \mathcal{O}_R^\Delta(g) \subset \mathcal{O}_R^\theta(g).$$

We also use Proposition 4.7 to view $\mathcal{F}_\theta$ as a subset of $\partial_\theta X$. Then Equation (5) and Proposition 4.7 imply that $\pi_\theta|_{\mathcal{F}_\Delta}$ coincides with the natural projection $\mathcal{F}_\Delta \to \mathcal{F}_\theta$ given by $g\,\mathsf{P}_\Delta \to g\,\mathsf{P}_\theta$.

Given a subgroup $\Gamma < \mathsf{G}$, we define its *conical limit set* in $\partial_\theta X$ by

$$\Lambda_\theta^{\text{con}}(\Gamma) := \left\{ \xi \in \partial_\theta X : \exists\, R > 0, \text{ escaping } \{\gamma_n\} \subset \Gamma \text{ s.t. } \xi \in \bigcap_{n \geq 1} \mathcal{O}_R^\theta(\gamma_n) \right\}, \tag{7}$$

following the classical definition of conical limit sets in rank one settings.

When G is of higher rank, the intersection $\bigcap_{n\geq 1} \mathcal{O}_R^\theta(\gamma_n)$ may not be a singleton, even after intersecting with the partial flag manifold $\mathcal{F}_\theta$. Moreover, the conical limit set $\Lambda_\theta^{\text{con}}(\Gamma)$, after intersecting with $\mathcal{F}_\theta$, may not be a subset of the limit set $\Lambda_\theta(\Gamma)$. See Example 5.8 below for detailed descriptions of them. Hence, in view of Lemma 5.6 below, we define the following smaller subset of conical limit set, which only involves shrinking shadows.

Definition 5.1. Given a subgroup $\Gamma < \mathsf{G}$, we call $\xi \in \partial_\theta X$ a *contracting conical limit point* of Γ if there exist $R > 0$ and a sequence $\{\gamma_n\} \subset \Gamma$ such that

$$\lim_{n \to +\infty} \min_{\alpha \in \theta} \alpha(\kappa(\gamma_n)) = +\infty \quad \text{and} \quad \xi \in \bigcap_{n \geq 1} \mathcal{O}_R^\theta(\gamma_n).$$

We denote by $\Lambda_\theta^{\text{concon}}(\Gamma)$ the *contracting conical limit set* of Γ, which is defined as the set of all contracting conical limit points of Γ.

For general groups, these limit sets have the following properties.

Proposition 5.2. *If $\Gamma < \mathsf{G}$ is a subgroup, then both $\Lambda_\theta^{\text{concon}}(\Gamma)$ and $\Lambda_\theta^{\text{con}}(\Gamma)$ are Γ-invariant subsets. Moreover, $\Lambda_\theta^{\text{concon}}(\Gamma)$ is a subset of the limit set $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$ introduced in Section 3.6.*

For transverse groups, one can say more.

Proposition 5.3. *If $\Gamma < \mathsf{G}$ is a non-elementary P_θ-transverse group, then:*

(1) $\Lambda_\theta^{\text{concon}}(\Gamma) = \Lambda_\theta^{\text{con}}(\Gamma)$. Hence $\Lambda_\theta^{\text{con}}(\Gamma)$ is a subset of the limit set $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$ introduced in Section 3.6.

(2) $\Lambda_\theta^{\mathrm{con}}(\Gamma)$ *coincides with the conical limit set in the convergence group sense (recall that* Γ *acts on* $\Lambda_\theta(\Gamma)$ *as a convergence group).*

In the next two subsections we establish some properties of these shadows and relate them to symmetric space shadows. Then in Section 5.3 we prove Propositions 5.2 and 5.3.

5.1. **Properties of shadows.** We record some properties of shadows.

Lemma 5.4. *For any* $g \in \mathsf{G}$ *and* $R > 0$*, there exists* $R' = R'(g, R) > 0$ *such that: if* $h \in \mathsf{G}$*, then*

$$g\,\mathcal{O}_R^\theta(h) \subset \mathcal{O}_{R'}^\theta(gh).$$

In particular, $\Lambda_\theta^{\mathrm{con}}(\Gamma)$ *and* $\Lambda_\theta^{\mathrm{concon}}(\Gamma)$ *are* Γ*-invariant.*

Proof. Fix $\xi \in \partial_\theta X$ with $h\xi \in \mathcal{O}_R^\theta(h)$. Then for each $\alpha \in \theta$,

$$\omega_\alpha \xi(h^{-1}o) > \omega_\alpha \kappa(h) - R.$$

Then by Lemma 4.1 and Property (R1), there exists $C = C(g) > 0$ such that

$$\left|\omega_\alpha \xi(h^{-1}o) - \omega_\alpha \xi(h^{-1}g^{-1}o)\right| < C \quad \text{and} \quad \left|\omega_\alpha \kappa(h) - \omega_\alpha \kappa(gh)\right| < C.$$

Hence, we have

$$\omega_\alpha \xi(h^{-1}g^{-1}o) > \omega_\alpha \kappa(gh) - R - 2C.$$

Setting $R' = R + 2C$, this implies $gh\xi \in \mathcal{O}_{R'}^\theta(gh)$, as desired. □

The next lemma shows that a shadow contains the "endpoint" of the associated group element, hence motivating the terminology.

Lemma 5.5. *For* $g \in \mathsf{G}$ *with* $\min_{\alpha\in\theta} \alpha(\kappa(g)) > 0$*, we have*

$$U_\theta(g) \in \mathcal{O}_R^\theta(g)$$

for all $R > 0$.

Proof. Fix a Cartan decomposition $g = ka\ell \in \mathsf{K}\mathsf{A}^+\mathsf{K}$. Then $U_\theta(g) = k\,\mathsf{P}_\theta$.

Fix any sequence $\{H_n\} \subset \mathfrak{a}^+$ such that $\alpha(H_n) \to +\infty$ for all $\alpha \in \Delta$. Then viewing $k\,\mathsf{P}_\Delta$ as an element of $\partial_\Delta X$, Proposition 4.7 implies that

$$(k\,\mathsf{P}_\Delta)(ho) = \lim_{n\to+\infty} b_{ke^{H_n}o}(ho) = \lim_{n\to+\infty} \kappa(h^{-1}ke^{H_n}) - \kappa(ke^{H_n})$$

for all $h \in \mathsf{G}$. If $a = e^H$ where $H \in \mathfrak{a}^+$, then

$$\begin{aligned} g^{-1}\cdot(k\,\mathsf{P}_\Delta)(g^{-1}o) &= (k\,\mathsf{P}_\Delta)(gg^{-1}o) - (k\,\mathsf{P}_\Delta)(go) \\ &= 0 - \lim_{n\to+\infty} \kappa(\ell^{-1}e^{-H}k^{-1}ke^{H_n}) - \kappa(ke^{H_n}) \\ &= -\lim_{n\to+\infty} H_n - H - H_n = H = \kappa(a) = \kappa(g). \end{aligned}$$

So $k\,\mathsf{P}_\Delta \in \mathcal{O}_R^\Delta(g)$, which implies that

$$U_\theta(g) = \pi_\theta(k\,\mathsf{P}_\Delta) \in \mathcal{O}_R^\theta(g).$$

□

For the next result, we fix any metric generating the topology on $\overline{X}^\theta$.

Lemma 5.6. *For a sequence* $\{g_n\} \subset \mathsf{G}$*, if* $\min_{\alpha\in\theta} \alpha(\kappa(g_n)) \to +\infty$*, then*

$$\operatorname{diam} \mathcal{O}_R^\theta(g_n) \to 0.$$

Proof. Fix $R > 0$ and a sequence $\{g_n\} \subset \mathsf{G}$ such that $\alpha(\kappa(g_n)) \to +\infty$ for all $\alpha \in \theta$. It suffices to consider the case where $g_n o \to \xi \in \partial_\theta X$.

Suppose the lemma fails. Then, after possibly passing to a subsequence of $\{g_n\}$, we can find a sequence $\{\xi_n\} \subset \partial_\theta X$ such that

$$g_n \xi_n \in \mathcal{O}_R^\theta(g_n) \text{ for all } n \in \mathbb{N} \quad \text{ and } \quad g_n \xi_n \to \eta \neq \xi.$$

After passing to a subsequence, we can suppose that $\frac{\Phi_\alpha(g_n)}{\|\Phi_\alpha(g_n)\|} \to S_\alpha$ in $\mathrm{End}(V_\alpha)$ for all $\alpha \in \theta$. Then by Lemma 4.6,

$$N_\alpha \omega_\alpha \xi(ho) = \log \left\| \Phi_\alpha(h^{-1}) S_\alpha \right\|$$

for all $h \in \mathsf{G}$ and $\alpha \in \theta$. Further, Lemma 3.5 implies that each S_α has rank one.

For each n and $\alpha \in \theta$, using Lemma 4.6 we can fix $T_\alpha^{\xi_n} \in \mathrm{End}(V_\alpha)$ with $\left\|T_\alpha^{\xi_n}\right\| = 1$ such that

$$\begin{aligned} N_\alpha \omega_\alpha g_n \xi_n(ho) &= N_\alpha \omega_\alpha \xi_n(g_n^{-1} ho) - N_\alpha \omega_\alpha \xi_n(g_n^{-1} o) \\ &= \log \left\| \Phi_\alpha(h^{-1}) \frac{\Phi_\alpha(g_n) T_\alpha^{\xi_n}}{\left\| \Phi_\alpha(g_n) T_\alpha^{\xi_n} \right\|} \right\| \end{aligned}$$

for all $h \in \mathsf{G}$. Passing to a subsequence, we can suppose that $T_\alpha^{\xi_n} \to T_\alpha$ in $\mathrm{End}(V_\alpha)$ for all $\alpha \in \theta$.

Since $g_n \xi_n \in \mathcal{O}_R^\theta(g_n)$, we have that

$$\omega_\alpha \xi_n(g_n^{-1} o) \geq \omega_\alpha \kappa(g_n) - R = \frac{1}{N_\alpha} \log \|\Phi_\alpha(g_n)\| - R.$$

So

$$\left\| \frac{\Phi_\alpha(g_n)}{\|\Phi_\alpha(g_n)\|} T_\alpha^{\xi_n} \right\| \geq e^{-N_\alpha R},$$

which implies that $S_\alpha T_\alpha \neq 0$. Then, since $g_n \xi_n \to \eta$, we then have

$$N_\alpha \omega_\alpha \eta(ho) = \log \left\| \Phi_\alpha(h^{-1}) \frac{S_\alpha T_\alpha}{\|S_\alpha T_\alpha\|} \right\|$$

for all $\alpha \in \theta$ and $h \in \mathsf{G}$. Notice that $\frac{S_\alpha T_\alpha}{\|S_\alpha T_\alpha\|}$ has rank one, operator norm one, and the same image as S_α. Since $\|S_\alpha\| = 1$ as well, we have $\frac{S_\alpha T_\alpha}{\|S_\alpha T_\alpha\|} = S_\alpha U$ for some orthogonal matrix $U \in \mathsf{SL}(V_\alpha)$ (recall that V_α has a fixed inner product). Thus

$$N_\alpha \omega_\alpha \eta(ho) = \log \left\| \Phi_\alpha(h^{-1}) S_\alpha \right\| = N_\alpha \omega_\alpha \xi(ho)$$

for all $\alpha \in \theta$ and $h \in \mathsf{G}$. Thus $\eta = \xi$ and we have a contradiction. □

5.2. **Shadows from symmetric spaces.** On $\mathcal{F}_\theta$, another natural definition of shadows involves balls and flats in the symmetric space. As before, let $X = \mathsf{G}/\mathsf{K}$ denote the symmetric space associated to G endowed with a symmetric metric and let $o = K \in X$.

For $R > 0$ and $g \in \mathsf{G}$, the *symmetric space shadow* $O_R^\theta(o, go) \subset \mathcal{F}_\theta$ of the ball $B_X(go, R) \subset X$ of radius $R > 0$ and center go is defined by

$$O_R^\theta(o, go) := \left\{ k\,\mathsf{P}_\theta \in \mathcal{F}_\theta : k \in \mathsf{K} \text{ and } k\mathsf{A}^+ o \cap B_X(go, R) \neq \emptyset \right\}.$$

Notice that $O_R^\theta(o, go)$ is an open subset of $\mathcal{F}_\theta$ while $\mathcal{O}_R^\theta(g)$ is an open subset of a larger space, namely $\partial_\theta X$. The following proposition relates the two shadows and part (2) implies that the contracting conical limit set could be defined using symmetric space shadows.

Proposition 5.7.

(1) *For any* $R > 0$*, there exists* $r > 0$ *such that*

$$O_R^\theta(o, go) \subset \mathcal{O}_r^\theta(g) \quad \text{for all} \quad g \in \mathsf{G}\,.$$

(2) *For any* $R > 0$*, there exists* $r > 0$ *such that if* $\{g_n\} \subset \mathsf{G}$ *is a sequence with* $\min_{\alpha\in\theta} \alpha(\kappa(g_n)) \to +\infty$ *and* $\bigcap_{n\geq 0} \mathcal{O}_R^\theta(g_n) \neq \emptyset$*, then*

$$\bigcap_{n\geq 0} \mathcal{O}_R^\theta(g_n) = \bigcap_{n\geq N_0} O_r^\theta(o, g_n o) \quad \text{for some } N_0.$$

Proof. We first prove the part (1). Fix $R > 0$. Then there exists $C > 0$ such that

$$\left\| B_\theta^{IW}(g, g^{-1}x) - \pi_\theta \kappa(g) \right\| \leq C$$

for all $g \in \mathsf{G}$ and $x \in O_R^\theta(o, go)$ ([LO23, Lemma 5.7], [KOW25b, Lemma 5.10]). Hence, there exists $r > 0$ such that for each $\alpha \in \theta$,

$$\omega_\alpha B_\theta^{IW}(g, g^{-1}x) > \omega_\alpha \kappa(g) - r$$

for all $g \in \mathsf{G}$ and $x \in O_R^\theta(o, go)$. By Proposition 4.7, this implies

$$O_R^\theta(o, go) \subset \mathcal{O}_r^\theta(g).$$

We now prove the part (2). We start by proving the following weaker claim:

Claim: For any $R > 0$, there exists $r > 0$ such that if $\{g_n\} \subset \mathsf{G}$ is a sequence with $\min_{\alpha\in\theta} \alpha(\kappa(g_n)) \to +\infty$ and $\bigcap_{n\geq 0} \mathcal{O}_R^\theta(g_n) \neq \emptyset$, then

$$\bigcap_{n\geq 0} \mathcal{O}_R^\theta(g_n) \subset \bigcap_{n\geq N_0} O_r^\theta(o, g_n o) \quad \text{for some } N_0.$$

Proof of Claim: Suppose not. Then for each $k \in \mathbb{N}$, there exist a sequence $\{g_{k,n}\} \subset \mathsf{G}$ with $\min_{\alpha\in\theta} \alpha(\kappa(g_{k,n})) \to +\infty$ as $n \to +\infty$ and $\{x_k\} = \bigcap_{n\in\mathbb{N}} \mathcal{O}_R^\theta(g_{k,n})$ while $x_k \notin O_k^\theta(o, g_{k,n}o)$ for infinitely many $n \in \mathbb{N}$. Lemmas 5.5 and 5.6 imply that $x_k \in \mathcal{F}_\theta$.

Then for each $k \in \mathbb{N}$, we can choose $n_k \in \mathbb{N}$ so that $x_k \notin O_k^\theta(o, g_{k,n_k}o)$ and $\min_{\alpha\in\theta} \alpha(\kappa(g_{k,n_k})) \geq k$ for all $k \in \mathbb{N}$. Setting $g_k := g_{k,n_k}$ for simplicity, we have

$$g_k^{-1}x_k \notin g_k^{-1} O_k^\theta(o, g_k o) \quad \text{for all } k \in \mathbb{N}\,.$$

After passing to a subsequence, we may assume that $g_k^{-1}x_k \to z \in \mathcal{F}_\theta$ and $U_{\mathrm{i}^*\theta}(g_k^{-1}) \to y \in \mathcal{F}_{\mathrm{i}^*\theta}$. By [KOW25a, Proposition 3.4], we have that y and z are not transverse.

On the other hand, since $x_k \in \mathcal{O}_R^\theta(g_k)$, Proposition 4.7 implies that for each $\alpha \in \theta$,

$$\omega_\alpha B_\theta^{IW}(g_k, g_k^{-1}x_k) = \omega_\alpha \left((g_k^{-1} \cdot x_k)(g_k^{-1}o) \right) > \omega_\alpha \kappa(g_k) - R.$$

For $\alpha \in \theta$ and $k \in \mathbb{N}$, fix a unit vector $v_{\alpha,k} \in V_\alpha$ with $\zeta_\alpha(g_k^{-1}x_k) = [v_{\alpha,k}]$. By Proposition 4.7,

$$\omega_\alpha B_\theta^{IW}(g_k, g_k^{-1}x_k) = \frac{1}{N_\alpha} \log \|\Phi_\alpha(g_k) v_{\alpha,k}\|\,.$$

Hence by Property (R1),

$$\left\| \frac{\Phi_\alpha(g_k)}{\|\Phi_\alpha(g_k)\|} v_{\alpha,k} \right\| > e^{-N_\alpha R}.$$

Taking the limit $k \to +\infty$, we may assume that $v_{\alpha,k} \to v_\alpha \in V_\alpha$ and $\frac{\Phi_\alpha(g_k)}{\|\Phi_\alpha(g_k)\|} \to T_\alpha \in \mathrm{End}(V_\alpha)$. Then we have

$$\|T_\alpha v_\alpha\| \geq e^{-N_\alpha R}.$$

In particular, $\zeta_\alpha(z) = [v_\alpha] \notin \ker T_\alpha = \zeta^*_\alpha(y)$ by Lemma 3.5. Since this holds for all $\alpha \in \theta$, y and z are transverse by Property (R3)(b). This is a contradiction, finishing the proof of the claim. ◀

Now suppose $\{g_n\}$ satisfies the hypothesis of part (2). By part (1) we can fix $R' > R$ such that $O^\theta_r(o, go) \subset \mathcal{O}^\theta_{R'}(g)$ for all $g \in \mathsf{G}$. Then

$$\bigcap_{n\geq 0} \mathcal{O}^\theta_R(g_n) \subset \bigcap_{n\geq N_0} O^\theta_r(o, g_n o) \subset \bigcap_{n\geq 0} \mathcal{O}^\theta_{R'}(g_n) = \bigcap_{n\geq 0} \mathcal{O}^\theta_R(g_n),$$

where the last equality uses Lemma 5.6. □

Example 5.8. Consider $\mathsf{G} = \mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R})$. Then

- the associated symmetric space is $\mathbb{H}^2 \times \mathbb{H}^2$ with the symmetric space distance $\mathrm{d}((x_1,x_2),(y_1,y_2)) = \sqrt{\mathrm{d}_{\mathbb{H}^2}(x_1,y_1)^2 + \mathrm{d}_{\mathbb{H}^2}(x_2,y_2)^2}$,
- $\Delta = \{\alpha_1, \alpha_2\}$, where α_1 and α_2 are simple roots for the first and the second $\mathsf{PSL}(2,\mathbb{R})$-components respectively, and
- the Furstenberg boundary is $\partial\,\mathbb{H}^2 \times \partial\,\mathbb{H}^2$.

Setting

$$\Gamma := \Gamma_0 \times \{\mathrm{id}\} < \mathsf{G}$$

where $\Gamma_0 < \mathsf{PSL}(2,\mathbb{R})$ is a cocompact lattice, we compute its limit set and conical limit set.

First, since all elements of Γ have identity on their second component, we have

$$\Lambda_\Delta(\Gamma) = \emptyset.$$

Fix a basepoint $o = (o_1, o_2) \in \mathbb{H}^2 \times \mathbb{H}^2$. Then for $R > 0$ and $\gamma = (\gamma_0, \mathrm{id}) \in \Gamma$, the symmetric space shadow is

$$O_R(o, \gamma o) = \left\{ (x,y) \in \partial\,\mathbb{H}^2 \times \partial\,\mathbb{H}^2 : \begin{array}{c} \exists \text{ geodesic ray from } o_1 \text{ to } x \text{ in } \mathbb{H}^2 \\ \text{intersecting } B_{\mathbb{H}^2}(\gamma_0 o_1, R) \end{array} \right\}.$$

In particular, $O_R(o, \gamma o) = O_R(o_1, \gamma_0 o_1) \times \partial\,\mathbb{H}^2$. Hence for any escaping $\{\gamma_n\} \subset \Gamma$ and $R > 0$, $\bigcap_{n\geq 1} O_R(o, \gamma_n o)$ is not a singleton as long as it is non-empty. In addition, by Proposition 5.7 and the fact that Γ_0 is a cocompact lattice, the above observation implies that

$$\Lambda^{\mathrm{con}}_\Delta(\Gamma) = \partial\,\mathbb{H}^2 \times \partial\,\mathbb{H}^2.$$

5.3. **Proofs of Propositions 5.2 and 5.3.**

Proof of Proposition 5.2. Lemma 5.4 implies that $\Lambda^{\mathrm{concon}}_\theta(\Gamma)$ and $\Lambda^{\mathrm{con}}_\theta(\Gamma)$ are Γ-invariant. Lemmas 5.5 and 5.6 imply that $\Lambda^{\mathrm{concon}}_\theta(\Gamma) \subset \Lambda_\theta(\Gamma)$. □

Proof of Proposition 5.3. The first assertion in part (1) follows from the definition of a transverse group and the "hence" part follows from Proposition 5.2 and the first assertion.

For part (2), first suppose that $x \in \Lambda^{\mathrm{con}}_\theta(\Gamma) = \Lambda^{\mathrm{concon}}_\theta(\Gamma)$. Then there exist $R > 0$ and a sequence $\{\gamma_n\} \subset \Gamma$ such that $\alpha(\kappa(\gamma_n)) \to +\infty$ for all $\alpha \in \theta$ and $x \in \bigcap_{n\geq 1} \mathcal{O}^\theta_R(\gamma_n)$. Then Lemmas 5.5 and 5.6 imply that $U_\theta(\gamma_n) \to x$. Then it follows from Proposition 5.7(2) that $x \in \bigcap_{n\geq 1} O^\theta_r(o, \gamma_n o)$ for some $r > 0$. By [KOW25b, Lemma 5.8], for any $y \in \mathcal{F}_{\mathrm{i}^*\theta}$ transverse to x, the sequence $\gamma_n^{-1}(x,y)$ converges to a transverse pair in $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}^*\theta}$. Since any two distinct points in $\Lambda_{\theta\cup\mathrm{i}^*\theta}(\Gamma)$ are transverse and the projections $\Lambda_{\theta\cup\mathrm{i}^*\theta}(\Gamma) \to \Lambda_\theta(\Gamma)$ and $\Lambda_{\theta\cup\mathrm{i}^*\theta}(\Gamma) \to \Lambda_{\mathrm{i}^*\theta}(\Gamma)$ are Γ-equivariant homeomorphisms (Observation 3.9), this implies that x is a conical limit point in the sense of the convergence action of Γ on $\Lambda_\theta(\Gamma)$.

Conversely, suppose that $x \in \Lambda_\theta(\Gamma)$ is a conical limit point in the sense of the convergence action of Γ on $\Lambda_\theta(\Gamma)$. Then there exist $a, b \in \Lambda_\theta(\Gamma)$ distinct and an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n^{-1}x \to a$ and $\gamma_n^{-1}y \to b$ for all $y \in \Lambda_\theta(\Gamma) \smallsetminus \{x\}$. Then Proposition 3.2 implies that $U_\theta(\gamma_n) \to x$. Further, since the projections $\Lambda_{\theta\cup\mathrm{i}^*\theta}(\Gamma) \to \Lambda_\theta(\Gamma)$ and $\Lambda_{\theta\cup\mathrm{i}^*\theta}(\Gamma) \to \Lambda_{\mathrm{i}^*\theta}(\Gamma)$ are Γ-equivariant homeomorphisms, $\gamma_n^{-1}(x,y)$ converges to a transverse pair in $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}^*\theta}$ for any $y \in \Lambda_{\mathrm{i}^*\theta}(\Gamma)$ which is transverse to x. By [KOW25b, Lemma 5.8], this implies that $x \in \bigcap_{n\geq 1} O_R^\theta(o, \gamma_n o)$ for some $R > 0$. Then by Proposition 5.7(1), $x \in \Lambda_\theta^{\mathrm{concon}}(\Gamma)$. □

6. Verifying the PS-system axioms

In this section, we consider a discrete subgroup $\Gamma < \mathsf{G}$ and verify the axioms of PS-systems for the boundaries $\overline{X}^\theta$.

Theorem 6.1. *Suppose $\theta \subset \Delta$ and $\phi \in \mathfrak{a}_\theta^*$. If $\Gamma < \mathsf{G}$ is strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible and μ is a coarse (Γ, ϕ, δ)-Patterson–Sullivan measure on $\partial_\theta X$, then $(\partial_\theta X, \Gamma, \phi \circ B_\theta, \mu)$ is a PS-system, with magnitude $\|\gamma\|_\phi := \phi(\kappa(\gamma))$ and the R-shadows $\mathcal{O}_R^\theta(\gamma)$ for each $\gamma \in \Gamma$. Moreover, (PS5) holds.*

For transverse groups, we can show that the system is well-behaved.

Theorem 6.2. *Suppose $\theta \subset \Delta$, $\phi \in \mathfrak{a}_\theta^*$, and $\Gamma < \mathsf{G}$ is a strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible P_θ-transverse group. Let μ be a coarse (Γ, ϕ, δ)-Patterson–Sullivan measure on $\partial_\theta X$. Then the PS-system $(\partial_\theta X, \Gamma, \phi \circ B_\theta, \mu)$ in Theorem 6.1 is well-behaved with respect to the trivial hierarchy $\mathcal{H}(R) \equiv \Gamma$.*

Using work in [CZZ24] we will show that for transverse groups, divergence of the Poincaré series implies that there is a unique PS-measure.

Theorem 6.3. *Suppose $\theta \subset \Delta$ and $\Gamma < \mathsf{G}$ is a strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible P_θ-transverse group. If $\phi \in \mathfrak{a}_\theta^*$, $\delta^\phi(\Gamma) < +\infty$, and*

$$\sum_{\gamma\in\Gamma} e^{-\delta^\phi(\Gamma)\phi(\kappa(\gamma))} = +\infty,$$

then there is a unique $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure μ on $\partial_\theta X$, the Γ-action on $(\partial_\theta X, \mu)$ is ergodic, and

$$\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 1$$

(in particular, μ is supported on $\mathcal{F}_\theta$).

Remark 6.4. Previously, Canary, Zhang, and the second author proved in [CZZ24] uniqueness for measures supported on the limit set $\Lambda_\theta(\Gamma) \subset \mathcal{F}_\theta$. When $\Gamma < \mathsf{G}$ is Zariski dense, the first author, Oh, and Wang previously proved uniqueness for measures supported on $\mathcal{F}_\theta$ in [KOW25b]. Since a Zariski dense subgroup is strongly $(\Phi_\alpha)_{\alpha\in\theta}$-irreducible, the above theorem generalizes this uniqueness result.

In the arguments that follow it will be helpful to have the following terminology. Given $\xi \in \partial_\theta X$, we say that $(T^\xi_\alpha)_{\alpha\in\theta} \in \prod_{\alpha\in\theta} \operatorname{End}(V_\alpha)$ *represents* ξ if

- $N_\alpha \omega_\alpha \xi(go) = \log \left\| \Phi_\alpha(g^{-1}) T^\xi_\alpha \right\|$ for all $g \in \mathsf{G}$ and all $\alpha \in \theta$,
- each T^ξ_α is a limit of elements of the form $\frac{1}{\|\Phi_\alpha(g)\|}\Phi_\alpha(g)$.

Lemma 4.6 implies that every element $\xi \in \partial_\theta X$ is represented by such a list, but the representation is not unique. For instance, one can always right-multiply by elements in $\Phi_\alpha(\mathsf{K})$.

6.1. **Proof of Theorem 6.1.** We verify Property (PS1), Property (PS2), and Property (PS5). Since Property (PS5) implies Property (PS3), this completes the proof of the theorem.

Property (PS1): Recall that for $g \in \mathsf{G}$ and $\xi \in \partial_\theta X$, $B_\theta(g,\xi) = \xi(g^{-1}o)$. Since $\xi(o) = 0$ for all $\xi \in \partial_\theta X$, Lemma 4.1 implies Property (PS1).

Property (PS2): By Lemma 4.6, for any $\xi \in \partial_\theta X$, $g \in \mathsf{G}$, and $\alpha \in \theta$ we have

$$\omega_\alpha B_\theta(g,\xi) = \omega_\alpha \xi(g^{-1}o) \le \omega_\alpha \kappa(g).$$

Further, for $\xi \in g^{-1}\,\mathcal{O}^\theta_R(g)$, we have

$$\omega_\alpha \kappa(g) - R \le \omega_\alpha \xi(g^{-1}o) = \omega_\alpha B_\theta(g,\xi).$$

Since $\phi \in \mathfrak{a}^*_\theta$ is a linear combination of the $\{\omega_\alpha : \alpha \in \theta\}$, this implies Property (PS2).

Property (PS5): Let $\{\gamma_n\} \subset \Gamma$ and $R_n > 0$ be sequences such that $R_n \to +\infty$ and

$$\left[\partial_\theta X \smallsetminus \gamma_n^{-1}\,\mathcal{O}^\theta_{R_n}(\gamma_n)\right] \to Z$$

with respect to the Hausdorff distance.

Then

$$\begin{aligned}
\partial_\theta X \smallsetminus \gamma_n^{-1}\,\mathcal{O}^\theta_{R_n}(\gamma_n) &= \bigcup_{\alpha\in\theta} \left\{\xi \in \partial_\theta X : \omega_\alpha \xi(\gamma_n^{-1}o) \le \omega_\alpha\kappa(\gamma_n) - R_n\right\} \\
&= \bigcup_{\alpha\in\theta} \left\{\xi \in \partial_\theta X : \left\| \frac{\Phi_\alpha(\gamma_n) T^\xi_\alpha}{\|\Phi_\alpha(\gamma_n)\|} \right\| \le e^{-N_\alpha R_n} \quad \begin{array}{c}\text{for all } T^\xi_\alpha \\ \text{representing } \xi\end{array} \right\}.
\end{aligned}$$

After passing to a subsequence, we can suppose that $\frac{\Phi_\alpha(\gamma_n)}{\|\Phi_\alpha(\gamma_n)\|} \to S_\alpha \in \operatorname{End}(V_\alpha)$ for each $\alpha \in \theta$. Then

$$Z \subset \bigcup_{\alpha\in\theta} \left\{\xi \in \partial_\theta X : S_\alpha T^\xi_\alpha = 0 \quad \begin{array}{c}\text{for all } T^\xi_\alpha \\ \text{representing } \xi\end{array} \right\}.$$

Now fix $h_1, \dots, h_m \in \Gamma$ and $\xi \in Z$. Fix a representative $(T^\xi_\alpha)_{\alpha\in\theta}$ of ξ. Using Lemma 3.5 and Lemma 3.8, we can fix $\gamma \in \Gamma$ such that

$$S_\alpha \Phi_\alpha(h_j^{-1}\gamma) T^\xi_\alpha \neq 0$$

for all $\alpha \in \theta$ and all $1 \le j \le m$. We claim that

$$\gamma\xi \notin \bigcup_{j=1}^m h_j Z.$$

Notice that

$$(h_j^{-1}\gamma\xi)(go) = \xi(\gamma^{-1}h_j go) - \xi(\gamma^{-1}h_j o)$$

and so

$$\left(T_\alpha^{h_j^{-1}\gamma\xi}\right)_{\alpha\in\theta} := \left(\frac{1}{\left\|\Phi_\alpha(h_j^{-1}\gamma)T_\alpha^\xi\right\|}\Phi_\alpha(h_j^{-1}\gamma)T_\alpha^\xi\right)_{\alpha\in\theta}$$

is a representative of $h_j^{-1}\gamma\xi$. Further,

$$S_\alpha T_\alpha^{h_j^{-1}\gamma\xi} \neq 0$$

for all $\alpha \in \theta$ and all $1 \leq j \leq m$. So

$$h_j^{-1}\gamma\xi \notin Z$$

for all $1 \leq j \leq m$. Hence

$$\gamma\xi \notin \bigcup_{j=1}^m h_j Z.$$

Thus Property (PS5) holds. □

6.2. **Proof of Theorem 6.2.** From Theorem 6.1 we know that Properties (PS1), (PS2), (PS3), and (PS5) hold. Property (PS6) follows from the definition of shadows. Property (PS8) follows from Lemma 5.6 and the definition of a transverse group. It remains to verify Properties (PS4) and (PS7).

We first verify Property (PS4).

Lemma 6.5. *For any $T > 0$, the set $\{\gamma \in \Gamma : \phi(\kappa(\gamma)) \leq T\}$ is finite.*

Proof. For Patterson–Sullivan measures supported on $\Lambda_\theta(\Gamma)$ this was verified in [BCZZ24a, Proposition 10.1] and we will reduce to this case.

Suppose for a contradiction that there exists $T > 0$ and an infinite sequence of distinct elements $\{\gamma_n\} \subset \Gamma$ with $\phi(\kappa(\gamma_n)) \leq T$ for all n. Passing to a subsequence we can suppose that $U_\theta(\gamma_n) \to x$. Then $x \in \Lambda_\theta(\Gamma)$. Fix $R > 0$ large enough to satisfy the Shadow Lemma (Proposition 2.2). Then there exists $\epsilon > 0$ such that

$$\mu(\mathcal{O}_R^\theta(\gamma_n)) \geq \epsilon$$

for all $n \geq 1$. Further, for any open set $\mathcal{U}$ containing x in $\partial_\theta X$, Lemma 5.5 and Lemma 5.6 imply that

$$\mathcal{O}_R^\theta(\gamma_n) \subset \mathcal{U}$$

for all n sufficiently large. So $\mu(\mathcal{U}) \geq \epsilon$. Since $\mathcal{U}$ is an arbitrary open set containing x, we must have $\mu(\{x\}) \geq \epsilon$. Then

$$\mu'(\cdot) := \frac{1}{\mu(\Gamma x)}\mu(\Gamma x \cap (\cdot))$$

defines a coarse (Γ, ϕ, δ)-Patterson–Sullivan measure supported on $\Lambda_\theta(\Gamma)$. Now by [BCZZ24a, Proposition 10.1], the set $\{\gamma \in \Gamma : \phi(\kappa(\gamma)) \leq T\}$ is finite, which is a contradiction. □

We establish two lemmas before proving Property (PS7). The following is a special case of [BCZZ24b, Proposition 3.3(7)].

Lemma 6.6. *For any $\{g_n\}, \{h_n\} \subset \Gamma$ such that $\phi(\kappa(g_n)) \leq \phi(\kappa(h_n))$ for all $n \geq 1$ and $\{g_n^{-1}h_n\}$ is escaping, we have*

$$\mathrm{d}(U_\theta(h_n^{-1}), U_\theta(h_n^{-1}g_n)) \to 0 \quad \text{as } n \to +\infty$$

where d *is any metric on $\mathcal{F}_\theta$ compatible to the standard topology on $\mathcal{F}_\theta$.*

Remark 6.7. If $\{g_n^{-1}h_n\}$ is escaping and $\phi(\kappa(g_n)) \leq \phi(\kappa(h_n))$ for all $n \geq 1$, then Lemma 6.5 implies that $\{h_n\}$ is escaping. Then by the definition of transverse groups,

$$\lim_{n\to+\infty} \alpha(\kappa(h_n^{-1})) = +\infty = \lim_{n\to+\infty} \alpha(\kappa(h_n^{-1}g_n))$$

for all $\alpha \in \theta$. Thus $U_\theta(h_n^{-1})$ and $U_\theta(h_n^{-1}g_n)$ are both well-defined for n sufficiently large.

Proof. Suppose not. Then we can find $\{g_n\}, \{h_n\} \subset \Gamma$ such that $\phi(\kappa(g_n)) \leq \phi(\kappa(h_n))$ for all $n \geq 1$, $\{g_n^{-1}h_n\}$ is escaping, and

$$\liminf_{n\to+\infty} \mathrm{d}(U_\theta(h_n^{-1}), U_\theta(h_n^{-1}g_n)) > 0.$$

Passing to a subsequence we can suppose that $U_\theta(h_n^{-1}) \to x$ and $U_\theta(h_n^{-1}g_n) \to y$ with $x \neq y$. Since $x, y \in \Lambda_\theta(\Gamma)$, we may assume that θ is symmetric by replacing θ with $\theta \cup \mathrm{i}^*\theta$ (see Observation 3.9), and hence we must have that x and y are transverse due to the P_θ-transversality of Γ.

Fix $\alpha \in \theta$. Passing to a subsequence we can suppose that $\frac{\Phi_\alpha(h_n)}{\|\Phi_\alpha(h_n)\|} \to T_\alpha$ and $\frac{\Phi_\alpha(h_n^{-1}g_n)}{\|\Phi_\alpha(h_n^{-1}g_n)\|} \to S_\alpha$ in $\mathrm{End}(V_\alpha)$. Lemma 3.5 implies that $\ker T_\alpha = \zeta_\alpha^*(x)$ and $\mathrm{im}\, S_\alpha = \zeta_\alpha(y)$. Since x is transverse to y, Property (R3) implies that $T_\alpha S_\alpha \neq 0$. Thus

$$0 < \|T_\alpha S_\alpha\| \leq 1.$$

Then by Property (R1),

$$\begin{aligned}\lim_{n\to+\infty} N_\alpha\omega_\alpha\left(\kappa(g_n) - \kappa(h_n) - \kappa(h_n^{-1}g_n)\right) &= \lim_{n\to+\infty} \log \frac{\left\|\Phi_\alpha(h_n)\Phi_\alpha(h_n^{-1}g_n)\right\|}{\|\Phi_\alpha(h_n)\| \left\|\Phi_\alpha(h_n^{-1}g_n)\right\|}\\ &= \log\|T_\alpha S_\alpha\|\end{aligned}$$

which is finite. Since $\alpha \in \theta$ was arbitrary and $\{\omega_\alpha\}_{\alpha\in\theta}$ is a basis for $\mathfrak{a}_\theta^*$, there exists $C > 0$ such that

$$-C \leq \phi\left(\kappa(g_n) - \kappa(h_n) - \kappa(h_n^{-1}g_n)\right) \leq C$$

for all $n \geq 1$.

Since the map $\gamma \in \Gamma \mapsto \phi(\kappa(\gamma)) \in \mathbb{R}$ is proper by Lemma 6.5,

$$0 \geq \lim_{n\to+\infty} \phi(\kappa(g_n)) - \phi(\kappa(h_n)) \geq -C + \lim_{n\to+\infty} \phi(\kappa(h_n^{-1}g_n)) = +\infty$$

and we have a contradiction. □

Lemma 6.8. *For every $R > 0$ there exists $R' > 0$ such that: if $\gamma_1, \gamma_2 \in \Gamma$, $\mathcal{O}_R^\theta(\gamma_1) \cap \mathcal{O}_R^\theta(\gamma_2) \neq \emptyset$, and $\phi(\kappa(\gamma_1)) \leq \phi(\kappa(\gamma_2))$, then*

$$\gamma_1^{-1}\,\mathcal{O}_R^\theta(\gamma_2) \subset \mathcal{O}_{R'}^\theta(\gamma_1^{-1}\gamma_2).$$

Proof. Suppose not. Then for every $n \geq 1$ we can fix $g_n, h_n \in \Gamma$ and $\xi_n \in \partial_\theta X$ such that $\mathcal{O}_R^\theta(g_n) \cap \mathcal{O}_R^\theta(h_n) \neq \emptyset$, $\phi(\kappa(g_n)) \leq \phi(\kappa(h_n))$, and

$$\xi_n \in h_n^{-1}\,\mathcal{O}_R^\theta(h_n) \smallsetminus h_n^{-1}g_n\,\mathcal{O}_n^\theta(g_n^{-1}h_n).$$

After passing to a subsequence, we can fix $\alpha \in \theta$ such that

$$\omega_\alpha\xi_n(h_n^{-1}o) > \omega_\alpha\kappa(h_n) - R \quad \text{and} \quad \omega_\alpha\xi_n(h_n^{-1}g_no) \leq \omega_\alpha\kappa(g_n^{-1}h_n) - n$$

for all $n \geq 1$.

For each n, fix $(T_\beta^{\xi_n})_{\beta\in\theta}$ representing ξ_n. Then by Property (R1),

$$\left\| \frac{\Phi_\alpha(h_n)}{\|\Phi_\alpha(h_n)\|} T_\alpha^{\xi_n} \right\| \geq e^{-N_\alpha R} \quad \text{and} \quad \left\| \frac{\Phi_\alpha(g_n^{-1}h_n)}{\|\Phi_\alpha(g_n^{-1}h_n)\|} T_\alpha^{\xi_n} \right\| \leq e^{-N_\alpha n}.$$

Passing to a subsequence, we can assume

$$\frac{\Phi_\alpha(h_n)}{\|\Phi_\alpha(h_n)\|} \to S_\alpha, \quad \frac{\Phi_\alpha(g_n^{-1}h_n)}{\|\Phi_\alpha(g_n^{-1}h_n)\|} \to S'_\alpha, \quad \text{and} \quad T_\alpha^{\xi_n} \to T_\alpha$$

in $\mathrm{End}(V_\alpha)$. Then we have

$$S_\alpha T_\alpha \neq 0 \quad \text{and} \quad S'_\alpha T_\alpha = 0.$$

The second equality implies that $S'_\alpha \notin \mathsf{SL}(V_\alpha)$ and hence $\{g_n^{-1}h_n\}$ must be escaping. Then by Lemmas 6.5 and 6.6 applied to $\theta \cup \mathrm{i}^*\theta$, we have

$$\mathrm{d}(U_{\theta\cup \mathrm{i}^*\theta}(h_n^{-1}), U_{\theta\cup \mathrm{i}^*\theta}(h_n^{-1}g_n)) \to 0.$$

Then Lemma 3.5 implies that $\ker S_\alpha = \ker S'_\alpha$, but this is impossible since $S_\alpha T_\alpha \neq 0$ and $S'_\alpha T_\alpha = 0$. □

We verify the two parts of Property (PS7) separately.

Lemma 6.9. *For every $R > 0$ there exists $R' > 0$ such that: if $\gamma_1, \gamma_2 \in \Gamma$, $\mathcal{O}_R^\theta(\gamma_1) \cap \mathcal{O}_R^\theta(\gamma_2) \neq \emptyset$, and $\phi(\kappa(\gamma_1)) \leq \phi(\kappa(\gamma_2))$, then*

$$\mathcal{O}_R^\theta(\gamma_2) \subset \mathcal{O}_{R'}^\theta(\gamma_1).$$

Proof. Suppose not. Then for every $n \in \mathbb{N}$ we can find $g_n, h_n \in \Gamma$ such that $\phi(\kappa(g_n)) \leq \phi(\kappa(h_n))$, $\mathcal{O}_R^\theta(g_n) \cap \mathcal{O}_R^\theta(h_n) \neq \emptyset$, and

$$\mathcal{O}_R^\theta(h_n) \not\subset \mathcal{O}_n^\theta(g_n).$$

Case 1: Assume that $\{g_n^{-1}h_n\}$ is finite. Then, passing to a subsequence, we can suppose that $\gamma := g_n^{-1}h_n$ for all n.

For each $n \in \mathbb{N}$, fix

$$\xi_n \in g_n^{-1}\,\mathcal{O}_R^\theta(h_n) \smallsetminus g_n^{-1}\,\mathcal{O}_n^\theta(g_n).$$

Passing to a subsequence, there exist $\alpha \in \theta$ such that

$$\omega_\alpha \gamma^{-1}\xi_n(\gamma^{-1}g_n^{-1}o) > \omega_\alpha\kappa(g_n\gamma) - R \quad \text{and} \quad \omega_\alpha \xi_n(g_n^{-1}o) \leq \omega_\alpha \kappa(g_n) - n$$

for all $n \in \mathbb{N}$. Noting that $\gamma^{-1}\xi_n(\gamma^{-1}g_n^{-1}o) = \xi_n(g_n^{-1}o) - \xi_n(\gamma o)$, the two sequences

$$\gamma^{-1}\xi_n(\gamma^{-1}g_n^{-1}o) - \xi_n(g_n^{-1}o) \quad \text{and} \quad \kappa(g_n\gamma) - \kappa(g_n)$$

are uniformly bounded by Lemma 4.1 and Property (R1). This yields a contradiction.

Case 2: Assume that $\{g_n^{-1}h_n\}$ is escaping. By Lemma 6.5 and Lemma 6.6, we have

$$\mathrm{d}(U_\theta(h_n^{-1}), U_\theta(h_n^{-1}g_n)) \to 0.$$

By Lemma 6.8, we can fix $R' > 0$ such that

$$g_n^{-1}\,\mathcal{O}_R^\theta(h_n) \subset \mathcal{O}_{R'}^\theta(g_n^{-1}h_n)$$

for all $n \in \mathbb{N}$. Since $\mathcal{O}_R^\theta(g_n) \cap \mathcal{O}_R^\theta(h_n) \neq \emptyset$ by the hypothesis, we also have $\mathcal{O}_{R'}^\theta(g_n^{-1}h_n) \cap g_n^{-1}\,\mathcal{O}_R^\theta(g_n) \neq \emptyset$ for all large $n \in \mathbb{N}$. Note that $\operatorname{diam} \mathcal{O}_{R'}^\theta(g_n^{-1}h_n) \to 0$ as $n \to +\infty$ by Lemma 5.6. This implies that for any small $r > 0$,

$$g_n^{-1}\,\mathcal{O}_R^\theta(h_n) \subset \mathcal{O}_{R'}^\theta(g_n^{-1}h_n) \subset \mathcal{N}_r(g_n^{-1}\,\mathcal{O}_R^\theta(g_n))$$

for all large $n \in \mathbb{N}$, where $\mathcal{N}_r(\cdot)$ denotes the r-neighborhood in $\partial_\theta X$.

Claim: There exists $r > 0$ such that

$$\mathcal{N}_r(g_n^{-1}\,\mathcal{O}_R^\theta(g_n)) \subset g_n^{-1}\,\mathcal{O}_n^\theta(g_n) \quad \text{for all large } n \in \mathbb{N}\,.$$

Proof of Claim: Suppose not. Then after passing to a subsequence, we have for all $n \in \mathbb{N}$ that

$$\mathcal{N}_{1/n}(g_n^{-1}\,\mathcal{O}_R^\theta(g_n)) \not\subset g_n^{-1}\,\mathcal{O}_n^\theta(g_n).$$

This implies that after passing to a subsequence there exist $\alpha \in \theta$ and sequences $\{\xi_n\}, \{\xi_n'\} \subset \partial_\theta X$ such that $\xi := \lim_{n\to+\infty} \xi_n = \lim_{n\to+\infty} \xi_n'$ and for all $n \in \mathbb{N}$,

$$\omega_\alpha \xi_n(g_n^{-1}o) \le \omega_\alpha \kappa(g_n) - n \quad \text{and} \quad \omega_\alpha \xi_n'(g_n^{-1}o) > \omega_\alpha \kappa(g_n) - R.$$

For each $n \in \mathbb{N}$, let $\{p_{n,k}\}, \{p_{n,k}'\} \subset \mathsf{G}$ be sequences such that $p_{n,k}o \to \xi_n$ and $p_{n,k}'o \to \xi_n'$ in $\overline{X}^\theta$ as $k \to +\infty$. Then by Lemma 4.6 and Property (R1), we have

$$\lim_{k\to+\infty} \left\| \frac{\Phi_\alpha(g_n)}{\|\Phi_\alpha(g_n)\|} \frac{\Phi_\alpha(p_{n,k})}{\|\Phi_\alpha(p_{n,k})\|} \right\| \le e^{-N_\alpha n} \quad \text{and} \quad \lim_{k\to+\infty} \left\| \frac{\Phi_\alpha(g_n)}{\|\Phi_\alpha(g_n)\|} \frac{\Phi_\alpha(p_{n,k}')}{\left\|\Phi_\alpha(p_{n,k}')\right\|} \right\| > e^{-N_\alpha R}$$

Hence, for each $n \in \mathbb{N}$, we can choose $p_n := p_{n,k_n}$ and $p_n' := p_{n,k_n'}$ for some $k_n, k_n' \in \mathbb{N}$ so that $p_n o \to \xi$ and $p_n' o \to \xi$ in $\overline{X}^\theta$, and

$$\lim_{n\to+\infty} \left\| \frac{\Phi_\alpha(g_n)}{\|\Phi_\alpha(g_n)\|} \frac{\Phi_\alpha(p_n)}{\|\Phi_\alpha(p_n)\|} \right\| = 0 \quad \text{and} \quad \lim_{n\to+\infty} \left\| \frac{\Phi_\alpha(g_n)}{\|\Phi_\alpha(g_n)\|} \frac{\Phi_\alpha(p_n')}{\|\Phi_\alpha(p_n')\|} \right\| > \frac{e^{-N_\alpha R}}{2}.$$

After passing to a subsequence, we may assume

$$\frac{\Phi_\alpha(g_n)}{\|\Phi_\alpha(g_n)\|} \to S, \quad \frac{\Phi_\alpha(p_n)}{\|\Phi_\alpha(p_n)\|} \to P, \quad \text{and} \quad \frac{\Phi_\alpha(p_n')}{\|\Phi_\alpha(p_n')\|} \to P' \quad \text{in } \operatorname{End}(V_\alpha).$$

Then it follows from $SP = 0$ that $\{g_n\}$ is escaping, and hence $\alpha(\kappa(g_n)) \to +\infty$. For a sequence $\{\ell_n\} \subset \mathsf{K}$ with $g_n \in \mathsf{KA}^+\ell_n$ for all $n \in \mathbb{N}$, we can assume $\ell_n \to \ell \in \mathsf{K}$ by passing to a subsequence. Then by Lemma 3.5 and Property (R3), we have $\Phi_\alpha(\ell^{-1})V_\alpha^- = \ker S$ and hence

$$\operatorname{im} P \subset \Phi_\alpha(\ell^{-1})V_\alpha^- \quad \text{and} \quad \operatorname{im} P' \not\subset \Phi_\alpha(\ell^{-1})V_\alpha^-.$$

Since $\|P\| \le 1$ and $\operatorname{im} P \subset \Phi_\alpha(\ell^{-1})V_\alpha^-$, we have for any $a \in \mathsf{A}^+$ that

$$\|\Phi_\alpha(a\ell)P\| \le e^{(N_\alpha\omega_\alpha - \alpha)(\kappa(a))}$$

by Properties (R1) and (R2). Since $\operatorname{im} P' \not\subset \Phi_\alpha(\ell^{-1})V_\alpha^-$, there exists a unit $v \in V_\alpha$ such that $\Phi_\alpha(\ell)P'v = u + w$ for some $u \in V_\alpha^+$ and $w \in V_\alpha^-$ with $u \neq 0$. Hence, we similarly have for any $a \in \mathsf{A}^+$ that

$$\|\Phi_\alpha(a\ell)P'\| \ge e^{N_\alpha\omega_\alpha(\kappa(a))}\,\|u\| - e^{(N_\alpha\omega_\alpha - \alpha)(\kappa(a))}\,\|w\|\,.$$

On the other hand, since $\{p_n o\}, \{p_n' o\} \subset \overline{X}^\theta$ converges to the same limit ξ, it follows from Lemma 4.6 that for any $a \in \mathsf{A}^+$,

$$\|\Phi_\alpha(a\ell)P\| = \|\Phi_\alpha(a\ell)P'\|\,,$$

and therefore

$$e^{(N_\alpha \omega_\alpha - \alpha)(\kappa(a))} \geq e^{N_\alpha \omega_\alpha(\kappa(a))} \|u\| - e^{(N_\alpha \omega_\alpha - \alpha)(\kappa(a))} \|w\| .$$

This implies

$$e^{-\alpha(\kappa(a))}(1 + \|w\|) \geq \|u\| .$$

Since this holds for any $a \in \mathsf{A}^+$ and $\|u\| > 0$, this is a contradiction and the claim follows. ◀

Now by the claim, we have

$$g_n^{-1} \mathcal{O}_R^\theta(h_n) \subset \mathcal{O}_{R'}^\theta(g_n^{-1}h_n) \subset \mathcal{N}_r(g_n^{-1} \mathcal{O}_R^\theta(g_n)) \subset g_n^{-1} \mathcal{O}_n^\theta(g_n)$$

for n large, which is a contradiction. This finishes the proof. □

Now we complete the proof of Property (PS7), and hence of Theorem 6.2, by showing the following.

Lemma 6.10. *For every $R > 0$ there exists $C > 0$ such that: if $\gamma_1, \gamma_2 \in \Gamma$, $\mathcal{O}_R^\theta(\gamma_1) \cap \mathcal{O}_R^\theta(\gamma_2) \neq \emptyset$, and $\phi(\kappa(\gamma_1)) \leq \phi(\kappa(\gamma_2))$, then*

$$\left|\phi\left(\kappa(\gamma_2)\right) - \left(\phi\left(\kappa(\gamma_1)\right) + \phi\left(\kappa(\gamma_1^{-1}\gamma_2)\right)\right)\right| \leq C.$$

Proof. Suppose not. Then for every $n \in \mathbb{N}$ we can find $g_n, h_n \in \Gamma$ such that $\phi(\kappa(g_n)) \leq \phi(\kappa(h_n))$, $\mathcal{O}_R^\theta(g_n) \cap \mathcal{O}_R^\theta(h_n) \neq \emptyset$, and

$$\left|\phi(\kappa(h_n)) - \left(\phi(\kappa(g_n)) + \phi(\kappa(g_n^{-1}h_n))\right)\right| > n.$$

First, it is easy to see that $\{g_n\}$ and $\{g_n^{-1}h_n\}$ are escaping sequences. Then we have $\phi(\kappa(h_n)) \geq \phi(\kappa(g_n)) \to +\infty$ by Lemma 6.5. In particular, $\{h_n\}$ is an escaping sequence.

For each $n \in \mathbb{N}$, fix $\xi_n \in \mathcal{O}_R^\theta(g_n) \cap \mathcal{O}_R^\theta(h_n)$. Then by Lemma 6.8, there exists $R' > 0$ such that $\xi_n \in g_n \mathcal{O}_{R'}^\theta(g_n^{-1}h_n)$, and hence

$$h_n^{-1}\xi_n \in (g_n^{-1}h_n)^{-1} \mathcal{O}_{R'}^\theta(g_n^{-1}h_n).$$

Therefore, together with Lemma 4.6, we have for each $\alpha \in \theta$ that

$$\begin{aligned}
\omega_\alpha \kappa(g_n) &\geq \omega_\alpha g_n^{-1}\xi_n(g_n^{-1}o) \geq \omega_\alpha \kappa(g_n) - R \\
\omega_\alpha \kappa(h_n) &\geq \omega_\alpha h_n^{-1}\xi_n(h_n^{-1}o) \geq \omega_\alpha \kappa(h_n) - R \\
\omega_\alpha \kappa(g_n^{-1}h_n) &\geq \omega_\alpha h_n^{-1}\xi_n((g_n^{-1}h_n)^{-1}o) \geq \omega_\alpha \kappa(g_n^{-1}h_n) - R'.
\end{aligned}$$

Notice that

$$\begin{aligned}
h_n^{-1}\xi_n((g_n^{-1}h_n)^{-1}o) &= h_n^{-1}\xi_n(h_n^{-1}g_n o) = \xi_n(h_n h_n^{-1} g_n o) - \xi_n(h_n o) \\
&= h_n^{-1}\xi_n(h_n^{-1}o) - g_n^{-1}\xi_n(g_n^{-1}o).
\end{aligned}$$

Hence for each $\alpha \in \theta$,

$$-R - R' \leq \omega_\alpha \kappa(h_n) - \omega_\alpha \kappa(g_n) - \omega_\alpha \kappa(g_n^{-1}h_n) \leq R.$$

Since ϕ is a linear combination of $\{\omega_\alpha : \alpha \in \theta\}$, this is a contradiction. □

6.3. **Proof of Theorem 6.3.** For Patterson–Sullivan-measures supported on $\Lambda_\theta(\Gamma)$, uniqueness and hence ergodicity was established in [CZZ24, Corollaries 12.1, 12.2]. Using this result and Proposition 5.2 it suffices to fix a $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure μ on $\partial_\theta X$ and show that $\mu(\Lambda_\theta^{\rm con}(\Gamma)) = 1$.

Theorem 2.3 implies that $\mu(\Lambda_\theta^{\rm con}(\Gamma)) > 0$ and Proposition 5.2 implies that $\Lambda_\theta^{\rm con}(\Gamma)$ is Γ-invariant. Suppose for a contradiction that $\mu(\Lambda_\theta^{\rm con}(\Gamma)) < 1$. Then

$$\mu'(\cdot) := \frac{1}{\mu(\partial_\theta X \smallsetminus \Lambda_\theta^{\rm con}(\Gamma))}\mu\big(\cdot \cap (\partial_\theta X \smallsetminus \Lambda_\theta^{\rm con}(\Gamma))\big)$$

is a $(\Gamma, \phi, \delta^\phi(\Gamma))$-Patterson–Sullivan measure μ on $\partial_\theta X$. So Theorem 2.3 applied to μ' implies that $\mu'(\Lambda_\theta^{\rm con}(\Gamma)) > 0$ which is impossible. Thus $\mu(\Lambda_\theta^{\rm con}(\Gamma)) = 1$ and hence uniqueness and ergodicity follow from [CZZ24, Corollaries 12.1, 12.2]. The "in particular" part is due to Proposition 5.3. □

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